

## Math 1210 Tutorial #10 solutions

1. There are two cases: if  $k = 0$ , then the R.E.F. is 
$$\left( \begin{array}{ccc|c} 1 & -1 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array} \right).$$

If  $k \neq 0$ , then the R.E.F. is 
$$\left( \begin{array}{ccc|c} 1 & -1 & k^2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{array} \right).$$
 Therefore, if  $k = 0$ , the answers are:

(a) 3 (b) 2 (c) 2 (d)  $3 - 2 = 1$  (e) infinitely many.

If  $k \neq 0$ , then:

(a) 3 (b) 3 (c) 2 (d) 0 because the system is inconsistent (e) 0.

2. (a) The coefficient matrix is 
$$\left( \begin{array}{ccccc} 1 & 2 & 1 & -1 & 0 \\ 2 & 1 & -3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 7 \end{array} \right).$$
 Reducing to R.R.E.F.:

$$\begin{aligned} & \xrightarrow{R_2 \rightarrow R_2 - 2R_1} \left( \begin{array}{ccccc} 1 & 2 & 1 & -1 & 0 \\ 0 & -3 & -5 & 2 & 0 \\ 0 & 0 & 0 & 1 & 7 \end{array} \right) \xrightarrow[\begin{array}{l} R_2 \rightarrow -\frac{1}{3}R_2 \\ R_1 \rightarrow R_1 - 2R_2 \end{array}]{R_1 \rightarrow R_1 - \frac{1}{3}R_3} \left( \begin{array}{ccccc} 1 & 0 & -\frac{7}{3} & \frac{1}{3} & 0 \\ 0 & 1 & \frac{5}{3} & -\frac{2}{3} & 0 \\ 0 & 0 & 0 & 1 & 7 \end{array} \right) \\ & \xrightarrow[\begin{array}{l} R_1 \rightarrow R_1 - \frac{1}{3}R_3 \\ R_2 \rightarrow R_2 + \frac{2}{3}R_3 \end{array}]{R_1 \rightarrow R_1 - \frac{1}{3}R_3} \left( \begin{array}{ccccc} 1 & 0 & -\frac{7}{3} & 0 & -\frac{7}{3} \\ 0 & 1 & \frac{5}{3} & 0 & \frac{14}{3} \\ 0 & 0 & 0 & 1 & 7 \end{array} \right) \end{aligned}$$

Free variables:  $z, t$ . Basic solutions:  $(x, y, z, w, t) = (\frac{7}{3}, -\frac{5}{3}, 1, 0, 0)$  and

$$(x, y, z, w, t) = (\frac{7}{3}, -\frac{14}{3}, 0, -7, 1).$$

(b)  $(x, y, z, w, t) = (2, 0, -1, 1, 1) + z(\frac{7}{3}, -\frac{5}{3}, 1, 0, 0) + t(\frac{7}{3}, -\frac{14}{3}, 0, -7, 1)$ . (Column vectors can also be used to write these solutions.)

3. *Solution 1.* The given points belong to the same line if and only if the vectors  $(x_1 - x_3, y_1 - y_3)$  and  $(x_2 - x_3, y_2 - y_3)$  belong to the same line, which means that they are linearly dependent.

This happens if and only if  $\det \begin{pmatrix} x_1 - x_3 & y_1 - y_3 \\ x_2 - x_3 & y_2 - y_3 \end{pmatrix} = 0$ . But this determinant is equal to the

determinant from the condition of the problem:

$$\det \begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix} = \det \begin{pmatrix} x_1 - x_3 & y_1 - y_3 & 0 \\ x_2 - x_3 & y_2 - y_3 & 0 \\ x_3 & y_3 & 1 \end{pmatrix} = \det \begin{pmatrix} x_1 - x_3 & y_1 - y_3 \\ x_2 - x_3 & y_2 - y_3 \end{pmatrix}.$$

*Solution 2.* The given determinant is zero if and only if the column vectors of the matrix are linearly dependent. This means that for some  $a, b, c$  not all equal to zero

$$a \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + b \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} + c \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

that is  $ax_j + by_j + c = 0$ ,  $j = 1, 2, 3$  (one can also think of the corresponding homogeneous system of equation on  $a, b, c$  which has a non-trivial solution if and only if the determinant is zero). But this means that all three points belong to the line with equation  $ax + by + c = 0$ .

4. Since  $A$  is invertible,  $A^T$  is invertible and  $(A^T)^{-1} = (A^{-1})^T$ . The product of invertible matrices is invertible, and we have  $(A^T B)^{-1} = B^{-1}(A^{-1})^T$ , and finally  $(A^T B C)^{-1} = C^{-1}(A^T B)^{-1} = C^{-1}B^{-1}(A^{-1})^T$ .

5. Indeed, since the matrix product satisfies  $(XY)Z = X(YZ)$  (associative), by the definition of inverse

$$A^{-1}(I + AB)A - I = A^{-1}IA + (A^{-1}A)BA - I = A^{-1}A + IBA - I = I + BA - I = BA.$$