

1. Base of induction $n=1$:

$$\text{LHS} = \frac{1}{1 \cdot 5} = \frac{1}{5}$$

$$\text{LHS} = \text{RHS}$$

$$\text{RHS} = \frac{1}{4 \cdot 1 + 1} = \frac{1}{5}$$

Hypothesis: Assume for some $n=k \geq 1$

$$\frac{1}{1 \cdot 5} + \frac{1}{5 \cdot 9} + \dots + \frac{1}{(4k-3)(4k+1)} = \frac{k}{4k+1}$$

We need to prove that for $n=k+1$ our equality is valid:

$$\frac{1}{1 \cdot 5} + \frac{1}{5 \cdot 9} + \dots + \frac{1}{(4k-3)(4k+1)} + \frac{1}{(4k+1)(4k+5)} = \frac{k+1}{4k+5}$$

Using hypothesis:

$$\text{LHS} = \frac{k}{4k+1} + \frac{1}{(4k+1)(4k+5)} = \frac{k(4k+5) + 1}{(4k+1)(4k+5)}$$

$$= \frac{4k^2 + 5k + 1}{(4k+1)(4k+5)}$$

$$\text{RHS} = \frac{k+1}{4k+5} = \frac{(4k+1)(k+1)}{(4k+1)(4k+5)} = \frac{4k^2 + 5k + 1}{(4k+1)(4k+5)}$$

So, $\text{LHS} = \text{RHS}$ which proves the statement for $n=k+1$.

So, by PMI, the identity is verified for all $n \geq 1$.

2. Base $n=1$:

$$7 \cdot 5^2 + 1 = 7 \cdot 25 + 1 = 176 = 8 \cdot 22 \text{ - divisible by 8.}$$

Hypothesis: Assume for some $n=k \geq 1$

$$8 \text{ divides } 7 \cdot 5^{2k} + 1.$$

We need to prove our statement for $n = k+1$, i.e. that 2

$$8 \text{ divides } 7 \cdot 5^{2(k+1)} + 1$$

We transform the last expression to factor $7 \cdot 5^{2k} + 1$:

$$7 \cdot 5^{2(k+1)} + 1 = 7 \cdot 5^{2k} \cdot 25 + 1 = 7 \cdot 5^{2k} \cdot 25 + 25 - 25 + 1$$

$$= 25 \underbrace{(7 \cdot 5^{2k} + 1)}_{\substack{\text{divisible by 8} \\ \text{by hypothesis}}} - 24 \rightarrow \text{divisible by 8}$$

So, 8 divides $7 \cdot 5^{2(k+1)} + 1$, and by PMI 8 divides $7 \cdot 5^{2n} + 1$ for any $n \geq 1$.

3. Since $2 = \frac{2}{2^0}$, we have $2 + \frac{3}{2^1} + \dots + \frac{22}{2^{20}} = \sum_{n=1}^{21} \frac{n+1}{2^{n-1}}$.

Alternatively, the following are also correct:

$$\sum_{n=0}^{20} \frac{n+2}{2^n} \quad \text{or} \quad \sum_{n=2}^{22} \frac{n}{2^{n-2}} \quad (\text{and other letters can be used instead of } n)$$

4. Since $\sum_{k=6}^{19} a_k = \sum_{k=1}^{19} a_k - \sum_{k=1}^5 a_k$, we have

$$\sum_{k=6}^{19} (2k - 3k^2 + k^3) = 2 \sum_{k=6}^{19} k - 3 \sum_{k=6}^{19} k^2 + \sum_{k=6}^{19} k^3$$

$$= 2 \left(\frac{19(19+1)}{2} - \frac{5(5+1)}{2} \right) - 3 \left(\frac{19(19+1)(2 \cdot 19+1)}{6} - \frac{5(5+1)(2 \cdot 5+1)}{6} \right)$$

$$+ \left(\frac{19^2 (19+1)^2}{4} - \frac{5^2 (5+1)^2}{4} \right) = 19 \cdot 20 - 5 \cdot 6 - 19 \cdot 10 \cdot 38$$

$$+ 5 \cdot 3 \cdot 11 + 19^2 \cdot 10^2 - 5^2 \cdot 3^2 \quad (\text{no need to compute further})$$

5. Base $n=1, n=2$: $3^1 = 3 > 1^2 = 1^2$ - true
 $3^2 = 9 > 4 = 2^2$ - true

Hypothesis: Assume that for $n=k \geq 2$ we know that

(1) $3^k > k^2$.

We need to prove our inequality for $n=k+1$, i.e.

(2). $3^{k+1} > (k+1)^2$.

To obtain (2) from (1), we need to multiply the inequality (1) by the inequality

$$3 > \left(\frac{k+1}{k} \right)^2$$

The last inequality is true since

$$\sqrt{3} > \frac{3}{2} \geq 1 + \frac{1}{k} = \frac{k+1}{k} \text{ for } k \geq 2.$$

This proves (2), and by PMI $3^n > n^2$ for all $n \geq 1$.

NOTE: Inductive step uses $k \geq 2$, so we have to check both $n=1$ and $n=2$ in the base of induction.