

In Exercises 1-4 express the complex numbers in simplified Cartesian form.

1. $\left(\frac{3}{5} + \frac{4}{5}i\right)^{39} \left(\frac{3}{5}i + \frac{4}{5}\right)^{39}$

Solution: There are several ways to simplify the given expression. Here is one:

$$\begin{aligned} \left(\frac{3}{5} + \frac{4}{5}i\right)^{39} \left(\frac{3}{5}i + \frac{4}{5}\right)^{39} &= \left(\frac{3}{5} + \frac{4}{5}i\right)^{39} \left[\left(\frac{3}{5} - \frac{4}{5}i\right)i\right]^{39} = \left[\left(\frac{3}{5} + \frac{4}{5}i\right)\left(\frac{3}{5} - \frac{4}{5}i\right)\right]^{39} i^{39} \\ &= \left[\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2\right]^{39} (i^{36} \cdot i^3) = 1^{39} \cdot (i^4)^9 \cdot (-i) = -i \end{aligned}$$

2. $\frac{(9e^{\frac{\pi i}{9}})(8e^{-\frac{5\pi i}{18}})}{(6e^{-\frac{\pi i}{3}})^2}$

Solution:
$$\frac{(9e^{\frac{\pi i}{9}})(8e^{-\frac{5\pi i}{18}})}{(6e^{-\frac{\pi i}{3}})^2} = \frac{72e^{\frac{\pi i}{9} - \frac{5\pi i}{18}}}{36e^{-\frac{2\pi i}{3}}} = 2e^{\frac{-\pi i}{6} + \frac{2\pi i}{3}} = 2e^{\frac{\pi i}{2}} = 2i$$

3. $\left[2\left(\cos \frac{3\pi}{7}\right) + 2\left(\sin \frac{3\pi}{7}\right)i\right]^7$

Solution:
$$\begin{aligned} &= \left[2\left(\cos \frac{3\pi}{7} + i \sin \frac{3\pi}{7}\right)\right]^7 = 2^7 \left(\cos \frac{3\pi}{7} + i \sin \frac{3\pi}{7}\right)^7 \\ &= 128 (\cos 3\pi + i \sin 3\pi) = -128 \end{aligned}$$

4. $(i - \sqrt{3})^{10}$

Solution: Since $|i - \sqrt{3}| = \sqrt{(-\sqrt{3})^2 + 1^2} = 2$, the exponential form of $i - \sqrt{3}$ is

$$-\sqrt{3} + i = 2\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = 2\left(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6}\right) = 2e^{\frac{5\pi i}{6}}.$$

Therefore

$$(i - \sqrt{3})^{10} = (2e^{\frac{5\pi i}{6}})^{10} = 2^{10} e^{\frac{25\pi i}{3}} = 2^{10} \left(\cos \frac{25\pi}{3} + i \sin \frac{25\pi}{3}\right) = 2^{10} \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = 512 + 512\sqrt{3}i$$

In Exercises 5-6 find all solutions of the equation.

5. $z^4 + 4z^2 + 16 = 0$

Solution: Set $y = z^2$. Then $y^2 + 4y + 16 = 0$. Hence

$$y = \frac{-4 \pm \sqrt{4^2 - 4 \cdot 16}}{2} = \frac{-4 \pm \sqrt{-48}}{2} = -2 \pm 2\sqrt{3}i$$

If $y = -2 + 2\sqrt{3}i$, then

$$z^2 = -2 + 2\sqrt{3}i = 4\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = 4e^{\frac{2\pi i}{3}}.$$

This leads to two solutions:

$$z = \pm 2e^{\frac{\pi i}{3}} = \pm 2\left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right) = \pm 2\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = \pm(1 + i\sqrt{3}).$$

If $y = -2 - 2\sqrt{3}i$, then

$$z^2 = -2 - 2\sqrt{3}i = 4 \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i \right) = 4e^{-\frac{2\pi i}{3}}.$$

This leads to two more solutions:

$$z = \pm 2e^{-\frac{\pi i}{3}} = \pm 2 \left(\cos \frac{-\pi i}{3} + i \sin \frac{-\pi i}{3} \right) = \pm 2 \left(\frac{1}{2} - i \frac{\sqrt{3}}{2} \right) = \pm (1 - i\sqrt{3}).$$

In total, there are four solutions: $z = \pm 1 \pm i\sqrt{3}$.

6. $z^6 - 7z^3 = 8$

Solution: The given equation is equivalent to $z^6 - 7z^3 - 8 = 0$, or

$$(z^3 - 8)(z^3 + 1) = 0.$$

Hence $z^3 = 8$ or $z^3 = -1$.

If $z^3 = 8$, then z can be any of the three cube roots of 8:

$$z = (8e^{2k\pi i})^{1/3} = 2e^{\frac{2k\pi i}{3}} \quad (k = 0, 1, 2).$$

This leads to three solutions: $z = 2$, $z = -1 + i\sqrt{3}$, and $z = -1 - i\sqrt{3}$.

If $z^3 = -1$, then z can be any of the three cube roots of -1 ($= e^{\pi i}$):

$$z = (e^{\pi i} e^{2k\pi i})^{1/3} = e^{\frac{(1+2k)\pi i}{3}} \quad (k = 0, 1, 2).$$

This leads to three more solutions: $z = \frac{1}{2} + \frac{\sqrt{3}}{2}i$, $z = -1$, and $z = \frac{1}{2} - \frac{\sqrt{3}}{2}i$.

In total, there are six solutions.