Discussiones Mathematicae
Graph Theory xx (xxxx) 1–15

FRACTIONAL REVIVAL OF THRESHOLD GRAPHS UNDER LAPLACIAN DYNAMICS

STEVE KIRKLAND

Department of Mathematics, University of Manitoba
Winnipeg, MB, Canada R3T 2N2

e-mail: stephen.kirkland@umanitoba.ca

AND

XIAOHONG ZHANG

Department of Mathematics, University of Manitoba
Winnipeg, MB, Canada R3T 2N2

e-mail: zhangx42@myumanitoba.ca

Abstract

We consider Laplacian fractional revival between two vertices of a graph X. Assume that it occurs at time τ between vertices 1 and 2. We prove that for the decomposition $L = \sum_{r=0}^{q} \theta_r E_r$ of the Laplacian matrix L of X, for each $r = 0, 1, \ldots, q$, either $E_r e_1 = E_r e_2$, or $E_r e_1 = -E_r e_2$, depending on whether $e^{i\tau\theta_r}$ equals to 1 or not. That is to say, vertices 1 and 2 are strongly cospectral with respect to L. We give a characterization of the parameters of threshold graphs that allow for Laplacian fractional revival between two vertices; those graphs can be used to generate more graphs with Laplacian fractional revival. We also characterize threshold graphs that admit Laplacian fractional revival within a subset of more than two vertices. Throughout we rely on techniques from spectral graph theory.

Keywords: Laplacian matrix, spectral decomposition, quantum information transfer, fractional revival.

2010 Mathematics Subject Classification: 05C50 15A18 81P45.

This paper is dedicated to Slobodan Simic
on the occasion of his 70th birthday.

1. INTRODUCTION

Transferring a quantum state from one location to another reliably, or generating entangled states, play important roles in quantum spin systems. We model a quantum spin system by an undirected weighted graph: assign a vertex to each spin, and two vertices are adjacent if and only if the two corresponding spins are interacting with each other, with the edge weight equal to the interaction strength between the two spins. The system evolves with time due to its own dynamics; for the one excitation subspace, the adjacency matrix of the graph serves as the Hamiltonian of the system under XY dynamics, and the Laplacian matrix of the graph serves as the Hamiltonian of the system under Heisenberg dynamics. Here we focus on the latter case, and refer to quantum state transfer on graphs instead of in a quantum system.

For a graph X on n vertices with labelling $\{1, \dots, n\}$, its *adjacency matrix* $A(X)$ is an n-by-n matrix with (j, k) entry 1 if vertices j and k are adjacent, and 0 otherwise. Its *Laplacian matrix* is $L = D - A$, where D is a diagonal matrix with j-th diagonal entry being the j-th row sum of A. Let $\mathcal{H}$ denote the Hamiltonian of the system (A or L, depending on the dynamics), and let $U(t) = e^{it\mathcal{H}}$. Then the *fidelity* of state transfer from vertex u to vertex v is given by $p_{u,v}(t) = |U(t)_{u,v}|^2$, and is a measurement of the closeness of the state at vertex v at time t to the state at vertex u at time 0. If there is some time $t_1 > 0$, such that $p_{u,v}(t_1) = 1$ for two distinct vertices u and v, then we say that there is *perfect state transfer (PST)* from u to v at time t_1. It means that, up to a phase factor, with probability 1 the state at vertex v at time t_1 is identical to the initial state at vertex u at time 0. There is a lot of research on perfect state transfer on graphs, including quantum state transfer properties with respect to graph operations, of weighting schema to obtain weighted graphs with PST where the unweighted ones do not, of adding potentials to graphs, and some special classes of graphs with PST; we refer the interested reader to [2, 4, 9, 10, 13, 15, 16]. Another phenomenon related to quantum state transfer is called fractional revival. If there is some time $t_2 > 0$ and two distinct vertices u and v, such that $U(t_2)e_u = \alpha e_u + \beta e_v$ for some $\alpha, \beta \in \mathbb{C}$ with $|\alpha|^2 + |\beta|^2 = 1$ and $\beta \neq 0$, we say there is *fractional revival* (FR) from u to v at time t_2. Further, if $|\alpha| = |\beta|$, the fractional revival is called *balanced* [7] (observe that FR generalizes PST). More generally, if there is some time $t_3 > 0$ and a proper subset S of $V(X)$, such that for any vertex $u \in S$, $U(t_3)_{u,v} = 0$ if $v \notin S$, and the unweighted graph associated to the submatrix $U(t_3)_{[S,S]}$ is connected, we say there is *generalized fractional revival between vertices in* S (here $U(t_3)_{[S,S]}$ is the submatrix of entries that lie in the rows and columns of $U(t_3)$ indexed by elements in S).

Fractional revival between two end vertices of a spin chain (where the underlying graph is a path) can also be used to transfer quantum states efficiently, and balanced fractional revival can be used to generate entangled states. For adjacency fractional revival to occur at the two end vertices of a quantum spin chain with weighted loops, the spectrum of the Hamiltonian $\mathcal{H} = A$ must take the form of a bi-lattice [11]. It is shown that spin chains with adjacency fractional revival can be obtained from isospectral deformations of spin chains with PST (a characterization of the spectrum of $\mathcal{H}$ for a spin chain to exhibit PST at the end vertices is known), and the deformation only changes the middle couplings (also weights of the loops on the middle two vertices of the path when n is even) of the chain with PST to get a chain with FR. In [5], a class of cubelike graphs and some weighted graphs obtained from hypercubes are found to exhibit fractional revival. In [7], some properties of adjacency fractional revival (Hamiltonian $\mathcal{H} = A$) on general graphs are studied; in particular, a characterization of fractional revival between cospectral vertices is given.

Not many graphs are known to exhibit fractional revival. Here we focus on Laplacian dynamics, and characterize the parameters of a family of graphs – threshold graphs – that admit fractional revival under Laplacian dynamics. With these threshold graphs, we can produce more graphs with Laplacian fractional revival. Recall that a *threshold graph* can be constructed from the one-vertex graph by repeatedly adding a single vertex of two possible types: an *isolated vertex*, i.e., a vertex without incident edges, or a *dominating vertex*, i.e., a vertex connected to all other vertices. A characterization of PST in threshold graphs is known (see Theorem 3 below), and consequently our results on FR in threshold graphs, which rely heavily on techniques from spectral graph theory, can be seen as an extension of that theorem.

The outline of the paper is as follows. In Section 2, we review almost equitable partitions of a graph, some basic graph theory, and related results about threshold graphs. In Section 3, we consider Laplacian fractional revival between two vertices of a graph X, where we deduce that the two vertices are strongly cospectral with respect to L. In Section 4, we characterize threshold graphs that admit (generalized) Laplacian fractional revival within a subset of the vertex set. In Section 5, we produce more graphs with Laplacian fractional revival by making use of threshold graphs.

2. PRELIMINARIES

Some graphs admit some special partitions of their vertex set, and these partitions play important roles in quantum state transfer under Laplacian dynamics. First we introduce the characteristic matrix of a partition of the vertex set $V(X)$ of the graph X, and a special partition of $V(X)$ that X may admit.

Definition [12]. If $\pi = (C_1, \dots, C_k)$ is a partition of $V(X)$, the *characteristic matrix* P of π is the $n \times k$ matrix

$$P_{j\ell} = \begin{cases} 1 & \text{if } v_j \in C_\ell, \\ 0 & \text{otherwise.} \end{cases}$$

If we scale each column of P so that its norm is 1, the resulting matrix is called the *normalized characteristic matrix* of the partition π, and is denoted by $\hat{P}$.

Definition [6]**.** For the graph $X = (V, E)$, a partition $\pi = (C_1, \dots, C_k)$ of its vertex set V, is called an *almost equitable partition* if $\forall j, \ell \in \{1, \dots, k\}$ with $j \neq \ell$, the number of neighbours of a vertex $v \in C_j$ has in the cell C_ℓ does not depend on the choice of v. The *generalized Laplacian matrix* $L(X)^\pi$ with respect the the almost equitable partition π is the $k \times k$ matrix such that

$$L(X)^\pi_{j,\ell} = \begin{cases} -c_{j\ell} & \text{if } j \neq \ell \\ s_j, & \text{otherwise,} \end{cases}$$

where $c_{j\ell}$ is the number of neighbours a vertex in cell C_j has in cell C_ℓ, and $s_j = \sum_{\ell \neq j} c_{j\ell}$.

If the condition in the definition of almost equitable partition above also holds whenever $j = \ell$, then this special almost equitable partition is called an *equitable partition*, which plays an important role in quantum state transfer under adjacency dynamics.

An almost equitable partition of a graph X has the following characterization by using its characteristic matrix and the Laplacian matrix of the graph X.

Proposition 1 [6]**.** *Let G be a graph, L its Laplacian matrix, $\pi = (C_1, \dots, C_k)$ a k-partition of $V(G)$ and P the characteristic matrix of π. Then π is an almost equitable partition if and only if there is a $k \times k$ matrix M such that*

$$LP = PM$$

If π is an almost equitable k-partition then M is the generalized Laplacian matrix $L(G)^\pi$.

Now we review some graph operations: complement, union and join.

Let $X = (V, E)$ denote the graph with vertex set V and edge set E. Then the *complement* X^c of X is the graph that has the same vertex set as X, and two vertices of X^c are adjacent if and only if they are not adjacent in X. Assume $X_1 = (V_1, E_1)$ and $X_2 = (V_2, E_2)$ are two graphs with disjoint vertex sets. Then the *union* $X_1 \cup X_2$ of X_1 and X_2 is the graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$, i.e., $X_1 \cup X_2 = (V_1 \cup V_2, E_1 \cup E_2)$. The *join* $X_1 \vee X_2$ of X_1 and X_2 is $X_1 \vee X_2 = (X_1^c \cup X_2^c)^c$, which is the graph obtained by taking the union of X_1 with X_2 first, then connecting every vertex of X_1 to every vertex of X_2.

By using the above two binary graph operations – union and join, we have the following characterization of connected threshold graphs, where K_p denotes the complete graph on p vertices, and O_p denotes the empty graph on p vertices.

Proposition 2 [17]**.** *Let X be a connected graph on at least two vertices. Then X is a connected threshold graph if and only if one of the following two conditions is satisfied: (1) there are indices $m_1, \dots, m_{2k} \in \mathbb{N}$ with $m_1 \geq 2$ such that $X = ((((O_{m_1} \vee K_{m_2}) \cup O_{m_3}) \vee K_{m_4}) \cdots) \vee K_{m_{2k}} \equiv \Gamma(m_1, \dots, m_{2k})$;*

(2) there are indices $m_1, \dots, m_{2k+1} \in \mathbb{N}$ *with* $m_1 \geq 2$ *such that* $X = ((((K_{m_1} \cup O_{m_2}) \vee K_{m_3}) \cup O_{m_4}) \cdots) \vee K_{m_{2k+1}} \equiv \Gamma(m_1, \dots, m_{2k+1})$*;*

The Laplacian PST properties of threshold graphs are known.

Theorem 3 [17]**.** *Let* X *be a threshold graph. When* $X \equiv \Gamma(m_1, \dots, m_{2k})$ *(resp.* $X \equiv \Gamma(m_1, \dots, m_{2k+1})$*), then there is PST between vertex* j *and* ℓ *at time* $t \in [0, 2\pi]$ *if and only if* $(j, \ell) = (1, 2)$ *and in addition:* $t = \pi/2$*;* $m_1 = 2$*;* $m_2 \equiv 2 \pmod 4$*, and* $m_j \equiv 0 \pmod 4$ *for* $j = 3, \dots, 2k$ *(resp.* $j = 3, \dots, 2k + 1$*).*

Throughout, we use $e_1, \dots, e_n$ to denote the standard basis vectors in the n-dimensional vector space, where for each $j = 1, \dots, n$, $e_j = (0, \dots, 0, 1, 0, \dots, 0)^T$. We use $J_{m,n}$ to denote the all ones matrix of size $m \times n$, use $\mathbf{1}_n$ to denote the all ones vector of size n, and use I_n to denote the identity matrix of size n. We denote a $p \times q$ zero matrix by $0_{p,q}$ and the zero vector in $\mathbb{C}^p$ by 0_p. Subscripts denoting the sizes of matrices and vectors will be suppressed when they are clear from the context.

3. LAPLACIAN FRACTIONAL REVIVAL BETWEEN TWO VERTICES

Assume that X is a graph on n vertices and that it admits Laplacian fractional revival from vertex u to vertex v at time τ. Without loss of generality, assume that vertices u and v are labelled 1 and 2, respectively. Then $U(\tau) = e^{i\tau L} = \begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix}$ for some complex symmetric unitary matrices U_1 of order 2 and U_2 of order $n - 2$, and the union of the spectrum of U_1 and the spectrum of U_2 gives the spectrum of $U(\tau)$. Denote the (j, ℓ) entry of U_1 by $U_{j,\ell}$, then for $j = 1, 2$, $e^{i\tau L} e_j = U_{1j} e_1 + U_{2j} e_2$. Now assume the spectral decomposition of L is $L = \sum_{r=0}^{q} \theta_r E_r$ with $\theta_0 = 0$. Then $e^{i\tau L} = \sum_{r=0}^{q} e^{i\tau\theta_r} E_r$, and $e^{i\tau L} e_u = \sum_{r=0}^{q} e^{i\tau\theta_r} E_r e_u$ for any vertex u of X. Therefore $\sum_{r=0}^{q} e^{i\tau\theta_r} E_r e_j = e^{i\tau L} e_j = U_{1j} e_1 + U_{2j} e_2$ for $j = 1, 2$. Premultiplying E_r on both sides of the equation, combined with the facts that $e^{i\tau L}$ and E_r commute, and that $E_r E_\ell = \delta_{r,\ell} E_r$, gives $e^{i\tau\theta_r} E_r e_j = U_{1j} E_r e_1 + U_{2j} E_r e_2$ for $j = 1, 2$. Putting them together, we have $[E_r e_1 \, E_r e_2](U_1 - e^{i\tau\theta_r} I) = 0$ for $r = 0, 1, \dots, q$. Therefore if $C_r = [E_r e_1 \, E_r e_2] \neq 0$, then $e^{i\tau\theta_r}$ is an eigenvalue of U_1, and any nonzero row of C_r is a real left eigenvector of U_1 associated to the eigenvalue $e^{i\tau\theta_r}$. In particular, for $\theta_0 = 0$, we have $C_0 = \frac{1}{n} J_{n,2} \neq 0$, and therefore $e^{i\tau\theta_0} = e^{i\tau 0} = 1$ is an eigenvalue of U_1. Furthermore, 1 is a simple eigenvalue of U_1, since the only 2-by-2 diagonalizable matrix that has 1 as a multiple eigenvalue is the identity matrix I_2.

Note that for a complex symmetric matrix, each of its real eigenvectors is a left eigenvector at the same time, and the real eigenvectors associated to distinct eigenvalues are orthogonal. To see this, assume U is a complex symmetric matrix, with a real eigenvector x associated to λ, and a real eigenvector y associated to $\mu \neq \lambda$. Taking the transpose of $Ux = \lambda x$ we have $x^T U = x^T U^T = (Ux)^T = \lambda x^T$, that is to say, x is also a left eigenvector of U. From $\lambda x^T y = (x^T U)y = x^T(Uy) = \mu x^T y$ and $\lambda \neq \mu$, we conclude that $x^T y = 0$, i.e., x and y are orthogonal to each other.

Now consider any eigenvalue θ_r. Then if $e^{i\tau\theta_r} \neq 1$, from the facts that U_1 is symmetric and that E_r is a real matrix for $r = 0, 1, \ldots, q$, we know $C_r \mathbf{1}_2 = [E_r e_1,\ E_r e_2]\mathbf{1}_2 = 0$, i.e., $E_r e_1 + E_r e_2 = 0$. Since 1 is a simple eigenvalue of U_1, we have that for each r such that $e^{i\tau\theta_r} = 1$, all the rows of C_r are scalar multiples of $\mathbf{1}_2^T$. That is to say, $[E_r e_1,\ E_r e_2] = [E_r e_1,\ E_r e_1]$, or $E_r e_1 = E_r e_2$. The following theorem summarizes those observations.

Theorem 4. *If there is Laplacian fractional revival between two vertices u and v at time τ in graph X, then vertices u and v are strongly cospectral with respect to the Laplacian matrix L. That is, if the spectral decomposition of L is $L = \sum_r \theta_r E_r$, then for each r, either $E_r e_u = E_r e_v$ (if $\frac{\tau\theta_r}{2\pi} \in \mathbb{Z}$) or $E_r e_u = -E_r e_v$ (if $\frac{\tau\theta_r}{2\pi} \notin \mathbb{Z}$) holds.*

While preparing this manuscript, we learned that Ada Chan and Jordan Teitelbaum [8] have also proved the necessity of strong cospectrality for Laplacian FR.

Remark 5. For generalized Laplacian fractional revival between $m \geq 3$ vertices, 1 is not necessarily a simple eigenvalue of U_1, but if it is, then with a similar argument as above, we have the following.
Assume X is a graph that admits generalized Laplacian fractional revival between vertices in $S = \{1, 2, \ldots, m\} \subset V(X)$ at time τ, and that $U_1 = U(\tau)_{[S,S]} = (e^{i\tau L})_{[S,S]}$ has 1 as a simple eigenvalue. Let $L = \sum_{r=0}^{q} \theta_r E_r$ be the spectral decomposition of the Laplacian matrix L of X. Then for each $r = 1, \ldots, m$, the vectors $E_r e_1,\ E_r e_2,\ \cdots,\ E_r e_m$ are linearly dependent, and either

$$E_r e_1 = E_r e_2 = \cdots = E_r e_m \text{ if } e^{i\tau\theta_r} = 1, \text{or} \tag{1}$$

$$E_r e_1 + E_r e_2 + \cdots + E_r e_m = 0 \text{ if } e^{i\tau\theta_r} \neq 1. \tag{2}$$

Example 6. Let X be the graph as shown in Figure 1, and write the spectral decomposition of its Laplacian as $L(X) = \sum_{r=0}^{4} \theta_r E_r$, with $\theta_0 = 0$, $\theta_1 = 1$, $\theta_2 = 3$, $\theta_3 = 4$, and $\theta_4 = 5$. There is Laplacian fractional revival between vertices v_1 and v_2, and generalized fractional revival between vertices $\{v_3, v_4, v_5, v_6\}$ at time $\frac{2\pi}{3}$. Direct observation shows that v_1 and v_2 are strongly cospectral with respect to L: $E_r e_1 = E_r e_2$ for $r = 0, 2$, $E_r e_1 = -E_r e_2$ for $r = 1, 3$, and $E_4 e_1 = E_4 e_2 = 0_6$, which is in accordance with Theorem 4. There is also generalized Laplacian fractional revival between vertices $\{v_1, v_4, v_5\}$, and between vertices $\{v_2, v_3, v_6\}$ at time π. Since 1 is a simple eigenvalue of $U_1 = U(\pi)_{[1,4,5],[1,4,5]}$, Remark 5 implies that $E_r e_1 = E_r e_4 = E_r e_5$ for $r = 0, 3$ ($e^{i\pi\theta_r} = 1$) and that $E_r e_1 + E_r e_4 + E_r e_5 = 0$ for $r = 1, 2, 4$ (since $e^{i\pi\theta_r} \neq 1$), which can be confirmed by checking the orthogonal projection matrices E_r directly.

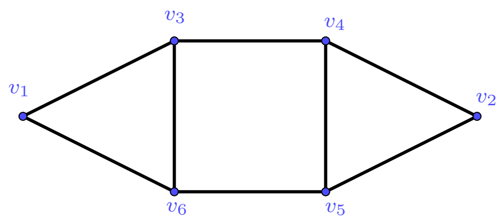

Figure 1

4. LAPLACIAN FRACTIONAL REVIVAL IN THRESHOLD GRAPHS

We will only give detailed consideration to connected threshold graphs of the form $\Gamma(m_1, m_2, \ldots, m_{2k})$ in this section; note that similar results hold for the connected threshold graphs $\Gamma(m_1, m_2, \ldots, m_{2k}, m_{2k+1})$, and we state them without proof.

As shown in [17], for the threshold graph $\Gamma(m_1, m_2, \ldots, m_{2k})$, its eigenvalues are:

$$\lambda_0 = 0, \tag{3}$$

$$\lambda_j = m_{j+1} + m_{j+3} + \cdots + m_{2k} \text{ for any odd integer } j \in \{1, \ldots, 2k\}, \tag{4}$$

$$\text{and } \lambda_j = \sigma_j + m_{j+2} + \cdots + m_{2k} \text{ for any even integer } j \in \{1, \ldots, 2k\}, \tag{5}$$

where $\sigma_j = m_1 + m_2 + \cdots + m_j$ for $j = 1, 2, \ldots, 2k$. The multiplicity of λ_j is

$$\begin{cases} 1 & \text{if } j = 0 \\ m_1 - 1, & \text{if } j = 1 \\ m_j & \text{otherwise.} \end{cases}$$

The orthogonal idempotents for L corresponding to $\lambda_0 = 0$, $\lambda = \lambda_1$ and $\lambda = \lambda_j$ for $j = 2, 3, \ldots, 2k$ are: $E_0 = \frac{1}{\sigma_{2k}} J_{\sigma_{2k},\sigma_{2k}}$,

$$E_1 = \begin{bmatrix} I_{m_1} - \frac{1}{m_1} J_{m_1,m_1} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix},$$

$$E_j = \begin{bmatrix} \frac{m_j}{\sigma_{j-1}\sigma_j} J_{\sigma_{j-1},\sigma_{j-1}} & -\frac{1}{\sigma_j} J_{\sigma_{j-1},m_j} & 0_{\sigma_{j-1},\sigma_{2k}-\sigma_j} \\ -\frac{1}{\sigma_j} J_{m_j,\sigma_{j-1}} & I_{m_j} - \frac{1}{\sigma_j} J_{m_j,m_j} & 0_{m_j,\sigma_{2k}-\sigma_j} \\ 0_{\sigma_{2k}-\sigma_j,\sigma_{j-1}} & 0_{\sigma_{2k}-\sigma_j,m_j} & 0_{\sigma_{2k}-\sigma_j,\sigma_{2k}-\sigma_j} \end{bmatrix}, \text{ respectively.}$$

We partition the vertex set of $\Gamma(m_1,\ldots,m_{2k})$ according to the indices $m_1,m_2,\ldots,m_{2k}$; denote the corresponding cells by $C_1,C_2,\ldots,C_{2k}$, and denote the partition by π.

Lemma 7. *If $\Gamma(m_1,\ldots,m_{2k})$ admits Laplacian fractional revival between two vertices u and v, then they must belong to the same cell of the partition π.*

Proof. From Theorem 4 we know that if there is fractional revival between two vertices u and v of $\Gamma(m_1,\ldots,m_{2k})$, then the two vertices are strongly cospectral with respect to L. Assume $u\in C_j$, $v\in C_\ell$, $j<\ell$, and u is the s-th entry of cell C_j. Then $E_je_v=0_{\sigma_{2k}}$ and for $e_s\in\mathbb{R}^{m_j}$, $E_je_u=\left[\begin{array}{c|c} e_s^T-\frac{1}{m_1}\mathbf{1}_{m_1}^T & 0_{\sigma_{2k}-m_1}^T\end{array}\right]^T$ if $j=1$; $E_je_u=\left[\begin{array}{c|c|c} -\frac{1}{\sigma_j}\mathbf{1}_{\sigma_{j-1}}^T & e_s^T-\frac{1}{\sigma_j}\mathbf{1}_{m_j}^T & 0_{\sigma_{2k}-\sigma_j}^T\end{array}\right]^T$ if $j>1$. In either case, u and v are not strongly cospectral with respect to L. Therefore u and v must be in the same cell of the partition π. ■

Lemma 8. *If $X=\Gamma(m_1,\ldots,m_{2k})$ admits Laplacian fractional revival between two vertices u and v, then $\{u,v\}=\{1,2\}$ and $m_1=2$.*

Proof. From Lemma 7 we know vertices u and v are in the same cell of π; assume $u,v\in C_j$, with u being the s-th vertex in C_j, and v the r-th vertex in C_j. Let $\sigma_0=0$, then $E_je_u=\left[\begin{array}{c|c|c} -\frac{1}{\sigma_j}\mathbf{1}_{\sigma_{j-1}}^T & (e_s-\frac{1}{\sigma_j}\mathbf{1}_{m_j})^T & 0_{\sigma_{2k}-\sigma_j}^T\end{array}\right]^T$ and $E_je_v=\left[\begin{array}{c|c|c} -\frac{1}{\sigma_j}\mathbf{1}_{\sigma_{j-1}}^T & (e_r-\frac{1}{\sigma_j}\mathbf{1}_{m_j})^T & 0_{\sigma_{2k}-\sigma_j}^T\end{array}\right]^T$, where $e_s,e_r\in\mathbb{R}^{m_j}$. By Theorem 4, Laplacian fractional revival between u and v implies $E_je_u=\pm E_je_v$, which is possible only if $j=1$ and $\sigma_1=m_1=2$. ■

Now we are going to characterize the parameters m_j such that Laplacian fractional revival occurs between vertices 1 and 2 in the graph $\Gamma(m_1,\ldots,m_{2k})$ by using the spectral decomposition of L shown at the beginning of this section. Since all the eigenvalues of L are integers, we know that L is periodic at all vertices at time 2π, i.e. $e^{2\pi iL}$ is a scalar multiple of the identity matrix (in fact it is the identity matrix here). In the following we will not consider this case.

Theorem 9. *The threshold graph $X=\Gamma(m_1,\ldots,m_{2k})$ admits Laplacian fractional revival between two vertices u and v at time τ if and only if*

i) $\{u,v\}=\{1,2\}$ *and* $m_1=2$, *and*

ii) *(a)* $m_1\frac{\tau}{\pi}=2\frac{\tau}{\pi}\notin\mathbb{Z}$

(b) $(m_1+m_2)\frac{\tau}{2\pi}, m_j\frac{\tau}{2\pi}\in\mathbb{Z}$ *for* $j=3,\ldots,2k$.

Proof. Assume that there is Laplacian fractional revival between vertices u and v at time $\tau>0$. Then Lemmas 7 and 8 imply that $i)$ holds. Using the spectral decomposition of

L we have

$$\begin{aligned}(e^{i\tau L})_{1,1} &= e^{i\tau\lambda_1}(1-\tfrac{1}{2}) + e^{i\tau\lambda_2}(\tfrac{m_2}{\sigma_1\sigma_2}) + e^{i\tau\lambda_3}(\tfrac{m_3}{\sigma_2\sigma_3}) + \cdots \\ &\quad + e^{i\tau\lambda_{2k-1}}(\tfrac{m_{2k-1}}{\sigma_{2k-2}\sigma_{2k-1}}) + e^{i\tau\lambda_{2k}}(\tfrac{m_{2k}}{\sigma_{2k-1}\sigma_{2k}}) + \tfrac{1}{\sigma_{2k}}, \\ (e^{i\tau L})_{1,2} &= e^{i\tau\lambda_1}(-\tfrac{1}{2}) + e^{i\tau\lambda_2}(\tfrac{m_2}{\sigma_1\sigma_2}) + e^{i\tau\lambda_3}(\tfrac{m_3}{\sigma_2\sigma_3}) + \cdots \\ &\quad + e^{i\tau\lambda_{2k-1}}(\tfrac{m_{2k-1}}{\sigma_{2k-2}\sigma_{2k-1}}) + e^{i\tau\lambda_{2k}}(\tfrac{m_{2k}}{\sigma_{2k-1}\sigma_{2k}}) + \tfrac{1}{\sigma_{2k}}, \\ (e^{i\tau L})_{1,w} &= e^{i\tau\lambda_j}(-\tfrac{1}{\sigma_j}) + e^{i\tau\lambda_{j+1}}(\tfrac{m_{j+1}}{\sigma_j\sigma_{j+1}}) + \cdots \\ &\quad + e^{i\tau\lambda_{2k}}(\tfrac{m_{2k}}{\sigma_{2k-1}\sigma_{2k}}) + \tfrac{1}{\sigma_{2k}} \text{ for } w \in C_j \text{ with } j = 2, \ldots, 2k.\end{aligned}$$

Since $(e^{i\tau L})_{1,w} = 0$ for $w \neq 1, 2$, then considering $w \in C_{2k}$, $w \in C_{2k-1}$, $\ldots, w \in C_3, w \in C_2$, we find that $\tau\sigma_{2k}$, τm_{2k}, $\tau(\sigma_{2k-2} + m_{2k})$, $\ldots$, $\tau(m_4 + m_6 + \ldots + m_{2k})$, and $\tau(\sigma_2 + m_4 + \ldots + m_{2k})$ are all even integer multiples of π, which is equivalent to the fact that τm_{2k}, τm_{2k-1}, τm_{2k-2}, $\ldots$, τm_3, and $\tau\sigma_2$ are all even integer multiples of π. In this case,

$$(e^{i\tau L})_{1,1} = \frac{1}{2}e^{i\tau m_2} + \frac{1}{2}, \text{and } (e^{i\tau L})_{1,2} = -\frac{1}{2}e^{i\tau m_2} + \frac{1}{2}. \tag{6}$$

Hence, if in addition,

- τm_2 and therefore $\tau m_1 = 2\tau$ is an even integer multiple of π, then the graph X is periodic at vertex 1 (and vertex 2);
- τm_2 and therefore $\tau m_1 = 2\tau$ is an odd integer multiple of π, then the graph X admits Laplacian perfect state transfer between vertices 1 and 2;
- τm_2 and therefore $\tau m_1 = 2\tau$ is not an integer multiple of π, then the graph X admits Laplacian fractional revival between vertices 1 and 2.

Therefore the conditions are necessary. It is straightforward to show that the conditions are sufficient. ■

With the same argument as above, we have the following.

Remark 10. The threshold graph $X = \Gamma(m_1, \ldots, m_{2k}, m_{2k+1})$ admits Laplacian fractional revival between two vertices u and v at time τ if and only if

i) $\{u, v\} = \{1, 2\}$ and $m_1 = 2$, and

ii) (a) $m_1\frac{\tau}{\pi} = 2\frac{\tau}{\pi} \notin \mathbb{Z}$

(b) $(m_1 + m_2)\frac{\tau}{2\pi}, m_j\frac{\tau}{2\pi} \in \mathbb{Z}$ for $j = 3, \ldots, 2k, 2k+1$.

Corollary 11. *There is balanced Laplacian fractional revival between vertices u and v in the threshold graph $X = \Gamma(m_1, \ldots, m_{2k})$ at time τ, if and only if*

i) $m_1 = 2$ *with* $\{u, v\} = \{1, 2\}$, *ii)* $\tau = \frac{2\ell+1}{4}\pi$ *for some non-negative integer* ℓ*,*

iii) $m_2 = \frac{2(2s+1)}{2\ell+1}$*, for the same integer* ℓ *as in* ii*), and for a non-negative integer* s *of distinct parity from* ℓ *such that* $(2\ell+1)|(2s+1)$ *(in fact when this is true, then* $\frac{2s+1}{2\ell+1} \equiv 3 \pmod 4$*), and*

iv) $m_j \equiv 0 \pmod 8$ *for* $j = 3, \dots, 2k$*.*

Proof. From Theorem 9 and equation (6), we know that if balanced fractional revival in X takes place between vertices u and v, then it is between vertices 1 and 2. In this case, $m_1 = 2$, $\cos(m_2\tau) = 0$, and $\tau(m_1+m_2), \tau m_3, \dots, \tau m_{2k}$ are all even integer multiples of π. Therefore $\tau m_2 = \frac{2s+1}{2}\pi$ for some integer s. Since $\tau(m_1+m_2)$ is an even integer multiple of π, we have $2\tau = \frac{2\ell+1}{2}\pi$ for some integer ℓ, where ℓ has different parity than s. Hence $\tau = \frac{2\ell+1}{4}\pi$ and $m_2 = \frac{2(2s+1)}{2\ell+1}$ for integers s and ℓ with distinct parity. Combining with the fact that τm_j is an even integer multiple of π for $j = 3, \dots, 2k$, we find that $m_j \equiv 0 \pmod 8$ for $j \geq 3$.
Conversely, if $m_j \equiv 0 \pmod 8$ for $j \geq 3$, and $\tau = \frac{2\ell+1}{4}\pi$ for some integer ℓ, then $m_j\tau = m_j\frac{2\ell+1}{4}\pi$ is an even integer multiple of π for $j \geq 3$. Furthermore, if $m_2 = \frac{2(2s+1)}{2\ell+1}$ for integer s of different parity than ℓ such that $(2\ell+1)|(2s+1)$, then $(m_1 + m_2)\tau = (s+\ell+1)\pi$ is an even integer multiple of π, and $\cos(m_2\tau) = \cos(\frac{2s+1}{2}\pi) = 0$. Again from Theorem 9 and equation (6), we know that there is balanced fractional revival in X between vertices 1 and 2 at time τ. ∎

Remark 12. There is balanced Laplacian fractional revival between vertices u and v in the threshold graph $X = \Gamma(m_1, \dots, m_{2k}, m_{2k+1})$ at time τ, if and only if

i) $m_1 = 2$ with $\{u, v\} = \{1, 2\}$,

ii) $\tau = \frac{2\ell+1}{4}\pi$ for some non-negative integer ℓ,

iii) $m_2 = \frac{2(2s+1)}{2\ell+1}$, for the same integer ℓ as in ii), and for a non-negative integer s of distinct parity from ℓ such that $(2\ell+1)|(2s+1)$ (in fact when this is true, then $\frac{2s+1}{2\ell+1} \equiv 3 \pmod 4$), and

iv) $m_j \equiv 0 \pmod 8$ for $j = 3, \dots, 2k, 2k+1$.

Remark 13. Since if there is PST between vertices u and v, then u and v are strongly cospectral [13], the proof of Theorem 9 can be used to prove Theorem 3: the second of the three cases in the proof gives us Theorem 3.

Now we address generalized Laplacian fractional revival within some subset of vertices in threshold graphs.

Theorem 14. *Consider the threshold graph $X = \Gamma(m_1, \dots, m_{2k})$, and let $C_\ell, \ell = 1, \dots, 2k$ denote the cells of the partition π of $V(X)$ according to the parameters $m_\ell, \ell = 1, \dots, 2k$. Then X admits generalized Laplacian fractional revival between vertices in $S \subset V(X)$ at some time $\tau > 0$ if and only if, for some integer $j < 2k$, $\tau m_{2k}, \tau m_{2k-1}, \dots, \tau m_{j+2}$ and $\tau\sigma_{j+1}$ are all even integer multiple of π, while τm_{j+1} is not. In this case, $S = C_1 \cup \dots \cup C_j$, and X is periodic at all vertices in the cells $C_{j+1}, \dots, C_{2k}$.*

Proof. Assume X admits generalized Laplacian fractional revival between vertices in S at time τ, with j being the largest index of the cells such that $S \cap C_j \neq \emptyset$. Let u be any vertex in $S \cap C_j$. Now

$$\begin{aligned}(e^{i\tau L})_{u,w} &= e^{i\tau\lambda_\ell}(-\tfrac{1}{\sigma_\ell}) + e^{i\tau\lambda_{\ell+1}}(\tfrac{m_\ell}{\sigma_\ell\sigma_{\ell+1}}) + \dots + e^{i\tau\lambda_{2k}}(\tfrac{m_{2k}}{\sigma_{2k-1}\sigma_{2k}}) + \tfrac{1}{\sigma_{2k}}\\ &= e^{i\tau\lambda_\ell}(-\tfrac{1}{\sigma_\ell}) + e^{i\tau\lambda_{\ell+1}}(\tfrac{1}{\sigma_\ell} - \tfrac{1}{\sigma_{\ell+1}}) + \dots + e^{i\tau\lambda_{2k}}(\tfrac{1}{\sigma_{2k-1}} - \tfrac{1}{\sigma_{2k}}) + \tfrac{1}{\sigma_{2k}},\end{aligned}$$

for any $w \in C_\ell$, with $\ell = j+1, \dots, 2k$, and

$$(e^{i\tau L})_{u,v} = e^{i\tau\lambda_j}(-\frac{1}{\sigma_j}) + e^{i\tau\lambda_{j+1}}(\frac{1}{\sigma_j} - \frac{1}{\sigma_{j+1}}) + \dots + e^{i\tau\lambda_{2k}}(\frac{1}{\sigma_{2k-1}} - \frac{1}{\sigma_{2k}}) + \frac{1}{\sigma_{2k}},$$

for any $v \in C_1 \cup C_2 \cup \dots \cup C_j$ with $v \neq u$, and

$$(e^{i\tau L})_{x,x} = e^{i\tau\lambda_\ell}(1 - \frac{1}{\sigma_\ell}) + e^{i\tau\lambda_{\ell+1}}(\frac{1}{\sigma_\ell} - \frac{1}{\sigma_{\ell+1}}) + \dots + e^{i\tau\lambda_{2k}}(\frac{1}{\sigma_{2k-1}} - \frac{1}{\sigma_{2k}}) + \frac{1}{\sigma_{2k}},$$

for any $x \in C_\ell$, with $\ell = 1, \dots, 2k$.
Since $(e^{i\tau L})_{u,w} = 0$ for $w \in C_{2k}, C_{2k-1}, \dots, C_{j+1}$, we find that

$$\frac{\tau m_{2k}}{2\pi}, \frac{\tau m_{2k-1}}{2\pi}, \dots, \frac{\tau m_{j+2}}{2\pi}, \frac{\tau\sigma_{j+1}}{2\pi} \in \mathbb{Z}. \tag{7}$$

In this case, we have

$$\begin{gathered}(e^{i\tau L})_{w,w} = 1, \text{for } w \in C_{j+1} \cup \ldots \cup C_{2k}\\ (e^{i\tau L})_{u,u} = e^{i\tau\lambda_j}(1 - \frac{1}{\sigma_j}) + \frac{1}{\sigma_j}, \text{and} \qquad (8)\\ (e^{i\tau L})_{u,v} = e^{i\tau\lambda_j}(-\frac{1}{\sigma_j}) + \frac{1}{\sigma_j} \text{for } v \in C_1 \cup \ldots \cup C_j \text{and } v \neq u.\end{gathered}$$

Therefore X is periodic at any vertex $w \in C_{j+1} \cup \ldots \cup C_{2k}$. The fact that u is involved in generalized Laplacian fractional revival implies that $|(e^{i\tau L})_{u,u}| \neq 1$. Combining with (7) and (8), we find $\frac{\tau m_{j+1}}{2\pi} \notin \mathbb{Z}$ irrespective of whether j is even or odd, and therefore $(e^{i\tau L})_{u,v} \neq 0$ for any $v \in C_1, \dots, C_{j-1}, C_j$ (if $(e^{itL})_{u,u} = 0$, then $\sigma_j = 2$, $j = 1$ and there is Laplacian PST between vertices 1 and 2, which is not the case we are considering). Hence $S = C_1 \cup \ldots \cup C_j$ and the conditions are necessary. The other direction follows directly. ■

Remark 15. For the threshold graph $X = \Gamma(m_1, \ldots, m_{2k}, m_{2k+1})$, let $C_\ell, \ell = 1, \ldots, 2k+1$ denote the cells of the partition of $V(X)$ according to the parameters $m_\ell, \ell = 1, \ldots, 2k+1$. Then X admits generalized Laplacian fractional revival between vertices in $S \subset V(X)$ at some time $\tau > 0$ if and only if, for some integer $j < 2k+1$, τm_{2k+1}, τm_{2k}, τm_{2k-1}, $\ldots$, τm_{j+2} and $\tau \sigma_{j+1}$ are all even integer multiples of π, while τm_{j+1} is not. In this case, $S = C_1 \cup \cdots \cup C_j$, and X is periodic at all vertices in the cells $C_{j+1}, \ldots, C_{2k}, C_{2k+1}$.

Example 16. Consider the threshold graph $X = \Gamma(2,2,2,2,4,4)$, direct computation shows that there is generalized Laplacian fractional revival between set $S = \{1, 2, \ldots, 6\}$ at $\tau = \pi/2$. The result agrees with the one stated in Theorem 14, since $\tau m_5 = \tau m_6$ and $\tau\sigma_4 = 8\tau$ are even integer multiples of π, while $\tau m_4 = \pi$ is not. Similarly $\Gamma(1,2,1,4)$ admits Laplacian fractional revival between the first 4 vertices at time $\tau = \frac{\pi}{4}$, and $\Gamma(2,2,6,2,4,4)$ admits Laplacian fractional revival between the first 10 vertices at time $\tau = \frac{\pi}{2}$.

Remark 17. Note that Theorem 14 implies Theorem 9, but the strong cospectrality of the two vertices involved in Laplacian fractional revival makes the proof more clear as shown in Theorem 9.

5. CONSTRUCTING GRAPHS WITH LAPLACIAN FRACTIONAL REVIVAL

More graphs with Laplacian fractional revival can be obtained from those threshold graphs that admit Laplacian fractional revival. For this result, we need to make use of almost equitable partitions of a graph. First note that apart from Proposition 1, there are other characterizations of an almost equitable partition of a graph. The proof is essentially the same as that for the characterization for equitable partitions [14], but we include it for completeness.

Proposition 18. *Suppose $\pi = (C_1, \ldots, C_k)$ is a partition of the vertices of the graph X, and that $\hat{P}$ is its normalized characteristic matrix. Denote the Laplacian of X by $L(X)$. Then the following are equivalent:*
(a) π is an almost equitable partition.
(b) The column space of $\hat{P}$ is $L(X)$-invariant.
(c) There is a matrix B of order $k \times k$ such that $L(X)\hat{P} = \hat{P}B$.
(d) $L(X)$ and $\hat{P}\hat{P}^T$ commute.

Proof. Assume P is the characteristic matrix of the partition π. From Theorem 1 we know that π is an almost equitable partition if and only if $L(X)P = PM$, i.e., the column space of P is $L(X)$-invariant. Since P and $\hat{P}$ have the same column space, it follows that (a) and (b) are equivalent.
Since (c) is an equivalent way of saying that the column space of $\hat{P}$ is $L(X)$- invariant, (b) and (c) are equivalent. Furthermore, $L(X)\hat{P} = \hat{P}B$ implies that $\hat{P}^T L(X)\hat{P} =$

$\hat{P}^T\hat{P}B = I_kB = B$, from which we see that the matrix B in (c) is symmetric. Now if (c) is true, and using the fact that B is symmetric, we have $L(X)\hat{P}\hat{P}^T = \hat{P}B\hat{P}^T = \hat{P}(\hat{P}B)^T = \hat{P}(L(X)\hat{P})^T = \hat{P}\hat{P}^TL(X)$, and therefore (c) implies (d). To prove that (d) implies (b), we note that if $L(X)$ commutes with a matrix S, then the column space of S is $L(X)$-invariant. Combined with the fact that $\hat{P}\hat{P}^T$ and $\hat{P}$ have the same column space, we get the desired result. ■

If a graph X_1 admits an equitable partition π_1 with vertices a and b being singletons, then $(e^{itA(X_1)})_{a,b} = (e^{it\widehat{A(X_1)}^{\pi_1}})_{\{a\},\{b\}}$, where $\widehat{A(X_1)}^{\pi_1} = \hat{P}A(X_1)\hat{P}$, with rows and columns indexed by the cells of the partition π_1, and the undirected weighted graph with adjacency matrix $\widehat{A(X_1)}^{\pi_1}$ is called the *symmetrized quotient graph of X with respect to* π_1 [4]. Now if a graph X admits an almost equitable partition, then a parallel result holds between $L(X)$ and $\widehat{L(X)}^{\pi}$ with exactly the same argument, where $\widehat{L(X)}^{\pi} = \hat{P}^TL(X)\hat{P}$ (note that $\widehat{L(X)}^{\pi}$ is not a Laplacian matrix in general).

Theorem 19. *Let $X = (V, E)$ be a graph with an almost equitable partition π where two distinct vertices a and b belong to singleton cells. Let $L(X)$ denote its Laplacian matrix. Let u, v be either a or b, then for any time t,*

$$(e^{itL(X)})_{u,v} = (e^{it\widehat{L(X)}^{\pi}})_{\{u\},\{v\}}$$

where $\{u\}$ and $\{v\}$ are the corresponding singleton cells of π, and are used to index the rows and columns of $\widehat{L(X)}^{\pi}$. Therefore, the system with Hamiltonian $L(X)$ has fractional revival (resp. perfect state transfer) from a to b at time t if and only if the system with Hamiltonian $\widehat{L(X)}^{\pi} = \hat{P}^TL(X)\hat{P}$ has fractional revival (resp. perfect state transfer) from $\{a\}$ to $\{b\}$ at time t.

The above result was used in an example in [1]. Now we can construct more graphs with Laplacian fractional revival (resp. Laplacian perfect state transfer) from given graphs.

Corollary 20. *Suppose that the graph $X = (V, E)$ has an almost equitable partition π of V, with vertices a and b belonging to singleton cells. If there is Laplacian fractional revival (resp. Laplacian perfect state transfer) from a to b in X, then for any graph Y obtained from X by adding or deleting any collection of edges within the cells of π, Y also admits Laplacian fractional revival (resp. Laplacian perfect state transfer) from a to b.*

Proof. The almost equitable partition of the vertex set of X is also an almost equitable partition of $V(Y)$. From the fact that $\hat{P}^TL(Y)\hat{P} = \hat{P}^TL(X)\hat{P}$ and Theorem 19, the result follows. ■

Remark 21. The partition π of a threshold graph according to the parameters m_j is an almost equitable partition, and so is any refinement of this partition. In particular, for a threshold graph X that admits Laplacian fractional revival at time τ, partitioning the cell $C_1 = \{1, 2\}$ of π into two smaller cells $C_{1,1} = \{1\}$ and $C_{1,2} = \{2\}$ and keeping all the other cells unchanged, results in the partition π', that is still an almost equitable partition of $V(X)$, but now the two vertices involved in Laplacian fractional revival are singletons. Therefore, we can produce more graphs with Laplacian fractional revival from the threshold graph X by adding or deleting edges inside the cells of the partition π' of $V(X)$. Similarly, if a threshold graph X admits generalized Laplacian fractional revival at time τ between vertices $\{1, \ldots, \ell\} = C_1 \cup \cdots \cup C_j$, where $C_1, \ldots, C_{2k}$ (C_{2k+1}) are the cells of the partition π, then the refinement π'' of π, which partitions $C_1 \cup \cdots \cup C_j$ into singletons as $\{1\}, \ldots, \{\ell\}$ and keeps the other cells of π unchanged, is still an almost equitable partition of $V(X)$, but with all the vertices involved in the revival as singletons. Again, adding or deleting vertices inside the cells of the partition π'' results in graphs that admit generalized Laplacian fractional revival between vertices $\{1, \ldots, \ell\}$ at time τ.

Example 22. For any threshold graph $X = \Gamma(m_1, \ldots, m_{2k})$ with Laplacian fractional revival (resp. Laplacian PST), and for odd integer $p > 1$, even integer $q \geq 2$, the graph Y obtained from X by adding edges in the induced subgraph O_{m_p} on cell C_p or deleting edges in the induced subgraph K_{m_q} on cell C_q of the equitable partition π, still admits Laplacian fractional revival between the two vertices, by Corollary 20 and Remark 21. For example, we know without calculations that the complete bipartite graph $K_{2,6}$ admits Laplacian fractional revival at time $\pi/4$ (and admits Laplacian PST at time $\pi/2$), since it can obtained from the threshold graph $O_2 \vee K_6$ (which admits Laplacian fractional revival at time $\pi/4$ by Theorem 9, and which admits Laplacian PST at time $\pi/2$ by Theorem 3) by removing all the edges inside K_6.

Acknowledgements

S.K. is supported by NSERC Discovery Grant RGPIN/6123-2014. X.Z. is supported by the University of Manitoba's Faculty of Science and Faculty of Graduate Studies.

REFERENCES

[1] R. Alvir, S. Dever, B. Lovitz, J. Myer, C. Tamon, Y. Xu, H. Zhan, *Perfect state transfer in Laplacian quantum walk*, J. Algebraic Combin. **43** (2016) 801–826.

[2] R.J. Angels-Canul, E. Norton, M. Opperman, C. Paribello, M. Russell, C. Tamon, *Quantum perfect state transfer on weighted join graphs*, International Journal of Quantum Information **07** (2009) 1429–1445.

[3] R. Bachman, E. Fredette, J. Fuller, M. Landry, M. Opperman, C. Tamon, A. Tollefson, *Perfect state transfer on quotient graphs*, Quantum Inf. Comput. **12** (2012) 293–313.

[4] M. Bašić, M.D. Petković, D. Stevanović, *Perfect state transfer in integral circulant graphs*, Appl. Math. Lett. **22** (2009) 1117–1121.

[5] P.-A. Bernard, A. Chan, É. Loranger, C. Tamon, L. Vinet, *A graph with fractional revival*, Phys. Lett. A **382** (2018) 259–264.

[6] D.M. Cardoso, C. Delorme, P. Rama, *Laplacian eigenvectors and eigenvalues and almost equitable partitions*, European J. Combin. **28** (2007) 665–673.

[7] A. Chan, G. Coutinho, C. Tamon, L.Vinet, H. Zhan, *Quantum fractional revival on graphs*, http://arxiv.org:1801.09654, 2018 .

[8] A. Chan, J. Teitelbaum, personal communication, 2018.

[9] W.-C. Cheung, C. Godsil, *Perfect state transfer in cubelike graphs*, Linear Algebra Appl. **435** (2011) 2468–2474.

[10] M. Christandl, N. Datta, T.C. Dorlas, A. Ekert, A. Kay, A.J. Landahl, *Perfect transfer of arbitrary states in quantum spin networks*, Phys. Rev. A **71** (2005) 032312.

[11] V.X. Genest, L. Vinet, A. Zhedanov, *Quantum spin chains with fractional revival*, Ann. Physics **371** (2016) 348–367.

[12] C. Godsil, Algebraic combinatorics (Chapman & Hall, New York, 1993).

[13] C. Godsil, *State transfer on graphs*, Discrete Math. **312** (2012) 129–147.

[14] C. Godsil, *When can perfect state transfer occur?*, Electron. J. Linear Algebra **23** (2012) 877–890.

[15] A. Kay, *A Review of Perfect State Transfer and its Application as a Constructive Tool*, Int. J. Quantum Inf. 8 **641** (2010) 641–676.

[16] M. Kempton, G. Lippner, S.-T. Yau, *Perfect state transfer on graphs with a potential*, Quantum Inf. Comput. **17** (2017) 303–327.

[17] S. Kirkland, S. Severini, *Spin-system dynamics and fault detection in threshold networks*, Phys. Rev. A **83** (2011) 012310.

[18] N.V.R. Mahadev, U.N. Peled, Threshold graphs and related topics (North-Holland Publishing Co., Amsterdam, 1995).