Two–Mode Networks Exhibiting Data Loss

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Abstract

Motivated by a question arising in the analysis of social networks, we investigate pairs of $(0, 1)$ matrices A, B such that $AA^T = BB^T$ and $A^T A = B^T B$. Using the techniques of combinatorial matrix theory, we show how the problem can be analysed in terms of certain linear systems. We construct two large infinite families of pairs of such matrices. One family has an amusing connection with regular tournament matrices, while the other is connected with a generalisation of Ryser's notion of an interchange for a $(0, 1)$ matrix. Not surprisingly, both families of matrices are highly structured.

1 Introduction and preliminaries

A two–mode network can be thought of as a rectangular $(0, 1)$ matrix in the following way: rows represent agents and columns represent events, with a 1 in the (i, j) position if agent i participates in event j, and a 0 in that position if not. In the sociology literature such matrices are analysed mathematically in order to yield insight on, for example, the relative importance of, or relationships between, the various agents. For instance, a classic example of a two–mode social network appears in [1], while the special issue [2] includes a variety of contemporary examples and approaches to two–mode social networks.

Given an $m \times n$ $(0, 1)$ matrix A that encodes the information in a two–mode network, one of the mathematical approaches to furnish such insights is to consider the related matrices AA^T (where both rows and columns correspond to agents) and A^TA (where both rows and columns correspond to events). Both AA^T and A^TA represent single–mode networks, and as pointed out in [3], analysis of this pair of single–mode networks offers a couple of advantages over direct analysis of A. First, both matrices are symmetric (unlike the original matrix A which is not symmetric in general), and so can be analysed by the wide array of mathematical techniques that is available for symmetric matrices. Second, since the rows and columns of the single–mode networks represent the same types of entities (i.e. they are all agents, or they are all events) the comparisons made between entities are perhaps more naturally made, as opposed to the 'apples by oranges' properties of two–mode networks. We refer the interested reader to [3] for a discussion of these and related issues.

The question has been posed explicitly in [3] as to whether knowledge of both AA^T and A^TA is sufficient to reconstruct A itself. To frame the question more mathematically, if we have $m \times n$ $(0, 1)$ matrices A and B such that $AA^T = BB^T$ and $A^TA = B^TB$, must it be the case that $A = B$? In general the answer is in the negative, and [3] alludes to the existence of a pair of distinct 19×19 matrices whose corresponding single–mode network matrices are the same; this is an example of data loss, whereby the pair of matrices AA^T and A^TA does not contain enough information to specify A.

In the present paper, our goal is to push farther in that direction by developing some tools that facilitate a more systematic study of pairs of $(0, 1)$ matrices A, B such that $AA^T = BB^T$ and $A^TA = B^TB$. Those tools will lead to the construction of two large infinite families of such pairs A, B, and as might be expected, both families are highly structured. Our techniques are linear algebraic, and make use of basic ideas in combinatorial matrix theory; we refer the interested reader to [4] for an introduction to that rich and active subject.

We begin with a basic and well–known observation. Suppose that A is a $(0, 1)$ matrix, and note that the diagonal entries of AA^T are simply the rows sums of A, and that the diagonal entries of A^TA are the column sums of A. Consequently, if we have two $m \times n$ $(0,1)$ matrices A, B such that $AA^T = BB^T$ and $A^TA = B^TB$, then necessarily A and B must have the same row sum vectors, and the same column sum vectors. Setting $E \equiv B - A$, it now follows readily that E is a $(0,1,-1)$ matrix with $E\mathbf{1} = 0$ and $\mathbf{1}^TE = 0^T$, where $\mathbf{1}$ denotes an all–ones vector of the appropriate size (which will always be clear from the context). That simple observation informs the approach in Sections 2 and 3.

We round out this Section with a preliminary result showing that, in some sense, the pairs of $(0,1)$ matrices A, B with $AA^T = BB^T$ and $A^TA = B^TB$ are not so common. Let $S_{m\times n}$ denote the set of all $m \times n$ $(0,1)$ matrices. Evidently the cardinality of $S_{m\times n}$ is 2^{mn}. We consider ordered pairs $(A, B) \in S_{m\times n} \times S_{m\times n}$; the following result gives an upper bound on the number of such ordered pairs such that $AA^T = BB^T$ and $A^TA = B^TB$.

Theorem 1.1. *The number of pairs $(A, B) \in S_{m\times n} \times S_{m\times n}$ such that $AA^T = BB^T$ and $A^TA = B^TB$ is bounded above by*

$$\min\left\{\left[\binom{2n}{n}\right]^m, \left[\binom{2m}{m}\right]^n\right\}.$$

Proof. Note that if $AA^T = BB^T$ for some $A, B \in S_{m\times n} \times S_{m\times n}$, then in particular, $A\mathbf{1} = B\mathbf{1}$. Thus, the number of pairs $(A, B) \in S_{m\times n} \times S_{m\times n}$ such that $AA^T = BB^T$ and $A^TA = B^TB$ is bounded above by the number of pairs (A, B) such that $A\mathbf{1} = B\mathbf{1}$.

Suppose for concreteness that we specify a row sum vector r. Then there are $\binom{n}{r_1}\binom{n}{r_2}\dots\binom{n}{r_m}$ matrices in $S_{m\times n}$ having row sum vector r. Hence there are $\binom{n}{r_1}^2\binom{n}{r_2}^2\dots\binom{n}{r_m}^2$ pairs of matrices $(A, B) \in S_{m\times n} \times S_{m\times n}$ such that $A\mathbf{1} = r = B\mathbf{1}$. It now follows that the number of pairs $(A, B) \in S_{m\times n} \times S_{m\times n}$ such that $A\mathbf{1} = B\mathbf{1}$ is given by

$$\sum_{0\le r_1,\dots,r_m\le n} \binom{n}{r_1}^2\binom{n}{r_2}^2\dots\binom{n}{r_m}^2.$$

Recalling that $\sum_{k=0}^{n}\binom{n}{k}^2 = \binom{2n}{n}$ (see [5]), we find that the number of pairs $(A, B) \in S_{m\times n} \times S_{m\times n}$ such that $A\mathbf{1} = B\mathbf{1}$ is given by $\left[\binom{2n}{n}\right]^m$. An analogous argument applies to pairs of matrices (A, B) such that $\mathbf{1}^TA = \mathbf{1}^TB$, and the conclusion follows. $\square$

Let $p(m,n)$ be the proportion of pairs of matrices $(A,B) \in S_{m\times n} \times S_{m\times n}$ such that $AA^T = BB^T$ and $A^TA = B^TB$.

Corollary 1.1.1. *As* $\max\{m,n\} \to \infty, p(m,n) \to 0$.

Proof. From Theorem 1.1, we find that

$$p(m,n) \le \min\left\{2^{-2mn}\left[\binom{2n}{n}\right]^m, 2^{-2mn}\left[\binom{2m}{m}\right]^n\right\}.$$

By a refinement of Stirling's approximation, we have, for each $k \in \mathbb{N}$,

$$\sqrt{2\pi}k^{k+\frac{1}{2}}e^{-k+\frac{1}{12k+1}} < k! < \sqrt{2\pi}k^{k+\frac{1}{2}}e^{-k+\frac{1}{12k}}$$

(see [6]). It now follows that $\binom{2k}{k} < \frac{2^k}{\sqrt{\pi k}}$, so that $2^{-2mn}\left[\binom{2n}{n}\right]^m < \frac{1}{(\pi n)^{\frac{m}{2}}}$ for any $m,n \in \mathbb{N}$. An analogous bound applies for $2^{-2mn}\left[\binom{2m}{m}\right]^n$, yielding $p(m,n) < \min\left\{\frac{1}{(\pi n)^{\frac{m}{2}}}, \frac{1}{(\pi m)^{\frac{n}{2}}}\right\}$. The conclusion follows. $\square$

From Corollary 1.1.1, we find that for large m or n, there is only a small probability that a randomly chosen pair of matrices $A, B \in S_{m\times n}$ has the property that $AA^T = BB^T$ and $A^TA = B^TB$.

2 A linear system approach

One way of approaching the question posed in Section 1 is as follows. Suppose that we are given an $m \times n$ $(0,1,-1)$ matrix E such that $E\mathbf{1} = 0, \mathbf{1}^TE = 0^T$. Determine the circumstances under which can we find a $(0,1)$ matrix A such that

$$\begin{aligned} &A + E \text{ is } (0,1),\\ &EA^T + AE^T + EE^T = 0 \text{ and}\\ &E^TA + A^TE + E^TE = 0. \end{aligned} \tag{1}$$

If our given E admits such an A, then setting $B = A + E$ we find readily that $AA^T = BB^T$ and $A^TA = B^TB$. This is the approach that we adopt in this Section.

The following result shows that the case for a general E can be reduced to the case that there are no zero rows or columns. The proof is straightforward, and is omitted.

Lemma 2.1. *Suppose that E is a $(0,1,-1)$ matrix such that $E\mathbf{1} = 0$ and $\mathbf{1}^T E = 0^T$. Suppose further that E has the form*

$$E = \left[\begin{array}{c|c} \tilde{E} & 0 \\ \hline 0 & 0 \end{array}\right].$$

Suppose that A is a $(0,1)$ matrix satisfying (1), and partitioned conformally with E as

$$A = \left[\begin{array}{c|c} A_{1,1} & A_{1,2} \\ \hline A_{2,1} & A_{2,2} \end{array}\right].$$

Then necessarily $\tilde{E}A_{1,1}^T + A_{1,1}\tilde{E}^T + \tilde{E}\tilde{E}^T = 0$, $\tilde{E}^T A_{1,1} + A_{1,1}^T \tilde{E} + \tilde{E}^T\tilde{E} = 0$, $\tilde{E}A_{2,1}^T = 0$ and $\tilde{E}^T A_{1,2} = 0$.

Here is one of the main results of this Section.

Theorem 2.1. *Suppose that $E = \left[\begin{array}{c} e_{i,j} \end{array}\right]$ is an $m \times n$ $(0,1,-1)$ matrix such that $E\mathbf{1} = 0$ and $\mathbf{1}^T E = 0^T$. For each pair of indices $i, j \in \{1, \dots, m\}$ with $i < j$, consider the following sets:*

$$\begin{array}{rcl} R_1(i,j) & = & \{l | e_{i,l} = 1, e_{j,l} = 0\}, \\ R_2(i,j) & = & \{l | e_{i,l} = 0, e_{j,l} = 1\}, \\ R_3(i,j) & = & \{l | e_{i,l} = -1, e_{j,l} = 0\}, \\ R_4(i,j) & = & \{l | e_{i,l} = 0, e_{j,l} = -1\}, \\ R_5(i,j) & = & \{l | e_{i,l} = 1, e_{j,l} = 1\}, \\ R_6(i,j) & = & \{l | e_{i,l} = -1, e_{j,l} = -1\}. \end{array}$$

Similarly, for each pair of indices $p, q \in \{1, \dots, n\}$ with $p < q$, consider the sets

$$\begin{array}{rcl} C_1(p,q) & = & \{l | e_{l,p} = 1, e_{l,q} = 0\}, \\ C_2(p,q) & = & \{l | e_{l,p} = 0, e_{l,q} = 1\}, \\ C_3(p,q) & = & \{l | e_{l,p} = -1, e_{l,q} = 0\}, \\ C_4(p,q) & = & \{l | e_{l,p} = 0, e_{l,q} = -1\}, \\ C_5(p,q) & = & \{l | e_{l,p} = 1, e_{l,q} = 1\}, \\ C_6(p,q) & = & \{l | e_{l,p} = -1, e_{l,q} = -1\}. \end{array}$$

There is an $m \times n$ $(0,1)$ matrix A such that $A + E$ is also $(0,1)$ with $AA^T = (A+E)(A+E)^T$ and $A^T A = (A+E)^T(A+E)$ if and only if *there is a* $(0,1)$ *solution to the following linear system:*

$$\begin{aligned}
&\sum_{l\in R_1(i,j)} a_{j,l} + \sum_{l\in R_2(i,j)} a_{i,l} - \sum_{l\in R_3(i,j)} a_{j,l} - \sum_{l\in R_4(i,j)} a_{i,l} = \\
&|R_6(i,j)| - |R_5(i,j)|, 1 \le i < j \le m, \\
&\sum_{l\in C_1(p,q)} a_{l,q} + \sum_{l\in C_2(p,q)} a_{l,p} - \sum_{l\in C_3(p,q)} a_{l,q} - \sum_{l\in C_4(p,q)} a_{l,p} = \\
&|C_6(p,q)| - |C_5(p,q)|, 1 \le p < q \le n.
\end{aligned} \tag{2}$$

Proof. Observe that the existence of a $(0,1)$ matrix A with the desired properties is equivalent to the existence of a $(0,1)$ matrix A satisfying (1).

Suppose that such an A exists. Applying the condition that $A+E$ is a $(0,1)$ matrix, we find that necessarily $a_{i,j} = 0$ whenever $e_{i,j} = 1$, and $a_{i,j} = 1$ whenever $e_{i,j} = -1$. Fix a pair of indices i,j with $1 \le i < j \le m$. Then the (i,j) entry of $EA^T + AE^T + EE^T$ is given by $\sum_{l=1}^n e_{i,l}a_{j,l} + \sum_{l=1}^n e_{j,l}a_{i,l} + \sum_{l=1}^n e_{i,l}e_{j,l}$. Setting $R_7(i,j) = \{l | e_{i,l} = 1, e_{j,l} = -1\}$ and $R_8(i,j) = \{l | e_{i,l} = -1, e_{j,l} = 1\}$, we find that

$$\begin{aligned}
&\sum_{l=1}^n e_{i,l}a_{j,l} = \\
&\sum_{l\in R_1(i,j)} e_{i,l}a_{j,l} + \sum_{l\in R_3(i,j)} e_{i,l}a_{j,l} + \sum_{l\in R_6(i,j)} e_{i,l}a_{j,l} + \sum_{l\in R_7(i,j)} e_{i,l}a_{j,l} = \\
&\sum_{l\in R_1(i,j)} a_{j,l} - \sum_{l\in R_3(i,j)} a_{j,l} - |R_6(i,j)| + |R_7(i,j)|.
\end{aligned}$$

Similarly we find that

$$\sum_{l=1}^n e_{j,l}a_{i,l} = \sum_{l\in R_2(i,j)} a_{i,l} - \sum_{l\in R_4(i,j)} a_{i,l} - |R_6(i,j)| + |R_8(i,j)|.$$

Considering the (i,j) entry of EE^T, it follows that $\sum_{l=1}^n e_{i,l}e_{j,l} = |R_5(i,j)| + |R_6(i,j)| - |R_7(i,j)| - |R_8(i,j)|$. Consequently, the (i,j) entry of $EA^T + AE^T + EE^T$ is equal to

$$\sum_{l\in R_1(i,j)} a_{j,l} + \sum_{l\in R_2(i,j)} a_{i,l} - \sum_{l\in R_3(i,j)} a_{j,l} - \sum_{l\in R_4(i,j)} a_{i,l} + |R_5(i,j)| - |R_6(i,j)|.$$

Fixing p, q with $1 \le p < q \le n$, an analogous argument shows that the (p, q) entry of $E^TA + A^TE + E^TE$ is equal to

$$\sum_{l \in C_1(p,q)} a_{l,q} + \sum_{l \in C_2(p,q)} a_{l,p} - \sum_{l \in C_3(p,q)} a_{l,q} - \sum_{l \in C_4(p,q)} a_{l,p} + |C_5(p,q)| - |C_6(p,q)|.$$

Consequently, if there is a $(0, 1)$ matrix with the desired properties, then the linear system (2) has a $(0, 1)$ solution.

The converse is straightforward. □

Henceforth we let J denote a (possibly rectangular) all–ones matrix; the dimensions will always be clear from the context.

Remark 2.1. Given any $(0, 1, -1)$ matrix E such that $E\mathbf{1} = 0$ and $\mathbf{1}^TE = 0^T$, it turns out that for the linear system (2), there is always an entrywise nonnegative solution. This is most easily seen by observing that if we set $B = \frac{1}{2}(J - E)$ (whose entries are in the set $\{0, 1, \frac{1}{2}\}$), then $EB^T + BE^T + EE^T = 0$ and $E^TB + B^TE + E^TE = 0$. Observe that in this setting, $B + E$ is also nonnegative.

Remark 2.2. Suppose that E is an $m \times n$ $(0, 1, -1)$ matrix with all row and column sums zero, and that r is the number of zero entries in E. It is straightforward to see that the linear system (2) has $\frac{m(m-1)+n(n-1)}{2}$ equations and r unknowns. Specifically, for each pair of indices (i, j) with $1 \le i < j \le m$, there is one equation corresponding to the sets $R_1(i, j), \ldots, R_6(i, j)$, and for each pair of indices (p, q) with $1 \le p < q \le n$, there is one equation corresponding to the sets $C_1(p, q), \ldots, C_6(p, q)$; the r unknowns are in one–to–one correspondence with the zero entries in E. Further, the coefficient matrix M of (2) can be constructed directly from E in the following manner. Fix a position (s, t) such that $e_{s,t} = 0$. For indices i, j with $1 \le i < j \le m$, the entry of M in the row corresponding to the sets $R_1(i, j), \ldots, R_6(i, j)$ and in the column corresponding to $e_{s,t}$ is equal to:
$e_{j,t}$ if $i = s$; $e_{i,t}$ if $j = s$; and 0 otherwise.
Similarly for indices p, q with $1 \le p < q \le n$, the entry of M in the row corresponding to the sets $C_1(p, q), \ldots, C_6(p, q)$ and in the column corresponding to $e_{s,t}$ is equal to:
$e_{s,q}$ if $p = t$; $e_{s,p}$ if $q = t$; and 0 otherwise.

For concreteness, we may order the r unknowns corresponding to the zeros in E lexicographically according to their positions in the matrix

E; this in turn yields an ordering of the columns of the matrix M. Similarly the equations in the linear system can be ordered with the equations corresponding to the sets $R_1(i,j),\dots,R_6(i,j)$ appearing first and ordered lexicographically according to the pairs (i,j), followed by the equations corresponding to the sets $C_1(p,q),\dots,C_6(p,q)$ ordered lexicographically according to the pairs (p,q); this yields an ordering of the rows of M. We adopt those orderings for the columns and rows of M in Examples 2.1 and 2.2.

Corollary 2.1.1. *Let E be an $m \times n$ $(0,1,-1)$ with $E\mathbf{1} = 0$ and $\mathbf{1}^T E = 0^T$. The following are equivalent:*
a) there is an $m \times n$ $(0,1)$ matrix A such that $A+E$ is $(0,1)$, $AA^T = (A+E)(A+E)^T$ and $A^TA = (A+E)^T(A+E)$;
b) the coefficient matrix of the linear system (2) has a $(1,-1)$ null vector.

Proof. Suppose that a) holds, and suppose that the number of zeros in E is r. By Theorem 2.1, there is a $(0,1)$ solution to (2), say $a \in \mathbb{R}^r$. It follows from Remark 2.1 that the vector $b = \frac{1}{2}\mathbf{1} \in \mathbb{R}^r$ is also a solution to (2). Evidently $a-b$ has entries $\frac{1}{2}$ or $-\frac{1}{2}$; moreover, as the difference between two particular solutions to (2), $a-b$ is necessarily a null vector for the corresponding coefficient matrix. We thus find that $2(a-b)$ is the desired $(1,-1)$ null vector.

The converse follows readily. □

Remark 2.3. Inspecting the proof of Corollary 2.1.1, we see that there is a one-to-one correspondence between $(1,-1)$ null vectors of the coefficient matrix for (2), and pairs of $(0,1)$ matrices $A, A+E$ such that $AA^T = (A+E)(A+E)^T$ and $A^TA = (A+E)^T(A+E)$.

Example 2.1. Consider the matrix

$$E = \begin{bmatrix} 1 & 1 & -1 & -1 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}.$$

Using Remark 2.2, we find that the coefficient matrix for (2) can be written as

$$M = \begin{bmatrix} 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & -1 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & -1 & 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

A straightforward computation shows that the null space of M consists only of the zero vector. From Corollary 2.1.1, we deduce that there is no $(0,1)$ matrix A such that $A+E$ is also $(0,1)$ with $AA^T = (A+E)(A+E)^T$ and $A^TA = (A+E)^T(A+E)$.

Example 2.2. Here we consider the matrix

$$E = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 0 & 1 \end{bmatrix}.$$

The coefficient matrix for the corresponding linear system (2) can be written as

$$M = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The null space of M contains eight $(1, -1)$ vectors, and this results in the following matrices $A_1, \ldots, A_8$, for which $A_j A_j^T = (A_j + E)(A_j + E)^T$ and $A_j^T A_j = (A_j + E)^T (A_j + E), j = 1, \ldots, 8$:

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix}, A_4 = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix},$$

$$A_5 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}, A_6 = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix},$$

$$A_7 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}, A_8 = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

In Section 3 we will consider a family of $(0, 1, -1)$ matrices with zero row and column sums that includes E, and prove a general result about that family.

Recall that an $n \times n$ $(0,1)$ matrix T is a *tournament matrix* provided that it satisfies the equation $T + T^T = J - I$. Each tournament matrix can be thought of as the record of outcomes of a round robin competition, or equivalently as the adjacency matrix of a *tournament*, i.e. a loop–free directed graph D with the property that for each pair of distinct vertices u, v, D contains precisely one of the arcs $u \to v$ and $v \to u$. A tournament matrix is *regular* provided that each of its row sums is equal to $\frac{n-1}{2}$; that condition immediately implies the every column sum is also equal to $\frac{n-1}{2}$. Evidently n must be odd in order for there to exist an $n \times n$ regular tournament matrix, and it is known that for any odd $n \geq 3$, there exists at least one regular tournament matrix of order n (also see Remark 2.4 below). It is straightforward to determine that if T is a regular tournament matrix, then $TT^T = T^T T$. We refer the interested reader to chapter 45 of [7] for further details on tournament matrices. Regular tournament matrices play a key role in the following result.

Theorem 2.2. *Suppose that $n \in \mathbb{N}$ and consider the following $(n+1) \times 2n$ matrix:*

$$E = \left[\begin{array}{c|c} -I & I \\ \hline \mathbf{1}^T & -\mathbf{1}^T \end{array}\right].$$

There is an $(n+1) \times 2n$ $(0,1)$ matrix A such that i) $A+E$ is also $(0,1)$, ii) $AA^T = (A+E)(A+E)^T$, and iii) $A^T A = (A+E)^T(A+E)$ if and only if n is odd.

When n is odd, each $(0,1)$ A satisfying i)–iii) has the form

$$A = \left[\begin{array}{c|c} T+I & T^T \\ \hline 0^T & \mathbf{1}^T \end{array}\right], \tag{3}$$

where T is a regular tournament matrix of order n. Conversely, for any regular tournament matrix T of order n, the matrix A of (3) satisfies i)–iii).

Proof. The case $n = 1$ is easily dealt with, so henceforth we consider the case that $n \geq 2$. Suppose that A is a $(0,1)$ matrix that satisfies i)–iii). Adopting the notation of Theorem 2.1, we find that $C_1(1,2) = \emptyset, C_2(1,2) = \emptyset, C_3(1,2) = \{1\}, C_4(1,2) = \{2\}, C_5(1,2) = \{n+1\}$, and

$C_6(1,2) = \emptyset$. Referring to (2) we thus find that $-a_{1,2} - a_{2,1} = -1$, that is, one of $a_{1,2}$ and $a_{2,1}$ is 0 and the other is 1. Arguing similarly with the sets $C_1(i,j), \ldots, C_6(i,j)$ for $1 \le i < j \le n$, it follows that necessarily the submatrix of A on rows $1, \ldots, n$ and columns $1, \ldots, n$ is of the form $T_1 + I$ for some $n \times n$ tournament matrix T_1. An analogous argument shows that the submatrix of A on rows $1, \ldots, n$ and columns $n+1, \ldots, 2n$ has the form T_2 for some $n \times n$ tournament matrix T_2.

Thus A has the general form

$$A = \left[\begin{array}{c|c} T_1 + I & T_2 \\ \hline 0^T & \mathbf{1}^T \end{array}\right],$$

while $A + E$ has the form

$$A + E = \left[\begin{array}{c|c} T_1 & T_2 + I \\ \hline \mathbf{1}^T & 0^T \end{array}\right].$$

Consequently we have

$$\begin{aligned}
AA^T &= \left[\begin{array}{c|c} T_1T_1^T + T_2T_2^T + J & T_2\mathbf{1} \\ \hline \mathbf{1}^T T_2^T & n \end{array}\right], \\
A^TA &= \left[\begin{array}{c|c} T_1^TT_1 + J & T_1^TT_2 + T_2 \\ \hline T_2^TT_1 + T_2^T & T_2^TT_2 + J \end{array}\right], \\
(A+E)(A+E)^T &= \left[\begin{array}{c|c} T_1T_1^T + T_2T_2^T + J & T_1\mathbf{1} \\ \hline \mathbf{1}^T T_1^T & n \end{array}\right], \\
(A+E)^T(A+E) &= \left[\begin{array}{c|c} T_1^TT_1 + J & T_1^TT_2 + T_1^T \\ \hline T_2^TT_1 + T_1 & T_2^TT_2 + J \end{array}\right].
\end{aligned}$$

From iii) we find that $T_2 = T_1^T$, while from ii), we have $T_2\mathbf{1} = T_1\mathbf{1}$. This last yields $(n-1)\mathbf{1} - T_1\mathbf{1} = T_2\mathbf{1} = T_1\mathbf{1}$, from which we deduce that $T_1\mathbf{1} = \frac{n-1}{2}\mathbf{1}$. Thus, n is odd and T_1 is regular. The form (3) now follows.

Finally, we note that if T is a regular tournament matrix of order n and A is given by (3), then

$$AA^T = \left[\begin{array}{c|c} 2TT^T + J & \frac{n-1}{2}\mathbf{1} \\ \hline \frac{n-1}{2}\mathbf{1}^T & n \end{array}\right] = (A+E)(A+E)^T, \text{ and}$$

$$A^TA = \left[\begin{array}{c|c} T^TT + J & (T^T)^2 + T^T \\ \hline T^2 + T & T^TT + J \end{array}\right] = (A+E)^T(A+E).$$

$\square$

Remark 2.4. A result of McKay [8] gives an asymptotic expression for the number of regular tournament matrices of odd order n. Specifically, as $n \to \infty$ (through odd values) then for any $\epsilon > 0$, the number of $n \times n$ regular tournament matrices is given by

$$t_n \equiv \left(\frac{2^{n+1}}{\pi n}\right)^{\frac{n-1}{2}} \left(\frac{n}{e}\right)^{\frac{1}{2}} \left(1 + O\left(n^{-\frac{1}{2}+\epsilon}\right)\right).$$

Thus we see that for each $(n+1) \times 2n$ matrix E of the form described in Theorem 2.2, there are t_n distinct pairs of $(0,1)$ matrices $A, A+E$ such that $AA^T = (A+E)(A+E)^T$ and $A^TA = (A+E)^T(A+E)$.

3 A permutation matrix approach

As noted in Section 1, if we have $(0,1)$ matrices A, B such that $AA^T = BB^T$ and $A^TA = B^TB$, then necessarily A and B have the same row sum vectors, and the same column sum vectors. A classic paper of Ryser [9] deals with an operation on $(0,1)$ matrices that is known as an *interchange*. For a $(0,1)$ matrix M we perform an interchange in one of two ways:

i) find a 2×2 submatrix of M equal to $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and replace it by $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$; or

ii) find a 2×2 submatrix of M equal to $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and replace it by $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Equivalently, we may effect an interchange by adding either $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$ or $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ to a suitable submatrix of M. Ryser [9] shows that for any pair of $(0,1)$ matrices M_1, M_2 having the same row sum vectors and the same column sum vectors, there is a sequence of interchanges that takes M_1 to M_2.

In this Section we generalise the notion of an interchange, and use it to construct many pairs of $(0,1)$ matrices $A, A+E$ satisfying (1). Here is the idea. Suppose that we have a $(0,1)$ matrix M with a $k \times k$ submatrix

S. Suppose further that there are permutation matrices Q_1, Q_2 of order k such that $S \geq Q_1$ (where the inequality holds entrywise) and $S \circ Q_2 = 0$ (where $\circ$ denotes the Hadamard product of matrices). Then we may replace the submatrix S of M by $S + Q_2 - Q_1$, and that operation yields another $(0,1)$ matrix $\tilde{M}$ having the same row and column sum vectors, respectively, as M does. Evidently this 'permutation exchange' operation coincides with an interchange in the case that $k = 2$.

In order to facilitate our analysis below, we begin in a simplified setting. Suppose that $k \in \mathbb{N}$ with $k \geq 2$, let P_k denote the $k \times k$ cyclic permutation matrix given by

$$P_k = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & & \ddots & \ddots & & \vdots \\ & & & & & \\ 0 & 0 & \dots & & 0 & 1 \\ 1 & 0 & & \dots & 0 & 0 \end{bmatrix}.$$

Our next few results discuss permutation exchanges in the special case that $Q_1 = I$ and $Q_2 = P_k$. Specifically, we determine, for the case of two matrices A and $\tilde{A}$ related by a permutation exchange with $Q_1 = I$ and $Q_2 = P_k$, when we have $AA^T = \tilde{A}\tilde{A}^T$ and $A^TA = \tilde{A}^T\tilde{A}$. Observe that without loss of generality we may assume that the submatrix S upon which the permutation exchange operates is the leading $k \times k$ submatrix of A, since for any permutation matrices R_1, R_2 of orders m, n respectively, we have $AA^T = \tilde{A}\tilde{A}^T$ and $A^TA = \tilde{A}^T\tilde{A}$ if and only if $R_1AR_2(R_1AR_2)^T = R_1\tilde{A}R_2(R_1\tilde{A}R_2)^T$ and $(R_1AR_2)^TR_1AR_2 = (R_1\tilde{A}R_2)^TR_1\tilde{A}R_2$.

We begin with a useful technical result.

Lemma 3.1. *Let A be an $n \times n$ $(0,1)$ matrix such that $A \geq I$ and $A \circ P_n = 0$. Let $B = A - I$ and $E = P_n - I$. We have $AA^T = (A+E)(A+E)^T$ and $A^TA = (A+E)^T(A+E)$ if and only if both $B(I - P_n^T)$ and $(I - P_n^T)B$ are skew-symmetric.*

Proof. From (1), we see that if $AA^T = (A+E)(A+E)^T$ then $EA^T + AE^T + EE^T = 0$. This last can be rewritten as $(P_n - I)(B^T + I) + (B + I)(P_n^T - I) + (P_n - I)(P_n^T - I) = 0$, which simplifies to the condition $(I - P_n)B^T + B(I - P_n^T) = 0$. A similar argument applies to the equation $A^TA = (A+E)^T(A+E)$, and the conclusion follows. $\square$

Next, we present a key result.

Theorem 3.1. *Let A be an $n \times n$ $(0,1)$ matrix $A \geq I$ and $A \circ P_n = 0$. Suppose that $E = P_n - I$. We have a) $AA^T = (A+E)(A+E)^T$ and b) $A^T A = (A+E)^T(A+E)$ if and only if there are scalars $\alpha_2, \dots, \alpha_{n-1} \in \{0,1\}$ with $\alpha_j = \alpha_{n+1-j}, j = 2, \dots, n-1$ such that $A = I + \sum_{j=2}^{n-1} \alpha_j P_n^j$.*

Proof. Set $B = A - I$. In the interests of notational simplicity, throughout the proof we suppress the dependence of P_n on n. For each $j = 2, \dots, n-1$, set $T_j = B \circ P^j$, and observe that $B = \sum_{j=2}^{n-1} T_j$. By Lemma 3.1, a) and b) hold if and only if $B(I - P^T)$ and $(I - P^T)B$ are both skew-symmetric.

Note that $BP^T = BP^{-1} = T_2P^{-1} + \sum_{j=3}^{n-1} T_j P^{-1} = T_2 P^{-1} + \sum_{j=2}^{n-2} T_{j+1} P^{-1}$, while $P^T B = P^{-1} B = P^{-1} T_2 + \sum_{j=2}^{n-2} P^{-1} T_{j+1}$. We note in passing that for each $j = 1, \dots, n-2$, both $T_{j+1}P^{-1}$ and $P^{-1}T_{j+1}$ only have nonzero entries in positions where P^j is positive.

We thus find that $B(I - P^{-1}) = -T_2P^{-1} + \sum_{j=2}^{n-2}(T_j - T_{j+1}P^{-1}) + T_{n-1}$. Further, $(B(I - P^{-1}))^T$ can be written as $(B(I - P^{-1}))^T = T_{n-1}^T + \sum_{j=2}^{n-2}(T_{n-j} - T_{n-j+1}P^{-1})^T - (T_2P^{-1})^T$. Observe that T_{n-1}^T is nonzero only in positions where P is nonzero, $(T_2P^{-1})^T$ is nonzero only in positions where P^{n-1} is nonzero, and for each $j = 2, \dots, n-2$, $(T_{n-j} - T_{n-j+1}P^{-1})^T$ is nonzero only in positions where P^j is nonzero. It now follows that $B(I - P^{-1})$ is skew-symmetric if and only if:
i) $T_2P^{-1} = T_{n-1}^T$ and ii) for each $j = 2, \dots, n-2, T_{j+1}P^{-1} - T_j = (T_{n-j} - T_{n-j+1}P^{-1})^T$. An analogous argument establishes that $(I - P^{-1})B$ is skew-symmetric if and only if:
iii) $P^{-1}T_2 = T_{n-1}^T$ and iv) for each $j = 2, \dots, n-2, P^{-1}T_{j+1} - T_j = (T_{n-j} - P^{-1}T_{n-j+1})^T$. Consequently, we find that a) and b) hold if and only if i)-iv) hold.

Suppose now that conditions i)-iv) hold. In particular, we have $T_2P^{-1} = T_{n-1}^T = P^{-1}T_2$. We claim that $T_2 = \alpha_2 P^2$ for some $\alpha_2 \in \{0,1\}$. To see the claim, observe that there is a diagonal matrix D such that $T_2 = DP^2$. Since $T_2P^{-1} = P^{-1}T_2$, we find that $DP = PD$; it is now straightforward to determine that all diagonal entries in D must be equal, from which the claim follows. Thus $T_2 = \alpha_2 P^2$ for some $\alpha_2 \in \{0,1\}$. Hence $T_{n-1}^T = \alpha_2 P$, so setting $\alpha_{n-1} = \alpha_2$, we then have $T_{n-1} = \alpha_{n-1}P^{n-1}$. Since $T_3P^{-1} - T_2 = (T_{n-2} - T_{n-1}P^{-1})^T$, we find that $T_3P^{-1} - \alpha_2P^2 = T_{n-2}^T - \alpha_2 P^2$; hence $T_3P^{-1} = T_{n-2}^T$. We find similarly that $P^{-1}T_3 = T_{n-2}^T$, which yields that for some $\alpha_3 \in$

$\{0,1\}, T_3 = \alpha_3 P^3$ and that $T_{n-2} = \alpha_3 P^{n-2} \equiv \alpha_{n-2} P^{n-2}$. Iterating the argument above now shows that if conditions i)-iv) hold, then there are constants $\alpha_2, \ldots, \alpha_{n-1} \in \{0,1\}$ such that $\alpha_j = \alpha_{n+1-j}, j = 2, \ldots, n-1$ and $T_j = \alpha_j P^j$ for all such j. The desired conclusion for B now follows.

Conversely, if $B = \sum_{j=2}^{n-1} \alpha_j P^j$, and $\alpha_j = \alpha_{n+1-j}, j = 2, \ldots, n-1$, then conditions i)-iv) are readily verified. $\square$

Remark 3.1. Suppose that $E = P_n - I$. From Theorem 3.1 we see that for any choice of $\alpha_2, \ldots, \alpha_{\lfloor \frac{n+1}{2} \rfloor} \in \{0,1\}$, we can then determine $\alpha_{\lfloor \frac{n+3}{2} \rfloor}, \ldots, \alpha_{n-1}$ in order to generate a pair of $(0,1)$ matrices $A, A + E$ with $AA^T = (A+E)(A+E)^T$ and $A^TA = (A+E)^T(A+E)$. Consequently we find that there are $2^{\lfloor \frac{n-1}{2} \rfloor}$ pairs of such $(0,1)$ matrices $A, A+E$.

Next, we consider a $(0,1)$ matrix A given as a 2×2 block partitioned matrix $A = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right]$, and an associated $(0,1,-1)$ matrix $E = \left[\begin{array}{c|c} P_p - I & 0 \\ \hline 0 & P_q - I \end{array}\right]$, where $p, q \in \mathbb{N}$. Suppose that both A and $A+E$ are $(0,1)$ matrices, and that we have

$$AA^T = (A+E)(A+E)^T \text{ and } A^TA = (A+E)^T(A+E). \qquad (4)$$

Considering the diagonal blocks in (4), it follows that $A_{11}A_{11}^T = (A_{11} + P_p - I)(A_{11} + P_p - I)^T$ and $A_{11}^T A_{11} = (A_{11} + P_p - I)^T (A_{11} + P_p - I)$, and that $A_{22}A_{22}^T = (A_{22} + P_q - I)(A_2 + P_q - I)^T$ and $A_{22}^T A_{22} = (A_{22} + P_q - I)^T (A_{22} + P_q - I)$. Evidently Theorem 3.1 can now be applied to A_{11} (with associated permutation matrix P_p), and to A_{22} (with associated permutation matrix P_q).

Next we consider the off–diagonal blocks of (4). It follows readily that

$$A_{12}(P_q^T - I) + (P_p - I)A_{21}^T = 0 \text{ and } (P_p^T - I)A_{12} + A_{21}^T(P_q - I) = 0. \quad (5)$$

We adopt the notation that for indices i, j, the (i,j) entry of A_{12} is $A_{12}(i,j)$; a similar notation applies to the entries of A_{21}. We can then rewrite (5) entry–by–entry as

$$A_{12}(i,j) - A_{12}(i,j+1) + A_{21}(j,i) - A_{21}(j,i+1) = 0 \text{ and} \qquad (6)$$
$$A_{12}(i,j) - A_{12}(i-1,j) + A_{21}(j,i) - A_{21}(j-1,i) = 0, \qquad (7)$$
$$\text{for all } i = 1, \ldots, p, j = 1, \ldots, q.$$

Here we note that in expressions of the form $A_{12}(k,l)$, we are to interpret k modulo p and l modulo q, while in expressions of the form $A_{21}(m,n)$, we are to interpret m modulo q and n modulo p.

Set $A_{21}(1,i) = x_i, i = 1,\dots,p$ and $A_{12}(i,1) = y_i, i = 1,\dots,p$. In the sequel, we will interpret the subscripts on the x_k's and y_k's as being taken modulo p.

The following sequence of lemmas will be used to establish the relationship between A_{12} and A_{21}. For Lemmas 3.2–3.3 and Corollaries 3.1.1 and 3.1.2, we assume that A_{12} and A_{21} are $(0,1)$ matrices satisfying (5).

Lemma 3.2. *For each $i = 1,\dots,p$ and $j = 2,\dots,q$, we have*

$$A_{12}(i,j) = \sum_{l=0}^{2j-3}(-1)^l x_{2+i-j+l} + \sum_{l=0}^{2j-4}(-1)^l y_{2+i-j+l} \tag{8}$$

and

$$A_{21}(j,i) = \sum_{l=0}^{2j-2}(-1)^l x_{1+i-j+l} + \sum_{l=0}^{2j-3}(-1)^l y_{1+i-j+l}. \tag{9}$$

Proof. We proceed by induction on j. Note that using (6) with $j = 1$ yields $A_{12}(i,2) = A_{12}(i,1) + A_{21}(1,i) - A_{21}(1,i+1) = x_i - x_{i+1} + y_i$, which agrees with (8) when $j = 2$. Next, we use (7) with $j = 2$ to find that $A_{21}(2,i) = A_{21}(1,i) + A_{12}(i-1,2) - A_{12}(i,2) = x_i + (x_{i-1} - x_i + y_{i-1}) - (x_i - x_{i+1} + y_i) = x_{i-1} - x_i + x_{i+1} + y_{i-1} - y_i$, which agrees with (9) when $j = 2$. This establishes the base case $j = 2$ for the induction.

Suppose now that for some $j_0 \geq 2$, (8) and (9) hold for $j = 2,\dots,j_0$. Using (6) with $j = j_0$ we find that $A_{12}(i,j_0+1) = A_{12}(i,j_0) + A_{21}(j_0,i) - A_{21}(j_0,i+1)$. Appealing to the induction hypothesis, we thus find that

$$\begin{aligned} A_{12}(i,j_0+1) = & \sum_{l=0}^{2j_0-3}(-1)^l x_{2+i-j_0+l} + \sum_{l=0}^{2j_0-4}(-1)^l y_{2+i-j_0+l} \\ & + \sum_{l=0}^{2j_0-2}(-1)^l x_{1+i-j_0+l} + \sum_{l=0}^{2j_0-3}(-1)^l y_{1+i-j_0+l} \\ & - \sum_{l=0}^{2j_0-2}(-1)^l x_{2+i-j_0+l} - \sum_{l=0}^{2j_0-3}(-1)^l y_{2+i-j_0+l}. \end{aligned}$$

This last now simplifies to

$$A_{12}(i, j_0+1) = \sum_{l=0}^{2j_0-1} (-1)^l x_{2+i-(j_0+1)+l} + \sum_{l=0}^{2j_0-2} (-1)^l y_{2+i-(j_0+1)+l},$$

which agrees with (8) when $j = j_0 + 1$.

Similarly, using (7) with $j = j_0 + 1$, we have $A_{21}(j_0 + 1, i) = A_{21}(j_0, i) + A_{12}(i - 1, j_0 + 1) - A_{12}(i, j_0 + 1)$. Using the induction hypothesis for A_{21} and the fact that we have already established (8) for $j_0 + 1$, we find that

$$\begin{aligned} A_{21}(j_0+1, i) = \sum_{l=0}^{2j_0-2} (-1)^l x_{1+i-j_0+l} + \sum_{l=0}^{2j_0-3} (-1)^l y_{1+i-j_0+l} \\ + \sum_{l=0}^{2j_0-1} (-1)^l x_{2+i-1-j_0-1+l} + \sum_{l=0}^{2j_0-2} (-1)^l y_{2+i-1-j_0-1+l} \\ - \sum_{l=0}^{2j_0-1} (-1)^l x_{1+i-j_0+l} - \sum_{l=0}^{2j_0-2} (-1)^l y_{1+i-j_0+l}. \end{aligned}$$

A little simplification now yields

$$A_{21}(j_0+1, i) = \sum_{l=0}^{2j_0} (-1)^l x_{1+i-(j_0+1)+l} + \sum_{l=0}^{2j_0-1} (-1)^l y_{1+i-(j_0+1)+l},$$

which completes the proof of the induction step. □

For each $k = 1, \ldots, p$, let $d_k = x_{k+1} - y_k$ where as usual the subscripts are interpreted modulo p. With this notation we have, in view of Lemma 3.2, that for each $i = 1, \ldots, p$ and $j = 2, \ldots, q$ $A_{12}(i, j) = x_{i+2-j} - \sum_{k=0}^{2j-4} (-1)^k d_{i+2-j+k}$ and $A_{21}(j, i) = x_{i+1-j} - \sum_{k=0}^{2j-3} (-1)^k d_{i+1-j+k}$.

Lemma 3.3. *Fix an index i between 1 and p, and suppose that $d_k \neq 0$ for some $k = 1, \ldots, p$. We have*

$$d_i = \begin{cases} 0 & if\ x_i \neq x_{i+1} \\ 1 & if\ x_i = x_{i+1} = 1 \\ -1 & if\ x_i = x_{i+1} = 0. \end{cases}$$

Proof. Suppose that $x_i = 1$ and $x_{i+1} = 0$. Then $d_i = 0 - y_i \in \{0, -1\}$. Since $A_{12}(i, 2) = x_i - d_i = 1 - d_i \in \{0, 1\}$ it must be the case that $d_i = 0$. Similarly, if $x_i = 0$ and $x_{i+1} = 1$, then $d_i = 1 - y_i \in \{0, 1\}$. As $A_{12}(i, 2) = x_i - d_i = -d_i \in \{0, 1\}$, we find that d_i must be 0. Thus, if $x_i \neq x_{i+1}$, then $d_i = 0$.

Next, suppose that $x_i = x_{i+1} = 1$. Since $x_{i+1} = 1$, we find as above that $d_i \in \{0, 1\}$. Suppose that $d_i = 0$. Recall that $d_k \neq 0$ for some k, and let t be the smallest positive index such that $d_i = d_{i+1} = \ldots = d_{i+t-1} = 0, d_{i+t} \neq 0$. If t is even, then note that $A_{12}(i + \frac{t}{2}, \frac{t+4}{2}) = x_i - \sum_{k=0}^{t}(-1)^k d_{i+k}$ while $A_{21}(\frac{t+2}{2}, i + \frac{t+2}{2}) = x_{i+1} - \sum_{k=0}^{t-1}(-1)^k d_{i+k+1}$; on the other hand if t is odd then $A_{21}(\frac{t+3}{2}, i + \frac{t+1}{2}) = x_i - \sum_{k=0}^{t}(-1)^k d_{i+k}$ while $A_{12}(i + \frac{t+1}{2}, \frac{t+3}{2}) = x_{i+1} - \sum_{k=0}^{t-1}(-1)^k d_{i+k+1}$. In either case, both $x_i - \sum_{k=0}^{t}(-1)^k d_{i+k}$ and $x_{i+1} - \sum_{k=0}^{t-1}(-1)^k d_{i+k+1}$ are in $\{0, 1\}$. Since $x_i - \sum_{k=0}^{t}(-1)^k d_{i+k} = 1 - (-1)^t d_{i+t}$, we deduce that $d_{i+t} = (-1)^t$. But then we have $x_{i+1} - \sum_{k=0}^{t-1}(-1)^k d_{i+k+1} = 1 - (-1)^{t-1}(-1)^t = 2$, a contradiction. Hence it must be the case that $d_i = 1$.

Finally, we suppose that $x_i = x_{i+1} = 0$. Since $x_i - d_i \in \{0, 1\}$, we find that $d_i \in \{0, -1\}$. If $d_i = 0$, let t be the smallest positive index such that $d_i = d_{i+1} = \ldots = d_{i+t-1} = 0, d_{i+t} \neq 0$. As above, both $x_i - \sum_{k=0}^{t}(-1)^k d_{i+k}$ and $x_{i+1} - \sum_{k=0}^{t-1}(-1)^k d_{i+k+1}$ are in $\{0, 1\}$. Since $x_i - \sum_{k=0}^{t}(-1)^k d_{i+k} = -(-1)^t d_{i+t}$, we find that $d_{i+t} = (-1)^{t+1}$. But then we have

$$x_{i+1} - \sum_{k=0}^{t-1}(-1)^k d_{i+k+1} = -(-1)^{t-1}(-1)^{t+1} = -1,$$

a contradiction. Hence $d_i = -1$. $\square$

Corollary 3.1.1. *Suppose that $d_k \neq 0$ for some $k = 1, \ldots, p$. Then $x_i + y_i = 1$ for $i = 1, \ldots, p$.*

Proof. Fix an index i between 1 and p. We consider three cases. First, suppose that $x_i \neq x_{i+1}$, so that necessarily $x_{i+1} = 1 - x_i$. Then by Lemma 3.3, $d_i = 0$. Then we have $y_i = x_{i+1} - d_i = x_{i+1} = 1 - x_i$. Next, suppose that $x_i = x_{i+1} = 1$; by Lemma 3.3 we then have $d_i = 1$, so that $y_i = x_{i+1} - d_i = 0$. Finally, suppose that $x_i = x_{i+1} = 0$. Then $d_i = -1$, so that $y_i = 1$. $\square$

Corollary 3.1.2. *Suppose that $d_k \neq 0$ for some $k = 1, \ldots, p$. Then $A_{12} = J - A_{21}^T$.*

Proof. For each $i = 1, \ldots, p$, we have $A_{12}(i,1) + A_{21}(1,i) = y_i + x_i = 1$, by Corollary 3.1.1. For each $j = 2, \ldots, q$, we have, by Lemma 3.2 and Corollary 3.1.1 that

$$\begin{aligned} A_{12}(i,j) + A_{21}(j,i) &= \\ & \sum_{l=0}^{2j-3} (-1)^l x_{2+i-j+l} + \sum_{l=0}^{2j-4} (-1)^l y_{2+i-j+l} \\ &+ \sum_{l=0}^{2j-2} (-1)^l x_{1+i-j+l} + \sum_{l=0}^{2j-3} (-1)^l y_{1+i-j+l} \\ &= -x_{i+j-1} + \sum_{l=0}^{2j-4} (-1)^l + x_{i+j-1} + \sum_{l=0}^{2j-3} (-1)^l \\ &= 1, \end{aligned}$$

as desired.

$\square$

We are now in a position to describe A_{12} and A_{21} when some d_k is nonzero.

Proposition 3.1. *Suppose that for some $k = 1, \ldots, p$ we have $d_k \neq 0$. Then the vector $x = \begin{bmatrix} x_1 \\ \vdots \\ x_p \end{bmatrix}$ satisfies $P_p^q x = x$ and $x_k + x_{k+1} \neq 1$. Further, $A_{12} = J - A_{21}^T$ and*

$$A_{21}^T = \left[\ x \mid P_p x \mid P_p^2 x \mid \ldots \mid P_p^{q-1} x\ \right].$$

Proof. From (9) and Corollary 3.1.1 we have

$$A_{21}(j,i) = \sum_{l=0}^{2j-2} (-1)^l x_{1+i-j+l} + \sum_{l=0}^{2j-3} (-1)^l y_{1+i-j+l} = x_{i+j-l} + \sum_{l=0}^{2j-3} (-1)^l$$
$$= x_{i+j-l}$$

whenever $j \geq 2$. It now follows that we may write A_{21}^T as

$$A_{21}^T = \left[\ x \mid P_p x \mid P_p^2 x \mid \ldots \mid P_p^{q-1} x\ \right].$$

Corollary 3.1.2 establishes the fact that $A_{12} = J - A_{21}^T$. Using that fact, in conjunction with (5), we deduce that $A_{21}^T P_q^T = P_p A_{21}^T$, which is equivalent to

$$\left[\ P_p x \mid P_p^2 x \mid \ldots \mid P_p^{q-1} x \mid x\ \right] = \left[\ P_p x \mid P_p^2 x \mid \ldots \mid P_p^{q-1} x \mid P_p^q x\ \right].$$

Hence $P_p^q x = x$. Finally we note that since $d_k \neq 0$ by hypothesis, we have $0 \neq x_{k+1} - y_k = x_k + x_{k+1} - 1$. $\square$

Here is the characterisation of A_{12} and A_{21} when each d_k is zero.

Proposition 3.2. *Suppose that $d_k = 0$ for $k = 1, \ldots, p$. Then the $(0,1)$ vector x satisfies $P_p^q x = x$. Further,*

$$A_{21}^T = \left[\ x \mid P_p^{-1} x \mid P_p^{-2} x \mid \ldots \mid P_p^{-q+1} x\ \right]$$

and

$$A_{12} = \left[\ P_p x \mid x \mid P_p^{-1} x \mid \ldots \mid P_p^{-q+2} x\ \right].$$

Proof. The formulas for A_{12} and A_{21}^T are readily established, so we need only show that $P_p^q x = x$. From (5) we find that $A_{12}(I - P_q^T) + (I - P_p)A_{21}^T = 0$. From the formulas for A_{12} and A_{21}^T, and considering the last column of $A_{12}(I - P_q^T) + (I - P_p)A_{21}^T$, it now follows that necessarily $P_p^q x = x$. $\square$

Remark 3.2. It is readily verified that if A_{12} and A_{21} are constructed as in Proposition 3.1, or as in Proposition 3.2, then the equations (5) are satisfied.

Remark 3.3. If p and q are relatively prime, it is straightforward to see that the only $(0,1)$ vectors that are solutions to the equation $P_p^q x = x$ are $x = 0$ and $x = \mathbf{1}$. Suppose now that $gcd(p,q) \equiv g \geq 2$, say with $p = ag$ for some $a \in \mathbb{N}$. Then the solution space to the equation $P_p^q x = x$ is spanned by the vectors $x(j) = \sum_{i=0}^{a-1} e_{j+iq}, j = 1, \ldots, g$ (which are obviously linearly independent). We deduce that the $(0,1)$ solutions to $P_p^q x = x$ are of the form $\sum_{j=1}^{g} a_j x(j)$, where $a_j \in \{0,1\}, j = 1, \ldots, g$. Hence there are 2^g $(0,1)$ vectors x such that $P_p^q x = x$.

Here is the main result of this Section. Observe that it is foreshadowed by Example 2.2.

Theorem 3.2. *Suppose that A is an $m \times n$ $(0,1)$ matrix, E is a $(0,1,-1)$ matrix such that $A+E$ is also $(0,1)$. Suppose further that*

$$E = \left[\begin{array}{ccc|c} P_{m_1}-I & & & \\ & \ddots & & 0 \\ & & P_{m_k}-I & \\ \hline & 0 & & 0 \end{array}\right]$$

and that

$$A = \left[\begin{array}{ccc|c} A_{11} & \dots & A_{1k} & \\ \vdots & \ddots & \vdots & B \\ A_{k1} & \dots & A_{kk} & \\ \hline & C & & D \end{array}\right],$$

where E and A have been partitioned conformally. Then A and E satisfy (4) *if and only if all of the following conditions hold.*

Condition 1– *For each $i = 1, \dots, k$, there are scalars $\alpha_{j,i} \in \{0,1\}, j = 2, \dots, m_i - 1$ such that $\alpha_{j,i} = \alpha_{m_i+1-j,i}, j = 2, \dots, m_i - 1$ and $A_{ii} = I + \sum_{j=2}^{m_i-1} \alpha_{j,i} P_{m_i}^j$.*

Condition 2– *There are $(0,1)$ vectors $u_1, \dots, u_k, v_1, \dots, v_k$ such that*

$$B = \left[\begin{array}{c} \mathbf{1}u_1^T \\ \hline \vdots \\ \hline \mathbf{1}u_k^T \end{array}\right] \text{ and } C = \left[\begin{array}{c|c|c} v_1\mathbf{1}^T & \dots & v_k\mathbf{1}^T \end{array}\right].$$

Condition 3– *For each pair of distinct indices i, j between 1 and k, there is a $(0,1)$ vector $x^{(i,j)}$ in $\mathbb{R}^{m_i}$ such that $P_{m_i}^{m_j} x^{(i,j)} = x^{(i,j)}$, and either*

a) $A_{ij} = J - A_{ji}^T, A_{ji}^T = \left[\begin{array}{c|c|c|c} x^{(i,j)} & P_{m_i}x^{(i,j)} & \dots & P_{m_i}^{m_j-1}x^{(i,j)} \end{array}\right]$

or

b) $A_{ij} = \left[\begin{array}{c|c|c|c} P_{m_i}x^{(i,j)} & x^{(i,j)} & \dots & P_{m_i}^{-m_j+2}x^{(i,j)} \end{array}\right]$ *and*

$A_{ji}^T = \left[\begin{array}{c|c|c|c} x^{(i,j)} & P_{m_i}^{-1}x^{(i,j)} & \dots & P_{m_i}^{-m_j+1}x^{(i,j)} \end{array}\right]$.

Proof. By considering the blocks of $(A+E)(A+E)^T$ and $(A+E)^T(A+E)$, it follows that (4) holds if and only if all of the following conditions are satisfied.

i) For each $i = 1, \dots, k$ we have $(A_{ii} + I - P_{m_i})(A_{ii} + I - P_{m_i})^T = A_{ii}A_{ii}^T$ and $(A_{ii} + I - P_{m_i})^T(A_{ii} + I - P_{m_i}) = A_{ii}^T A_{ii}$.

ii)

$$C\begin{bmatrix} I-P_{m_1}^T & & \\ & \ddots & \\ & & I-P_{m_k}^T \end{bmatrix} = 0 \text{ and}$$

$$\begin{bmatrix} I-P_{m_1}^T & & \\ & \ddots & \\ & & I-P_{m_k}^T \end{bmatrix} B = 0.$$

iii) For each pair of distinct indices i, j between 1 and k, $A_{ij}(I-P_{m_j}^T)+(I-P_{m_i})A_{ji}^T = 0$ and $(I-P_{m_i}^T)A_{ij} + A_{ji}^T(I-P_{m_j}) = 0$.
Applying Theorem 3.1, we find that i) is equivalent to Condition 1, while it is readily established that ii) is equivalent to Condition 2. The equivalence of iii) with Condition 3 follows from Propositions 3.1, 3.2 and Remark 3.2. □

Remark 3.4. In this remark we maintain the notation of Theorem 3.2. From Remark 3.1 we find that for each $i = 1, \ldots, k$ there are $2^{\lfloor \frac{m_i-1}{2} \rfloor}$ matrices A_{ii} satisfying Condition 1. It is clear that there are $2^{k(n-\sum_{i=1}^k m_i)}$ matrices B satisfying Condition 2 and $2^{k(m-\sum_{i=1}^k m_i)}$ matrices C satisfying Condition 2. Next, fix a pair of distinct indices $i, j \in \{1, \ldots, k\}$, and let $g_{ij} = gcd(m_i, m_j)$. It follows from Remark 3.3 and Theorem 3.2 that there are $2^{g_{ij}+1}$ pairs of matrices A_{ij}, A_{ji} satisfying Condition 3 for those indices.

Assembling these observations we find that there are

$$\Pi_{i=1}^k 2^{\lfloor \frac{m_i-1}{2} \rfloor} \times \Pi_{1\le i<j\le k} 2^{g_{ij}+1} \times 2^{k(m+n-2\sum_{i=1}^k m_i)}$$

matrices satisfying each of Conditions 1–3.

Remark 3.5. Suppose that we have two $m \times n$ $(0,1)$ matrices $A, \tilde{A}$ that are related by performing a permutation exchange on an $r \times r$ submatrix S of A. As explained in the earlier part of this Section, there is no loss of generality in assuming that S consists of the leading $r \times r$ principal submatrix of A. For concreteness, we suppose that we have $r \times r$ permutation matrices Q_1, Q_2 such that $S \ge Q_1$ and $S \circ Q_2 = 0$, and that in order to generate $\tilde{A}$ from A, we replace S by $S + Q_2 - Q_1$.

Let R be the $m \times m$ permutation matrix given by $R = \left[\begin{array}{c|c} Q_1^T & 0 \\ \hline 0 & I \end{array}\right]$, set $B = RA$ and $\tilde{B} = R\tilde{A}$. Evidently $AA^T = \tilde{A}\tilde{A}^T$ and $A^TA = \tilde{A}^T\tilde{A}$ if and only if $BB^T = \tilde{B}\tilde{B}^T$ and $B^TB = \tilde{B}^T\tilde{B}$. Observe also that

$$\tilde{B} = B + \left[\begin{array}{c|c} Q_1^TQ_2 - I & 0 \\ \hline 0 & 0 \end{array}\right] \equiv B + E.$$

Next, we recall that the permutation matrix $Q_1^TQ_2$ is itself permutationally similar to direct sum of cyclic permutation matrices, i.e. there is an $r \times r$ permutation matrix U and indices $m_1, \dots, m_k \in \mathbb{N}$ such that

$$UQ_1^TQ_2U^T = \begin{bmatrix} P_{m_1} & & \\ & \ddots & \\ & & P_{m_k} \end{bmatrix}.$$

Consequently, by performing a suitable permutation similarity transformation on B and $\tilde{B}$ simultaneously (which only affects the first r rows and columns of B and $\tilde{B}$) we see that our perturbing matrix E can be taken to have the form

$$\left[\begin{array}{ccc|c} P_{m_1} - I & & & \\ & \ddots & & 0 \\ & & P_{m_k} - I & \\ \hline & 0 & & 0 \end{array}\right]$$

that appears as part of the hypothesis of Theorem 3.2.

It now follows that by performing suitable row and column permutations on both A and $\tilde{A}$, the question of whether or not $AA^T = \tilde{A}\tilde{A}^T$ and $A^TA = \tilde{A}^T\tilde{A}$ is resolved by an application of Theorem 3.2.

4 Conclusion

Corollary 1.1.1 makes it clear that examples of two–mode networks that exhibit data loss are rare from a probabilistic standpoint. Nevertheless, by using tools from combinatorial matrix theory, we are able to construct large infinite families of two–mode networks such that the corresponding incidence matrix A cannot be reconstructed from knowledge of AA^T and A^TA (see Theorem 2.2 and Remark 2.4, as well as Theorem 3.2 and Remark 3.4). Given a $(0, 1, -1)$ matrix E with zero row and column sums, Corollary 2.1.1 and Remark 2.2 provide a linear algebraic technique for determining whether or not there is a pair of

$(0,1)$ matrices A, B whose difference is E such that $AA^T = BB^T$ and $A^TA = B^TB$. However, we have not addressed the more subtle question of how one might determine, for a given $(0,1)$ matrix A whether or not it exhibits data loss; we note in passing that [3] describes how the singular value decomposition can be used to provide some insight on that question. Finally we note that the families of matrices arising in Theorems 2.2 and 3.2 possess a tremendous amount of structure. It remains to be seen whether there are two–mode networks that arise in empirical settings where data loss takes place.

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