Random Walk Centrality and a Partition of Kemeny's Constant

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Abstract

We consider an accessibility index for the states of a discrete–time, ergodic, homogeneous Markov chain on a finite state space; this index is naturally associated with the random walk centrality introduced in [8] for a random walk on a connected graph. We observe that the vector of accessibility indices provides a partition of Kemeny's constant for the Markov chain. We provide three characterisations of this accessibility index: one in terms of the first return time to the state in question, and two in terms of the transition matrix associated with the Markov chain. Several bounds are provided on the accessibility index in terms of the eigenvalues of the transition matrix and the stationary vector, and the bounds are shown to be tight. The behaviour of the accessibility index under perturbation of the transition matrix is investigated, and examples exhibiting some counter–intuitive behaviour are presented. Finally, we characterise the situation in which the accessibility indices for all states coincide.

This paper is warmly dedicated to the memory of Miroslav Fiedler, whose mathematical legacy continues to inspire.

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1 Introduction and preliminaries

An $n \times n$ matrix is *stochastic* if it is entrywise nonnegative and in addition each of its row sums is 1. Stochastic matrices are at the centre of the analysis of Markov chains, and both are well–studied. One of the simplest examples of a Markov chain is that of a random walk on a connected undirected graph $\mathcal{G}$: the states of the Markov chain are the vertices of $\mathcal{G}$, and from state i, corresponding to a vertex of degree d_i say, transitions are only possible to the neighbours of i in the graph, each with probability $\frac{1}{d_i}$. Denote the adjacency matrix of $\mathcal{G}$ by A, and set $D = diag(A\mathbf{1})$, where $\mathbf{1}$ is the all–ones vector of the appropriate order, and for a vector $x \in \mathbb{R}^n, diag(x)$ is the diagonal matrix each of whose diagonal entries are the corresponding entries of x. Observe that for each $i = 1, \ldots, n, d_{i,i}$ is the degree of vertex i. It follows that the transition matrix for the random walk on $\mathcal{G}$ is given by $D^{-1}A$. In this paper we assume familiarity with both stochastic matrices and Markov chains; we refer the interested reader to [9] for necessary background. We also make use of basic concepts and notation in graph theory, and we direct the reader to [2] for that material.

In [8], the authors consider a random walk on a connected undirected graph, say with $n \times n$ transition matrix T. Denoting the stationary vector for T by $w^\top$, [8] introduces two quantities: the *characteristic relaxation time of vertex* k, $\tau_k \equiv \sum_{j=0}^{\infty}((T^j)_{k,k} - w_k)$, and the *random walk centrality of vertex* $k, C_k \equiv \frac{w_k}{\tau_k}$. Observe that the series for τ_k converges only if T is primitive. While there is a good deal of empirical work on random walk centrality (indeed [8] has been cited hundreds of times) there is a paucity of literature analysing random walk centrality from a rigorous mathematical perspective. Our goal in this paper is to investigate a quantity that is closely related to the random walk centrality, and place it in the larger context of a time–homogeneous Markov chain on a finite state space whose transition matrix is irreducible (that is, the directed graph associated with the transition matrix is strongly connected). In particular, several connections with the existing mathematical literature will be made that may inform further research on random walk centrality.

Suppose for concreteness that T is an irreducible stochastic matrix of order n with stationary vector $w^\top$ and mean first passage matrix M. We define the nonnegative vector $\alpha \in \mathbb{R}^n$ via the equation

$$\alpha^\top = w^\top M - \mathbf{1}^\top \tag{1}$$

(throughout we suppress the explicit dependence of α on T). For each $k = 1, \ldots, n$ the k–th entry α_k is the *accessibility index* of state k in the Markov chain corresponding to T. The accessibility index α_k admits a natural interpretation: since $\alpha_k = \sum_{j \neq k} w_j m_{j,k}$, we see that α_k is the expected time, starting from stationarity, that the Markov chain is first in state k (here we take the convention that if state k is the initial state, then the Markov chain is first in state k at time 0). Note further that since $Mw = (K+1)\mathbf{1}$, where K is Kemeny's constant for the Markov chain (see [6]), we have $\alpha^\top w = w^\top M w - \mathbf{1}^\top w = K w^\top \mathbf{1} = K$. Consequently, we may think of the vector α as a partition of Kemeny's constant K, since the weighted average of α (with weights given by the stationary vector w) yields K.

Our first result gives three characterisations of the accessibility index.

Theorem 1.1. *Let T be an irreducible stochastic matrix of order n with stationary vector $w^\top$, and denote $I - T$ by Q. The for each $k = 1, \ldots, n$, we have the following:*
a) $\alpha_k = \frac{q^\#_{k,k}}{w_k}$, where $Q^\#$ denotes the group generalised inverse of Q;
b) $\alpha_k = \frac{r_k^\top (I - T_{(k)})^{-2}\mathbf{1}}{1 + r_k^\top (I - T_{(k)})^{-1}\mathbf{1}}$, where $T_{(k)}$ is the matrix formed from T by deleting its k–th row and column, and $r_k^\top$ is the vector formed from $e_k^\top T$ by deleting its k–th entry;

c) $\alpha_k = \frac{E(R_k^2)}{2E(R_k)} - \frac{1}{2}$, *where* R_k *is the mean first return time to state* k, *and* $E(\cdot)$ *denotes the expected value.*

Proof. a) Let M be the mean first passage matrix for T, and let $Q^\#_{dg} = diag(\begin{bmatrix} q^\#_{1,1} & \dots & q^\#_{n,n} \end{bmatrix})$. Then from Theorem 8.4.1 of [1] we find that $M = (I - Q^\# + JQ^\#_{dg})W^{-1}$, where J is the all–ones matrix of the appropriate order, and $W = diag(w)$. Hence $\alpha^\top = w^\top M - \mathbf{1}^\top = w^\top(I - Q^\# + JQ^\#_{dg})W^{-1} - \mathbf{1}^\top$. Since $w^\top Q^\# = 0^\top$, we find that $\alpha^\top = \mathbf{1}^\top Q^\#_{dg} W^{-1}$. The conclusion now follows readily.
b) From Proposition 2.5.1 of [4] we find that

$$q^\#_{k,k} = \frac{r_k^\top (I - T_{(k)})^{-2}\mathbf{1}}{(1 + r_k^\top (I - T_{(k)})^{-1}\mathbf{1})^2},$$

while from [3] we have $w_k = \frac{1}{1 + r_k^\top (I - T_{(k)})^{-1}\mathbf{1}}$, and the desired formula follows.
c) It is well–known that $w_k = \frac{1}{E(R_k)}$, from which we deduce that $E(R_k) = 1 + r_k^\top (I - T_{(k)})^{-1}\mathbf{1}$. Next we observe that $(I - T_{(k)})^{-2} = \sum_{m=0}^\infty (m+1)(T_{(k)})^m$, so that $r_k^\top (I - T_{(k)})^{-2}\mathbf{1} = \sum_{m=0}^\infty (m+1) r_k^\top (T_{(k)})^m \mathbf{1}$. It is straightforward to determine that for each $m \geq 0$, $r_k^\top (T_{(k)})^m \mathbf{1} = \Pr\{R_k \geq m+2\}$, where $\Pr\{\cdot\}$ denotes the probability of an event. Consequently, we find that

$$\begin{aligned} r_k^\top (I - T_{(k)})^{-2}\mathbf{1} = \sum_{m=0}^\infty (m+1)\Pr\{R_k \geq m+2\} &= \sum_{m=0}^\infty (m+1) \sum_{l=k+2}^\infty \Pr\{R_k = l\} \\ = \sum_{l=2}^\infty \frac{l(l-1)}{2} \Pr\{R_k = l\} &= \frac{1}{2} E(R_k^2 - R_k). \end{aligned}$$

Hence we find from b) that $\alpha_k = \frac{E(R_k^2 - R_k)}{2E(R_k)} = \frac{E(R_k^2)}{2E(R_k)} - \frac{1}{2}$, as desired. □

Corollary 1.1.1. *Maintaining the notation of Theorem 1.1, when* T *is primitive, we have that for each* $k = 1, \dots, n, \alpha_k = \frac{1}{C_k}$.

Proof. From Theorem 8.3.1 of [1], we have $Q^\# = \sum_{j=0}^\infty (T^j - \mathbf{1}w^\top)$, from which we readily deduce that $q^\#_{k,k} = \tau_k, k = 1, \dots, n$. The conclusion now follows from Theorem 1.1 a). □

From Corollary 1.1.1 it is evident that any results proven about the accessibility index can be readily interpreted in terms of the random walk centrality. In the remainder of the paper we frame our results in terms of the α_ks, as they are somewhat more convenient to work with than the C_ks.

Remark 1.1. Suppose that T is an irreducible stochastic matrix of order n with stationary vector $w^\top$ and accessibility vector α. Let $\tilde{w}$ denote the vector formed from w by deleting its last entry. Referring to Observation 2.3.4 of [4], we find that $(I-T)^{\#}$ is given by

$$\left(\tilde{w}^\top (I-T_{(n)})^{-1}\mathbf{1}\right)\mathbf{1}w^\top +$$
$$\left[\begin{array}{c|c} (I-T_{(n)})^{-1} - \mathbf{1}\tilde{w}^\top (I-T_{(n)})^{-1} - (I-T_{(n)})^{-1}\mathbf{1}\tilde{w}^\top & -w_n (I-T_{(n)})^{-1}\mathbf{1} \\ \hline -\tilde{w}^\top (I-T_{(n)})^{-1} & 0 \end{array}\right].$$

Consequently we find that for each $j = 1, \ldots, n-1$,

$$\alpha_j = \alpha_n + \frac{(I-T_{(n)})^{-1}_{j,j} - \tilde{w}^\top (I-T_{(n)})^{-1} e_j - w_j e_j^\top (I-T_{(n)})^{-1}\mathbf{1}}{w_j}. \qquad (2)$$

We round out this section by considering the accessibility index for two families of well–structured examples. For an undirected graph $\mathcal{G}$ we write $i \sim j$ to indicate that vertices i and j are adjacent.

Example 1.1. Let $\mathcal{T}$ be a weighted tree on n vertices, and for each edge e of $\mathcal{T}$, let $\theta(e)$ denote the corresponding edge weight. Let A be the adjacency matrix of the weighted tree $\mathcal{T}$ – i.e. for each $i, j = 1, \ldots, n$, $a_{i,j} = 0$ if i and j are not adjacent, while if i is adjacent to j, $a_{i,j} = \theta(e)$, where e is the edge between i and j. Let $d = A\mathbf{1}$ denote the corresponding vector of row sums. Set $D = diag(d)$, and note that the transition matrix of the random walk on $\mathcal{T}$ is given by $T = D^{-1}A$, which has stationary vector $\frac{1}{\sum_j d_j} d^\top$. For each $k = 1, \ldots, n, \alpha_k = \frac{1}{\sum_j d_j} d_{(k)}^\top (I - D_{(k)}^{-1} A_{(k)})^{-1}\mathbf{1} = \frac{1}{\sum_j d_j} d_{(k)}^\top (D_{(k)} - A_{(k)})^{-1} d_{(k)}$. (Here $d_{(k)}$ denotes the vector formed from d by deleting its k–th entry.)

For each edge e of $\mathcal{T}$ and each $k = 1, \ldots, n$, let $S_k(e)$ denote the set of vertices in the connected component of $\mathcal{T} \setminus e$ that does not include vertex k, and let $\sigma_k(e)$ be its indicator vector. It is straightforward to show that $(D_{(k)} - A_{(k)})^{-1} = \sum_{e\in\mathcal{T}} \frac{1}{\theta(e)} \sigma_k(e)\sigma_k(e)^\top$. Consequently, we find that for each $k = 1, \ldots, n$,

$$\alpha_k = \frac{\sum_{e\in\mathcal{T}} \frac{1}{\theta(e)} (d_{(k)}^\top \sigma_k(e))^2}{\sum_j d_j} = \frac{\sum_{e\in\mathcal{T}} \frac{1}{\theta(e)} (\sum_{j\in S_k(e)} d_j)^2}{\sum_j d_j}.$$

Suppose now that vertices k and l are adjacent in $\mathcal{T}$, and let $\hat{e}$ denote the edge between them. For any edge $e \neq \hat{e}$ in $\mathcal{T}$, we have $S_k(e) = S_l(e)$, from which it follows that $\alpha_k - \alpha_l = \frac{1}{\theta(\hat{e})\sum_j d_j}((\sum_{j\in S_k(\hat{e})} d_j)^2 - (\sum_{j\in S_l(\hat{e})} d_j)^2)$. Since $\sum_j d_j = \sum_{j\in S_k(\hat{e})} d_j + \sum_{j\in S_l(\hat{e})} d_j$, it now follows that

$$\alpha_k - \alpha_l = \frac{1}{\theta(\hat{e})\sum_j d_j}(\sum_j d_j)(2\sum_{j\in S_k(\hat{e})} d_j - \sum_j d_j).$$

In particular we find that $\alpha_k < \alpha_l$ if and only if $\sum_{j\in S_k(\hat{e})} d_j < \frac{\sum_j d_j}{2}$.

We claim that either there is a unique vertex k such that $\alpha_k = \min \alpha_j$, or there are two such vertices k_1, k_2 and they are necessarily adjacent in $\mathcal{T}$. To see the claim, suppose to the contrary that there are two vertices, k_0, k_d such that $\alpha_{k_0} = \alpha_{k_d} = \min \alpha_j$, where $k_0 \sim k_1 \sim \ldots \sim k_d$ is the path from k_0 to k_d, and where $d \geq 2$. For each $j = 1, \ldots . d-1$, let S_j denote the component of $\mathcal{T} \setminus \{k_{j-1} \sim k_j, k_j \sim k_{j+1}\}$ containing vertex k_j. Similarly, let S_0 denote the component of $\mathcal{T} \setminus \{k_0 \sim k_1\}$ containing vertex k_0 and S_d be the component of $\mathcal{T} \setminus \{k_{d-1} \sim k_d\}$ containing vertex k_d. Since $\alpha_{k_0} \leq \alpha_{k_1}$, we find from the above that

$$2\left(\sum_{j=1}^{d} \sum_{l \in S_j} d_l\right) \leq \sum_{j=0}^{d} \sum_{l \in S_j} d_l,$$

so that $\sum_{j=1}^{d} \sum_{l \in S_j} d_l \leq \sum_{l \in S_0} d_l$. Since $\alpha_{k_d} \leq \alpha_{k_{d-1}}$, we find similarly that $\sum_{j=0}^{d-1} \sum_{l \in S_j} d_l \leq \sum_{l \in S_d} d_l$. Summing these two inequalities now yields

$$\sum_{l \in S_0} d_l + 2\sum_{j=1}^{d-1} \sum_{l \in S_j} d_l + \sum_{l \in S_d} d_l \leq \sum_{l \in S_0} d_l + \sum_{l \in S_d} d_l,$$

which is impossible. We thus conclude that either there is a unique vertex minimising α_j, or there are just two such vertices, which are necessarily adjacent.

Finally we claim that $\max \alpha_j$ is attained only at a pendent vertex of $\mathcal{T}$. To see the claim, suppose to the contrary that for vertex m of degree $k \geq 2$ we have $\alpha_m = \max \alpha_j$, and denote the edges incident with m by $e_1, \ldots, e_k$. Considering the edges incident with m, we find that for $j = 1, \ldots, k, 2\sum_{l \in S_m(e_j)} d_l \geq \sum_{p=1}^{k} \sum_{l \in S_m(e_p)} d_l + d_m$. Summing these inequalities and recalling that $d_m = k$, we have $2\sum_{p=1}^{k} \sum_{l \in S_m(e_p)} d_l \geq k\sum_{p=1}^{k} \sum_{l \in S_m(e_p)} d_l + k^2$, a contradiction since $k \geq 2$. Consequently, α_j is maximised at a pendent vertex, as claimed.

For a strongly connected directed graph $\mathcal{H}$, we can define a random walk on $\mathcal{H}$ analogously with the undirected case. Again the transition matrix of such a random walk is given by $D^{-1}A$, where A is the adjacency matrix of $\mathcal{H}$ and $D = diag(A\mathbf{1})$. Next we consider an example of a random walk on a particular family of tournaments.

Example 1.2. Suppose that $n \geq 3$, and consider the tournament on n vertices whose adjacency matrix is given by

$$A = \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 & 1 \\ 1 & 0 & 0 & \ldots & 0 & 0 \\ 1 & 1 & 0 & \ldots & 0 & 0 \\ \vdots & & \ddots & \ddots & & \vdots \\ 1 & 1 & \ldots & 1 & 0 & 0 \\ 0 & 1 & \ldots & 1 & 1 & 0 \end{bmatrix}.$$

We now form the transition matrix T for the simple random walk on this tournament, i.e.

$$T = \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 & 1 \\ 1 & 0 & 0 & \ldots & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & \ldots & 0 & 0 \\ \vdots & & \ddots & \ddots & & \vdots \\ \frac{1}{n-2} & \frac{1}{n-2} & \ldots & \frac{1}{n-2} & 0 & 0 \\ 0 & \frac{1}{n-2} & \ldots & \frac{1}{n-2} & \frac{1}{n-2} & 0 \end{bmatrix}.$$

It is straightforward to verify that stationary vector for T is given by

$$w^\top = \frac{1}{(n-1)\sum_{j=1}^{n-2}\frac{1}{j} + n - 2}\left[\begin{array}{cccccccc} n-2 & \frac{n-1}{2} & \frac{n-1}{3} & \ldots & \frac{n-1}{n-2} & \frac{n-1}{n-1} & n-2 \end{array}\right].$$

In particular the maximum entries are $w(1)$ and $w(n)$, while the entries $w(2), \ldots, w(n-1)$ are decreasing, with the pattern $w(j) = \frac{\text{constant}}{j}$. Our goal is to compute α for this example.

A computation reveals that

$$(I - T_{(n)})^{-1} = \begin{bmatrix} 1 & 0 & 0 & & \ldots & 0 & 0 \\ 1 & 1 & 0 & & \ldots & 0 & 0 \\ 1 & \frac{1}{2} & 1 & 0 & \ldots & 0 & 0 \\ 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \ddots & & \vdots \\ 1 & \frac{1}{2} & \frac{1}{3} & \ldots & \frac{1}{n-3} & 1 & 0 \\ 1 & \frac{1}{2} & \frac{1}{3} & \ldots & \frac{1}{n-3} & \frac{1}{n-2} & 1 \end{bmatrix}.$$

We deduce that

$$e_1^\top (I - T_{(n)})^{-1}\mathbf{1} = 1, e_j^\top (I - T_{(n)})^{-1}\mathbf{1} = 1 + \sum_{l=1}^{j-1}\frac{1}{l},$$

for $j = 2, \ldots, n-1$. Letting $\tilde{w}$ be the vector formed from w by deleting its last entry, we find similarly that $\tilde{w}^\top (I - T_{(n)})^{-1} e_1 = 1 - w_n = 1 - \frac{n-2}{(n-1)\sum_{j=1}^{n-2}\frac{1}{j}+n-2}$ and

$$\tilde{w}^\top (I - T_{(n)})^{-1} e_j = w_j \left(1 + \sum_{l=j+1}^{n-1}\frac{1}{l}\right) = \frac{n-1}{j((n-1)\sum_{j=1}^{n-2}\frac{1}{j} + n - 2)}\left(1 + \sum_{l=j+1}^{n-1}\frac{1}{l}\right),$$

for $j = 2, \ldots, n-1$.

It now follows that $\alpha_n = \tilde{w}^\top (I - T_{(n)})^{-1}\mathbf{1} = 2 - 3w_1 + \sum_{l=3}^{n-1}\frac{w_2+\ldots+w_{l-1}}{l}$. Referring to (2), we see that for each $j = 1, \ldots, n-1$,

$$\alpha_j = \alpha_n + \frac{(I - T_{(n)})^{-1}_{j,j} - \tilde{w}^\top (I - T_{(n)})^{-1} e_j - w_j e_j^\top (I - T_{(n)})^{-1}\mathbf{1}}{w_j}.$$

Substituting $j = 1$ yields $\alpha_1 = \alpha_n + \frac{1-(1-w_n)-w_1}{w_1} = \alpha_n$.

For $j = 2, \ldots, n-1$, note that

$$(I - T_{(n)})^{-1}_{j,j} - \tilde{w}^\top (I - T_{(n)})^{-1} e_j - w_j e_j^\top (I - T_{(n)})^{-1}\mathbf{1} = 1 - w_j \left(1 + \sum_{l=j+1}^{n-1} \frac{1}{l}\right) - w_j \left(1 + \sum_{l=1}^{j-1} \frac{1}{l}\right) = 1 - w_j \left(2 + \sum_{l=1}^{n-1} \frac{1}{l} - \frac{1}{j}\right).$$

Hence we have

$$\frac{(I - T_{(n)})^{-1}_{j,j} - \tilde{w}^\top (I - T_{(n)})^{-1} e_j - w_j e_j^\top (I - T_{(n)})^{-1}\mathbf{1}}{w_j} =$$

$$\frac{(n-2)j}{n-1} + j \sum_{l=1}^{n-2} \frac{1}{l} - 2 - \sum_{l=1}^{n-1} \frac{1}{l} + \frac{1}{j} = j \left(\frac{n-2}{n-1} + \sum_{l=1}^{n-2} \frac{1}{l}\right) - \left(2 + \sum_{l=1}^{n-1} \frac{1}{l}\right) + \frac{1}{j}.$$

It now follows that for $j = 2, \ldots, n-1$,

$$\alpha_j = \alpha_n + j \left(\frac{n-2}{n-1} + \sum_{l=1}^{n-2} \frac{1}{l}\right) - \left(2 + \sum_{l=1}^{n-1} \frac{1}{l}\right) + \frac{1}{j}.$$

In particular we find that for $j = 2, \ldots, n-1, \alpha_j$ is an increasing function of j that is close to being linear.

2 Bounds on the accessibility index

In the restricted setting of a random walk on a connected graph [8] considers an expression for τ_k in terms of eigenvalues and eigenvectors associated with the transition matrix T. Denoting the eigenvalues of T by $1 \equiv \lambda_1, \lambda_2, \ldots, \lambda_n$, and setting $\gamma = \max\{|\lambda_2|, \ldots, |\lambda_n|\}$, [8] states that

$$\tau_k \approx \frac{a_k b_k}{|\ln \gamma|}, \tag{3}$$

where a and $b^\top$ are right and left eigenvectors of T corresponding to the eigenvalue associated with γ, normalised so that $b^\top a = 1$. (It seems that there may be an implicit assumption that the eigenvalue associated with γ is a simple eigenvalue of T here.)

Example 2.1. Let $n \in \mathbb{N}$ and consider the undirected graph $\mathcal{G}$ on $n+3$ vertices formed from $K_{1,n+2}$ by adding a single edge. With a suitable labelling of the vertices, the transition matrix for the random walk on $\mathcal{G}$ is given by

$$T = \left[\begin{array}{cc|cccc|c} 0 & \frac{1}{2} & 0 & \ldots & & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \ldots & & 0 & \frac{1}{2} \\ \hline 0 & 0 & 0 & \ldots & & 0 & 1 \\ \vdots & \vdots & \vdots & \ddots & & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & & 0 & 1 \\ \hline \frac{1}{n+2} & \frac{1}{n+2} & & \frac{1}{n+2}\mathbf{1}^\top & & & 0 \end{array}\right].$$

It turns out that the eigenvalues of T are given by $1, -\frac{1}{2}, 0$ (with multiplicity $n-1$) and $-\frac{1}{4} \pm \frac{1}{2}\sqrt{\frac{1}{4}+\frac{2n}{n+2}}$. Hence, in the notation above we have $\gamma = |\lambda|$, where $\lambda = -\frac{1}{4} - \frac{1}{2}\sqrt{\frac{1}{4}+\frac{2n}{n+2}}$. We have the following right and left eigenvectors respectively, corresponding to λ and partitioned conformally with T:

$$x = \left[\begin{array}{c} \frac{1}{1-2\lambda}\mathbf{1} \\ \hline -\frac{1}{\lambda}\mathbf{1} \\ \hline -1 \end{array}\right], y^\top = \left[\begin{array}{c|c|c} \frac{2}{(n+2)(1-2\lambda)}\mathbf{1}^\top & -\frac{1}{(n+2)\lambda}\mathbf{1}^\top & -1 \end{array}\right].$$

A short computation shows that $y^\top x = 1 + \frac{n}{(n+2)\lambda^2} + \frac{4}{(n+2)(1-2\lambda)^2}$, and it now follows that for $k = n+3$, the right side of (3) is equal to

$$\frac{1}{\left(1 + \frac{n}{(n+2)\lambda^2} + \frac{4}{(n+2)(1-2\lambda)^2}\right)\left|\ln\left(\frac{1}{4}+\frac{1}{2}\sqrt{\frac{1}{4}+\frac{2n}{n+2}}\right)\right|}.$$

On the other hand, using Proposition 2.5.1 of [4], we find that $(n+3, n+3)$ entry of $(I-T)^\#$, which coincides with τ_{n+3}, is equal to $\frac{(n+2)(n+8)}{(2n+6)^2}$.

Note that as $n \to \infty$, $\frac{1}{4}+\frac{1}{2}\sqrt{\frac{1}{4}+\frac{2n}{n+2}} \to 1$, so that the right side of (3) diverges to infinity. On the other hand, $n \to \infty$, $\frac{(n+2)(n+8)}{(2n+6)^2} \to \frac{1}{4}$, so that the left side of (3) converges to $\frac{1}{4}$. Evidently the approximation provided by (3) is not especially accurate for this family of examples.

Motivated by (3) and Example 2.1, in this section we provide several eigenvalue–eigenvector–based bounds on α_k, and hence on τ_k. We begin with the following simple lower bound on the accessibility index.

Proposition 2.1. *For each $k = 1, \ldots, n, \alpha_k \geq 1 - w_k$; equality holds for state k if and only if the directed graph of T is a directed star, possibly with a loop at the centre vertex, with k as the centre vertex.*

Proof. Without loss of generality, we take $k = n$. From (1) we have $\alpha_n = \tilde{w}^\top(I - T_{(n)})^{-1}\mathbf{1}$, where $\tilde{w}$ is obtained from w by deleting its last entry. Since $(I - T_{(n)})^{-1} \geq I$, we find that $\alpha_n \geq \tilde{w}^\top\mathbf{1} = 1 - w_n$, establishing the inequality. From this argument we deduce that $\alpha_n = 1 - w_n$ if and only if $\tilde{w}^\top T\mathbf{1} = 0$, i.e. if and only if $T_{(n)} = 0$. Evidently $T_{(n)} = 0$ if and only if directed graph of T is a directed star with centre at vertex n, possibly with a loop at vertex n.

□

Remark 2.1. Let T be an irreducible stochastic matrix of order n, denote the eigenvalues of T by $1 \equiv \lambda_1, \lambda_2, \ldots, \lambda_n$, and denote the stationary vector of T by $w^\top$. Meyer [7] has shown that for $Q = I - T$, we have

$$|q^\#_{j,j}| < \frac{(n-1)}{\Pi_{k=2}^n(1-\lambda_k)}, j = 1, \ldots, n \tag{4}$$

(the argument establishing this fact is embedded in the proof of that paper's main result, Theorem 2.1). From (4) we thus see that for any $j = 1, \ldots, n$, $\alpha_j < \frac{(n-1)}{w_j \Pi_{k=2}^n (1-\lambda_k)}$.

The following example shows that for diagonal entries of $Q^{\#}$, the upper bound $\frac{(n-1)}{\Pi_{k=2}^n (1-\lambda_k)}$ cannot be improved.

Example 2.2. Suppose that $0 < t < 1$, and that $u \in \mathbb{R}^{n-1}$ is a nonnegative nonzero vector such that $u^\top \mathbf{1} \leq 1$. Consider the matrix T given by

$$T = \left[\begin{array}{cccccc|c} 0 & 0 & \ldots & & 0 & t & 1-t \\ 1 & 0 & 0 & & \ldots & 0 & 0 \\ 0 & 1 & 0 & & \ldots & 0 & 0 \\ \vdots & & \ddots & & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & & 1 & 0 & 0 \\ \hline & & u^\top & & & & 1 - u^\top \mathbf{1} \end{array}\right],$$

and observe that T is irreducible.

Denoting the eigenvalues of T by $1 \equiv \lambda_1, \lambda_2, \ldots, \lambda_n$, it is shown in [4] (see formula (5.38)) that

$$\Pi_{k=2}^n (1 - \lambda_k) = 1 - t + (n-1) \sum_{k=1}^{n-1} u_k - (1-t) \sum_{k=1}^{n-2} (n-k-1) u_k. \tag{5}$$

Next, we set $Q = I - T$ and consider $q^{\#}_{n,n}$. It is straightforward to determine that the inverse of the leading principal submatrix of order $n-1$ of Q is given by

$$\begin{bmatrix} 1 & t & t & \ldots & t \\ 1 & 1 & t & \ldots & t \\ \vdots & & \ddots & \ddots & \vdots \\ 1 & 1 & \ldots & 1 & t \\ 1 & 1 & \ldots & 1 & 1 \end{bmatrix}.$$

An uninteresting computation now reveals that

$$q^{\#}_{n,n} = \frac{\sum_{i=1}^{n-1} u_i \left[\frac{i(i+1)}{2} + t(\frac{n(n-1)}{2} + i(n-2) - i^2) + \frac{t^2(n-2-i)(n-1-i)}{2} \right]}{(1 - t + \sum_{i=1}^{n-1} u_i [i + t(n-i-1)])^2}. \tag{6}$$

From (6) and (5) we find that $\frac{q^{\#}_{n,n} \Pi_{k=2}^n (1-\lambda_k)}{n-1}$ is given by $\frac{a(t)}{b(t)}$, where

$a(t) =$

$$\left(1 - t + (n-1) \sum_{k=1}^{n-1} u_k - (1-t) \sum_{k=1}^{n-2} (n-k-1) u_k \right) \times$$

$$\sum_{i=1}^{n-1} u_i \left[\frac{i(i+1)}{2} + t \left(\frac{n(n-1)}{2} + i(n-2) - i^2 \right) + \frac{t^2(n-2-i)(n-1-i)}{2} \right]$$

and $b(t) = (n-1)\left(1 - t + \sum_{i=1}^{n-1} u_i[i + t(n-i-1)]\right)^2$. Referring to (7), we find that as $t \to 1^-$, $\frac{q^{\#}_{n,n}\Pi_{k=2}^{n}(1-\lambda_k)}{n-1} \to 1$. In particular we see that for the diagonal entries of $Q^{\#}$, the upper bound of (4) is sharp.

The inequality (4) and Example 2.2 provides a bound on the accessibility index for any irreducible stochastic matrix T; next we turn our attention to the case that T corresponds to a reversible Markov chain. Recall that for an irreducible stochastic matrix T with stationary vector w, the corresponding Markov chain is said to be *reversible* if $W^{\frac{1}{2}}TW^{\frac{-1}{2}}$ is symmetric, where $W = diag(w)$. The proof of the following result is essentially given in [5]; we give a shortened argument here.

Theorem 2.1. *Suppose that T is an irreducible transition matrix of order n that is associated with a reversible Markov chain. Denote the eigenvalues of T by $1 \equiv \lambda_1 > \lambda_2 \geq \lambda_3 \geq \ldots \geq \lambda_n$ and let w be the stationary vector for T. For each $k = 1, \ldots, n$, we have*

$$\frac{1-w_k}{w_k(1-\lambda_n)} \leq \alpha_k \leq \frac{1-w_k}{w_k(1-\lambda_2)}. \tag{7}$$

In the case that $\lambda_2 = \lambda_n$, equality holds throughout (7); this is equivalent to T having the form $t\mathbf{1}w^\top + (1-t)I$ for some $0 \leq t \leq \frac{1}{1-\min_j w_j}$.

Suppose now that $\lambda_2 > \lambda_n$. Equality holds in the upper bound if and only if there is a permutation matrix P with $Pe_1 = e_k$ such that $P^\top TP$ has the form

$$\left[\begin{array}{c|c} 1 - \frac{\alpha(1-w_k)}{w_k} & \frac{\alpha}{w_k}\overline{w}^\top \\ \hline \alpha\mathbf{1} & (1-\alpha)S \end{array}\right], \tag{8}$$

where:
i) $\overline{w}$ is the vector formed from w by deleting its k–th entry;
ii) $0 < \alpha < 1$;
iii) S is stochastic;
iv) letting $\overline{W} = diag(\overline{w})$, S is such that $\overline{W}^{\frac{1}{2}}S\overline{W}^{-\frac{1}{2}}$ is symmetric with eigenvalues $1 \equiv \mu_1 \geq \mu_2 \geq \ldots \geq \mu_{n-1}$; and
v) $1 - \frac{\alpha}{w_k} \geq (1-\alpha)\mu_2$.

Equality holds in the lower bound in (7) if and only if there is a stochastic matrix S such that T can be permuted to the form (8) where:
vi) $0 < \alpha \leq 1$;
vii) $\overline{W}$ is as in i);
viii) $\overline{W}^{\frac{1}{2}}S\overline{W}^{-\frac{1}{2}}$ is symmetric with eigenvalues $1 \equiv \mu_1 \geq \mu_2 \geq \ldots \geq \mu_{n-1}$; and
ix) $1 - \frac{\alpha}{w_k} \leq (1-\alpha)\mu_{n-1}$.

Proof. Since T corresponds to a reversible Markov chain, the matrix $A = W^{\frac{1}{2}}TW^{\frac{-1}{2}}$ is symmetric, where $W = diag(w)$. Further, $v_1 \equiv W^{\frac{1}{2}}\mathbf{1}$ is the Perron vector of A having 2–norm equal to 1. Set $Q = I - A$. In order to prove the theorem, it suffices to show that for each $k = 1, \ldots, n$, $\frac{1-w_k}{1-\lambda_n} \le q^{\#}_{k,k} \le \frac{1-w_k}{1-\lambda_2}$, then characterise the equality cases.

Let $v_2, \ldots, v_n$ denote an orthonormal collection of eigenvectors corresponding to $\lambda_2, \ldots, \lambda_n$, respectively. For any such k we have

$$q^{\#}_{k,k} = \sum_{j=2}^{n} \frac{(e_k^\top v_j)^2}{1-\lambda_j} \le \frac{1}{1-\lambda_2} \sum_{j=2}^{n} (e_k^\top v_j)^2 = \frac{1}{1-\lambda_2}(1-(e_k^\top v_1)^2) = \frac{1-w_k}{1-\lambda_2}.$$

An analogous argument shows that $\frac{1-w_k}{1-\lambda_n} \le q^{\#}_{k,k}, k = 1, \ldots, n$. Evidently if $\lambda_2 = \lambda_n$, then equality must hold throughout (7), and this is readily seen to be equivalent to the condition that for some $0 \le t \le \frac{1}{1-\min_j w_j}, T = t\mathbf{1}w^\top + (1-t)I$.

Henceforth we assume that $\lambda_2 > \lambda_n$, and without loss of generality, we take $k = 1$. Suppose that $\frac{1-w_1}{1-\lambda_2} = q^{\#}_{1,1}$. Examining the argument above, it must be the case that for each j such that $\lambda_2 > \lambda_j, e_1^\top v_j = 0$. Further, by taking linear combinations of the orthonormal basis of the λ_2–eigenspace for A if necessary, we can assume without loss of generality that in fact $e_1^\top v_j = 0$ for $j = 3, \ldots, n$. We partition off the first row and columns of A, writing A as

$$A = \left[\begin{array}{c|c} a & u^\top \\ \hline u & M \end{array}\right].$$

For each $j = 3, \ldots, n$, let z_j be formed from v_j by deleting the first entry. Evidently $Mz_j = \lambda_j z_j$ and $u^\top z_j = 0, j = 3, \ldots, n$. It now follows that u must be an eigenvector of M, say with $Mu = (1-\alpha)u$ for some $0 < \alpha \le 1$. Write $M = (1-\alpha)Y$, and denote the eigenvalues of Y by $\mu_1 \ge \mu_2 \ge \ldots \ge \mu_{n-1}$,

Since v_1 and v_2 are orthogonal to $v_3, \ldots, v_n$, it follows that for some $s, t \in \mathbb{R}, v_1, v_2$ are scalar multiples of the vectors

$$\left[\begin{array}{c} s \\ \hline u \end{array}\right], \left[\begin{array}{c} t \\ \hline u \end{array}\right],$$

respectively. From the fact that the Perron value of A is 1, it follows that $s = \alpha$ and $a = 1 - \frac{u^\top u}{\alpha}$. From this we find that $t = -\frac{u^\top u}{\alpha}$ and $\lambda_2 = 1 - \alpha - \frac{u^\top u}{\alpha}$. Next, observe that since $v_1 = W^{\frac{1}{2}}\mathbf{1}$ is a scalar multiple of $\left[\begin{array}{c} \alpha \\ \hline u \end{array}\right]$, it now follows that $u = \frac{\alpha}{\sqrt{w_1}}\overline{W}^{\frac{1}{2}}\mathbf{1}$. Consequently, we find that $u^\top u = \frac{\alpha^2(1-w_1)}{w_1}$, and so $a = 1 - \frac{\alpha(1-w_1)}{w_1}$ while $\lambda_2 = 1 - \frac{\alpha}{w_1}$. Since the eigenvalues of A are $1, \lambda_2$ and $(1-\alpha)\mu_2, \ldots, (1-\alpha)\mu_{n-1}$, it must be the case that $1 - \frac{\alpha}{w_1} \ge (1-\alpha)\mu_2$. Observe that this last condition cannot hold if $\alpha = 1$, so in fact we have $0 < \alpha < 1$.

Assembling the observations above, we find that

$$A = \left[\begin{array}{c|c} 1 - \frac{\alpha(1-w_1)}{w_1} & \frac{\alpha}{\sqrt{w_1}}\mathbf{1}^\top\overline{W}^{\frac{1}{2}} \\ \hline \frac{\alpha}{\sqrt{w_1}}\overline{W}^{\frac{1}{2}}\mathbf{1} & (1-\alpha)Y \end{array}\right].$$

Recalling that $T = W^{-\frac{1}{2}} A W^{\frac{1}{2}}$, (8) now follows. The converse is straightforward.

Finally, an analogous argument establishes the characterisation of equality in the lower bound in (7); observe that in this case, the value $\alpha = 1$ is admissible. $\square$

Remark 2.2. Suppose that $\mathcal{G}$ is a connected undirected graph on n vertices with adjacency matrix A, and let D denote the diagonal matrix of vertex degrees. Then $T = D^{-1}A$ is the transition matrix of the random walk on $\mathcal{G}$. In this remark, we identify the graphs for which the corresponding random walk yields equality in either the left–hand or right–hand inequality in (2.1). Note that T has just two distinct eigenvalues if and only if $\mathcal{G} = K_n$, and evidently equality holds throughout (2.1) in that case.

Suppose now that $\mathcal{G} \neq K_n$, and that equality holds in either of the inequalities of (7), say with $k = 1$. Necessarily T is of the form given in (8). Partitioning A conformally with T, and writing D as $\left[\begin{array}{c|c} d_1 & 0^\top \\ \hline 0 & \overline{D} \end{array}\right]$, we have

$$T = \left[\begin{array}{c|c} 0 & \frac{1}{d_1} r^\top \\ \hline \overline{D}^{-1} r & \overline{D}^{-1}\overline{A} \end{array}\right],$$

where $\overline{A}$ is the adjacency matrix of the subgraph $\overline{\mathcal{G}}$ of $\mathcal{G}$ induced by vertices $2, \ldots, n$, and where the vector r is 1 or 0 in position $j-1$ according as vertex j is adjacent to vertex 1 or not, $j = 2, \ldots, n$. Since $\overline{D}^{-1} r$ must be a multiple of the all ones vector (and so in particular must be positive), we deduce that $r = \mathbf{1}$. Hence $\overline{D}^{-1}\mathbf{1} = \alpha \mathbf{1}$ for some $0 < \alpha \leq 1$; since $\overline{D} = D_0 + I$, where D_0 is the diagonal matrix of vertex degrees of $\overline{\mathcal{G}}$, it follows that in fact $\overline{\mathcal{G}}$ must be regular, say of degree δ. If $\delta = 0$, then $\overline{\mathcal{G}}$ is an empty graph and we find that $\mathcal{G} = K_{1,n-1}$; it is readily seen that equality holds in the lower bound in (2.1) in that case.

Suppose henceforth that $\delta \geq 1$. Since $r = \mathbf{1}, d_1 = n-1$, and it follows now that $w_1 = \frac{1}{\delta+2}$. Since $1 - \frac{\alpha(1-w_1)}{w_1} = 0$, we deduce that $\alpha = \frac{1}{\delta+1}$. Denote the eigenvalues of $\overline{A}$ by $\gamma_1 \equiv \delta \geq \gamma_2 \geq \ldots \geq \gamma_{n-1}$, so that $\mu_j = \frac{\gamma_j}{\delta}, j = 2, \ldots, n-1$. The conditions $1 - \frac{\alpha}{w_1} \geq (1-\alpha)\mu_2$ and $1 - \frac{\alpha}{w_1} \geq (1-\alpha)\mu_{n-1}$ are thus equivalent to $\gamma_2 \leq -1$ and $\gamma_{n-1} \geq -1$, respectively. It is well–known that $\gamma_2 \leq -1$ if and only if $\overline{\mathcal{G}} = K_{n-1}$ and that $\gamma_{n-1} \geq -1$ if and only if $\overline{\mathcal{G}} = \frac{n-1}{2}K_2$. In former case we have $\mathcal{G} = K_n$, and in the latter we have $\mathcal{G} = K_1 \vee \frac{n-1}{2}K_2$, where '$\vee$' denotes the join operation for graphs. Observe that for $\mathcal{G} = K_1 \vee \frac{n-1}{2}K_2$, equality holds in the lower bound in (2.1).

Consequently there are just three families of graphs for which the random walks yield equality in (7): complete graphs, stars, and graphs of the form $K_1 \vee \ell K_2$.

3 Behaviour of the accessibility index

Suppose that we have an irreducible stochastic matrix T and a corresponding accessibility index α_k. Intuitively, one may expect that if we decrease a transition probability into state k (with a compensating increase of another transition probability), then the accessibility index for state k will increase. Our next result confirms that intuition in the case that α_k is sufficiently small.

Theorem 3.1. *Suppose that T is an irreducible stochastic matrix of order n, and that for some $1 \le k \le n, \alpha_k < 2$. Fix indices $1 \le i, j \le n, i, j \ne k$, and suppose that $t_{i,j}, t_{i,k} > 0$, consider the family of stochastic matrices $T(\epsilon) = T + \epsilon e_i(e_j - e_k)^\top, \epsilon \in (-t_{i,j}, t_{i,k})$ and denote the corresponding accessibility indices by $\alpha_k(\epsilon)$. Then $\frac{d\alpha_k(\epsilon)}{d\epsilon}|_{\epsilon=0} > 0$.*

Proof. Without loss of generality, we take $k = n$. From Theorem 1.1 b), we find, keeping the notation of that result, that for all sufficiently small $\epsilon > 0$,

$$\alpha_n(\epsilon) = \frac{r_n^\top (I - T_{(n)} - \epsilon e_i e_j^\top)^{-2}\mathbf{1}}{1 + r_n^\top (I - T_{(n)} - \epsilon e_i e_j^\top)^{-1}\mathbf{1}}.$$

From the Sherman–Morrison formula, we find that

$$(I - T_{(n)} - \epsilon e_i e_j^\top)^{-1} = (I - T_{(n)})^{-1} + \frac{\epsilon}{1 - \epsilon e_j^\top (I - T_{(n)})^{-1} e_i}(I - T_{(n)})^{-1} e_i e_j^\top (I - T_{(n)})^{-1}.$$

Set $g(\epsilon) \equiv 1 + r_n^\top (I - T_{(n)} - \epsilon e_i e_j^\top)^{-1}\mathbf{1}$ and $f(\epsilon) \equiv r_n^\top (I - T_{(n)} - \epsilon e_i e_j^\top)^{-2}\mathbf{1}$, so that $\alpha_n(\epsilon) = \frac{f(\epsilon)}{g(\epsilon)}$. It now follows that

$$g(\epsilon) = 1 + r_n^\top (I - T_{(n)})^{-1}\mathbf{1} + \frac{\epsilon}{1 - \epsilon e_j^\top (I - T_{(n)})^{-1} e_i} r_n^\top (I - T_{(n)})^{-1} e_i e_j^\top (I - T_{(n)})^{-1}\mathbf{1},$$

and that

$$f(\epsilon) = r_n^\top (I - T_{(n)})^{-2}\mathbf{1} + \frac{\epsilon}{1 - \epsilon e_j^\top (I - T_{(n)})^{-1} e_i}(r_n^\top (I - T_{(n)})^{-2} e_i e_j^\top (I - T_{(n)})^{-1}\mathbf{1} + r_n^\top (I - T_{(n)})^{-1} e_i e_j^\top (I - T_{(n)})^{-2}\mathbf{1}) + \left(\frac{\epsilon}{1 - \epsilon e_j^\top (I - T_{(n)})^{-1} e_i}\right)^2 (e_j^\top (I - T_{(n)})^{-2} e_i)(r_n^\top (I - T_{(n)})^{-1} e_i e_j^\top (I - T_{(n)})^{-1}\mathbf{1}).$$

Straightforward computations show that

$$\frac{df(\epsilon)}{d\epsilon}|_{\epsilon=0} = r_n^\top (I - T_{(n)})^{-2} e_i e_j^\top (I - T_{(n)})^{-1}\mathbf{1} + r_n^\top (I - T_{(n)})^{-1} e_i e_j^\top (I - T_{(n)})^{-2}\mathbf{1},$$

while

$$\frac{dg(\epsilon)}{d\epsilon}|_{\epsilon=0} = r_n^\top (I - T_{(n)})^{-1} e_i e_j^\top (I - T_{(n)})^{-1}\mathbf{1}.$$

Hence we have

$$\frac{df(\epsilon)}{d\epsilon}|_{\epsilon=0} g(0) - f(0)\frac{dg(\epsilon)}{d\epsilon}|_{\epsilon=0} =$$
$$g(0)\frac{dg(\epsilon)}{d\epsilon}|_{\epsilon=0}\left(\frac{r_n^\top(I-T_{(n)})^{-2}e_i}{r_n^\top(I-T_{(n)})^{-1}e_i} + \frac{e_j^\top(I-T_{(n)})^{-2}\mathbf{1}}{e_j^\top(I-T_{(n)})^{-1}\mathbf{1}} - \frac{f(0)}{g(0)}\right). \quad (9)$$

Evidently $\frac{r_n^\top(I-T_{(n)})^{-2}e_i}{r_n^\top(I-T_{(n)})^{-1}e_i}, \frac{e_j^\top(I-T_{(n)})^{-2}\mathbf{1}}{e_j^\top(I-T_{(n)})^{-1}\mathbf{1}} \geq 1$, while $\frac{f(0)}{g(0)} = \alpha_n(0) < 2$ by hypothesis. Consequently $\frac{df(\epsilon)}{d\epsilon}|_{\epsilon=0} g(0) - f(0)\frac{dg(\epsilon)}{d\epsilon}|_{\epsilon=0} > 0$, and the conclusion follows readily. □

Example 3.1. Suppose that $n \in \mathbb{N}$ with $n \geq 4$, that $0 < a < 1$ and $0 < t, b < 1$. Consider the $n \times n$ matrix

$$T = \left[\begin{array}{c|c|c|c} 0 & 0_{n-3}^\top & a & 1-a \\ \hline 0_{n-3} & (1-t)I_{n-3} & 0_{n-3} & t\mathbf{1}_{n-3} \\ \hline 0 & 0_{n-3}^\top & 1 & 0 \\ \hline b & \frac{1-b}{n-2}\mathbf{1}_{n-3}^\top & \frac{1-b}{n-2} & 0 \end{array}\right]$$

(here subscripts on matrices and vectors denote their orders). Keeping the notation of Theorem 3.1, and considering $i = 1, j = n-1$, we want to look at the effect on α_n of decreasing $t_{1,n}$ and increasing $t_{1,n-1}$ by the corresponding amount. Thus, for $\epsilon \in (-a, 1-a)$, we let $T(\epsilon) = T + \epsilon e_1(e_{n-1} - e_n)^\top$, and denote the corresponding accessibility index for state n by $\alpha_n(\epsilon)$. As in the proof of Theorem 3.1, we find that $\frac{d\alpha_n(\epsilon)}{d\epsilon}|_{\epsilon=0}$ is negative, zero, or positive according as $\frac{r_n^\top(I-T_{(n)})^{-2}e_i}{r_n^\top(I-T_{(n)})^{-1}e_i} + \frac{e_j^\top(I-T_{(n)})^{-2}\mathbf{1}}{e_j^\top(I-T_{(n)})^{-1}\mathbf{1}} - \alpha_n(0)$ is negative, zero, or positive, respectively.

It it straightforward to determine that $(I - T_{(n)})^{-1} = \left[\begin{array}{c|c|c} 1 & 0_{n-3}^\top & a \\ \hline 0_{n-3} & \frac{1}{t}I_{n-3} & 0_{n-3} \\ \hline 0 & 0_{n-3}^\top & 1 \end{array}\right]$ and $(I - T_{(n)})^{-2} = \left[\begin{array}{c|c|c} 1 & 0_{n-3}^\top & 2a \\ \hline 0_{n-3} & \frac{1}{t^2}I_{n-3} & 0_{n-3} \\ \hline 0 & 0_{n-3}^\top & 1 \end{array}\right]$. We find readily that $\frac{r_n^\top(I-T_{(n)})^{-2}e_1}{r_n^\top(I-T_{(n)})^{-1}e_1} = 1, \frac{e_{n-1}^\top(I-T_{(n)})^{-2}\mathbf{1}}{e_{n-1}^\top(I-T_{(n)})^{-1}\mathbf{1}} = 1$, while

$$\alpha_n = \frac{r_n^\top(I-T_{(n)})^{-2}\mathbf{1}}{1 + r_n^\top(I-T_{(n)})^{-1}\mathbf{1}} = \frac{b(1+2a) + \frac{(1-b)(n-3)}{(n-2)t^2} + \frac{1-b}{n-2}}{1 + b(1+a) + \frac{(1-b)(n-3)}{(n-2)t} + \frac{1-b}{n-2}}.$$

It now follows that $\alpha_n \geq 2$ if and only if $(2n-3+(n-3)b)t^2 + 2(n-3)(1-b)t - (n-3)(1-b) \leq 0$, – i.e. if and only if

$$t \leq t_0 \equiv \frac{-(n-3)(1-b) + \sqrt{(n-3)^2(1-b)^2 + (n-3)(1-b)(2n-3+(n-3)b)}}{2n-3+(n-3)b}$$

(observe that $t_0 > 0$). In particular, we find that if $t < t_0$, then decreasing $t_{1,n}$ and increasing $t_{1,n-1}$ has the (counterintuitive) effect of *decreasing* α_n. It is also not so difficult to show that if $t = t_0$, then moving weight between $t_{1,n}$ and $t_{1,n-1}$ in either direction does not affect the value of α_n. Thus we see that in some sense, the hypothesis that $\alpha_n < 2$ of Theorem 3.1 cannot be weakened without affecting the conclusion.

Next we provide a result parallel to Theorem 3.1 for the case that $i = k$.

Theorem 3.2. *Suppose that T is an irreducible stochastic matrix of order n, and that for some $1 \le k \le n, \alpha_k < 1$. Fix an index $j \ne k$, and suppose that $t_{k,j}, t_{k,k} > 0$, consider the family of stochastic matrices $T(\epsilon) = T + \epsilon e_k(e_j - e_k)^\top, \epsilon \in (-t_{k,j}, t_{k,k})$, with corresponding accessibility indices $\alpha_k(\epsilon)$. Then $\frac{d\alpha_k(\epsilon)}{d\epsilon}|_{\epsilon=0} > 0$.*

Proof. Without loss of generality we assume that $k = n$. Referring to Theorem 1.1 b), and keeping the notation of that result, we have for all sufficiently small $\epsilon > 0$,

$$\alpha_n(\epsilon) = \frac{(r_n + \epsilon e_j)^\top (I - T_{(n)})^{-2}\mathbf{1}}{1 + (r_n + \epsilon e_j)^\top (I - T_{(n)})^{-1}\mathbf{1}}.$$

A straightforward computation shows that

$$\frac{d\alpha_n(\epsilon)}{d\epsilon}|_{\epsilon=0} = \left(\frac{e_j^\top (I - T_{(n)})^{-1}\mathbf{1}}{1 + r_n^\top (I - T_{(n)})^{-1}\mathbf{1}} \right) \left(\frac{e_j^\top (I - T_{(n)})^{-2}\mathbf{1}}{e_j^\top (I - T_{(n)})^{-1}\mathbf{1}} - \alpha_n(0) \right).$$

Since $e_j^\top (I - T_{(n)})^{-2}\mathbf{1} \ge e_j^\top (I - T_{(n)})^{-1}\mathbf{1}$, the conclusion follows readily. $\square$

Example 3.2. Suppose that $0 \le t < 1$ and $a, b > 0$ with $a + b < 1$. Consider the following family of stochastic matrices of order $n \ge 3$, parameterised by $\epsilon \in (-a, b)$:

$$T(\epsilon) = \left[\begin{array}{c|c|c} 0 & 0_{n-2}^\top & 1 \\ \hline 0_{n-2} & (1-t)I_{n-2} & t\mathbf{1}_{n-2} \\ \hline a & \frac{1-a-b}{n-2}\mathbf{1}_{n-2}^\top & b \end{array} \right] + \epsilon e_n(e_1 - e_n)^\top.$$

We consider the corresponding accessibility index $\alpha_n(\epsilon)$. Using the technique of Theorem 3.2, we find that $\frac{d\alpha_n(\epsilon)}{d\epsilon}|_{\epsilon=0} < 0$ if and only if $\alpha_n(0) > 1$. This last is in turn equivalent to the condition $(1-t)(1-a-b) > t^2$. Thus we find that when t is sufficiently small we have the surprising effect that decreasing the (n, n) entry of $T(0)$ and increasing its $(n, 1)$ entry will decrease the accessibility index for state n.

Our next example illustrates the fact that, in general, the stationary vector and the accessibility vector can exhibit different qualitative behaviour.

Example 3.3. Suppose that $n \in \mathbb{N}$ with $n \ge 3$, and define $T(x) \equiv \frac{1}{n}J + x(e_1 - e_2)(e_1 - e_2)^\top, x \in [-\frac{1}{n}, \frac{1}{n}]$. Evidently each such $T(x)$ is doubly stochastic, so that $w^\top = \frac{1}{n}\mathbf{1}^\top$. It can be verified that $(I - T(x))^\# = I - \frac{1}{n}J + \frac{x}{1-2x}(e_1 - e_2)(e_1 - e_2)^\top$, from which we find that $\alpha = (n-1)\mathbf{1} + \frac{nx}{1-2x}(e_1 + e_2)$. Thus we see that while the stationary distribution is insensitive to the value of x, the accessibility index for states 1 and 2 are increasing as functions of x on the interval $[-\frac{1}{n}, \frac{1}{n}]$.

As observed in section 1, for an irreducible stochastic matrix T with stationary vector $w^\top$ and accessibility vector $\alpha, \alpha^\top w = K$, where K is Kemeny's constant for the corresponding Markov chain. In particular it follows that $\max_k \alpha_k \geq K \geq \min_k \alpha_k$, with equality holding in either the left–hand or the right–hand inequality if and only if $\alpha = K\mathbf{1}$. Motivated by this simple observation, we turn our attention to characterising the situation in which the accessibility vector is a scalar multiple of $\mathbf{1}$.

Theorem 3.3. *Let T be an irreducible stochastic matrix of order $n \geq 2$ have stationary vector $w^\top$. Form $\tilde{w}$ from w by deleting its last entry. Then α is a scalar multiple of $\mathbf{1}$ if and only if we have*

$$(I - T_{(n)})^{-1}_{j,j} = \tilde{w}^\top (I - T_{(n)})^{-1} e_j + w_j e_j^\top (I - T_{(n)})^{-1}\mathbf{1}, j = 1, \ldots, n-1. \tag{10}$$

When (10) holds, $\alpha = (\tilde{w}^\top (I - T_{(n)})^{-1}\mathbf{1})\mathbf{1}$.

Proof. From Remark 1.1 and (2)), we see that $\alpha_n = (\tilde{w}^\top (I - T_{(n)})^{-1}\mathbf{1})$ and thus we find that all entries of α coincide with α_n if and only if the diagonal entries of $(I - T_{(n)})^{-1} - \mathbf{1}\tilde{w}^\top (I - T_{(n)})^{-1} - (I - T_{(n)})^{-1}\mathbf{1}\tilde{w}^\top$ are all zero. This last is readily seen to be equivalent to (10). □

Example 3.4. Suppose that $n \in \mathbb{N}$ with $n \geq 5$, fix $a \in [0, 1), b \in (0, \frac{1}{n-1}]$, and consider the $n \times n$ stochastic matrix T given by

$$T = \left[\begin{array}{c|c} \frac{a}{n-2}(J - I) & (1-a)\mathbf{1} \\ \hline b\mathbf{1}^\top & 1 - (n-1)b \end{array}\right].$$

Note that necessarily we take $b \leq \frac{1}{n-1}$. The stationary vector for T is readily seen to be $w^\top = \frac{1}{(n-1)b+1-a}\left[\begin{array}{c|c} b\mathbf{1}^\top & 1-a \end{array}\right]$. In this example, we use Theorem 3.3 to determine the conditions on a and b which ensure that α is a multiple of $\mathbf{1}$.

We have $I - T_{(n)} = (1 + \frac{a}{n-2})I - \frac{a}{n-2}J$, so that $(I - T_{(n)})^{-1} = \frac{n-2}{n-2+a}(I + \frac{a}{(n-2)(1-a)}J)$. It now follows that (10) holds if and only if

$$1 + \frac{a}{(n-2)(1-a)} = \frac{2b}{(n-1)b + 1 - a}\left(1 + \frac{(n-1)a}{(n-2)(1-a)}\right). \tag{11}$$

Rearranging (11) and simplifying, it now follows that (10) holds if and only if

$$b = \frac{(1-a)(n-2-(n-3)a)}{(n^2-4n+5)a - (n^2-5n+6)}. \tag{12}$$

Recalling that we must have $b \in (0, \frac{1}{n-1}]$, we find that when (10) holds, it must also be the case that

$$\frac{(1-a)(n-2-(n-3)a)}{(n^2-4n+5)a - (n^2-5n+6)} \leq \frac{1}{n-1}.$$

Rearranging this last inequality yields $((n-1)a-(n-2))((n-3)a-2(n-2)) \leq 0$. Hence, in order that all entries in α are equal, we must also have $\frac{n-2}{n-1} \leq a < 1$. We thus deduce that α is a multiple of the all ones vector if and only if $\frac{n-2}{n-1} \leq a < 1$ and b is given by (12). When both conditions are met, we have

$$\alpha_j = \frac{(n-1)(n-2-(n-3)a)}{2(1-a)(n-2+a)}$$

for $j = 1, \ldots, n$. This common value is of course Kemeny's constant for T.

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