ALGEBRAIC CONNECTIVITY OF k-CONNECTED GRAPHS

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Abstract. Let G be a $k-$connected graph with $k \geq 2$. A hinge is a subset of k vertices whose deletion from G yields a disconnected graph. We consider the algebraic connectivity and Fiedler vectors of such graphs, paying special attention to the signs of the entries in Fielder vectors corresponding to vertices in a hinge, and to vertices in the connected components at a hinge. The results extend those in [6], [7] and [10].

Keywords: Algebraic connectivity, Fiedler vector.

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1. Introduction

Throughout this paper, a graph is a pair of sets $G = (V, E)$, where the elements of E are two elements subsets of V. The elements of V are vertices of the graph and the elements of E are its edges.

Given a graph $G = (V, E)$ on n vertices, the *Laplacian matrix* of G is the matrix of order n given by $L(G) = [l_{ij}]$, where $l_{ij} = -1$ if $v_i v_j \in E$, $l_{ii} = d(v_i)$ and $l_{ij} = 0$ for the remaining entries. In the survey [11], some known results about Laplacian matrices are exhibited.

In [6], Fiedler has shown that a graph is connected if and only if the second smallest Laplacian eigenvalue is positive. This eigenvalue is called the *algebraic connectivity*, and plays a fundamental role in the field of spectral graph theory. We denote the algebraic connectivity of G by $a(G)$ and refer to [1] and [8], where surveys of old and new results on this spectral parameter are presented.

An eigenvector associated with the algebraic connectivity is called a *Fiedler vector*, after the pioneering work of Fiedler in [7]; these eigenvectors have proven to be very useful tools in many areas of pure and applied sciences. An account of Fiedler's influence on spectral graph theory is given in [12].

Labeling the vertices of G by $v_1, v_2, \ldots, v_n$ and denoting a Fiedler vector by $y = [y_i]$, the coordinates of y can be assigned naturally to the vertices of G: the coordinate y_i labels the vertex v_i. (We also use the alternate notation $y(v_i)$ for y_i.) This assignment has been called a *characteristic valuation* and Fiedler noticed that it induces partitions of the vertices of G that are naturally connected clusters, important for applications and for characterizing the graph structure. As an example, Fiedler [6] shows that for any Fiedler vector y, the subgraph induced by $\{v_i \in V | y_i \geq 0\}$ is connected. In light of such results, Pothen, Simon and Liu [13] suggested a spectral graph partitioning algorithm based on the entries of a Fiedler vector.

After some terminology and notation is introduced, we recount some of Fiedler's findings, in order to state the kind of results we discuss in this paper.

If G is a connected graph, a vertex v is called an *articulation point* if the graph $G \setminus v$, formed from G by deleting v and all edges incident with it, is disconnected. A graph G is said to be *k-connected* ($k \in \mathbb{N}$), if $|G| > k$ and $G \setminus X$ is connected for each set $X \subset V$ with $|X| < k$. In other words, no two vertices can be separated by the removal of less than k vertices (and their incident edges). A *block* in G is any maximal induced connected subgraph with no articulation points. A vertex v is called a *characteristic vertex* of G with respect to the Fiedler vector y if $y(v) = 0$ and there is a vertex u adjacent to v such that $y(u) \neq 0$. An edge uv is called a *characteristic edge* of G with respect to y if $y(u)y(v) < 0$.

Fiedler's monotonicity theorem [7] shows (among other results) that, given a connected graph G and characteristic valuation of its vertices, precisely one of the following cases occurs.

Case A: There is exactly one block C in G which contains both positively and negatively valuated vertices. Every other block is either a positive block (i.e. all vertices have positive valuation), or a negative block, or a zero block.

Case B: No block of G contains both positively and negatively valuated vertices. In this case, there exists a unique characteristic vertex z. This vertex z is a point of articulation. Each block of G is (with the exception of z) either a positive block, or is (with the exception of z) a negative block, or is a zero block.

(We note in passing that in [10] it is shown that cases A and B occur independently of the choice of the Fiedler vector – i.e. either case A holds for every Fiedler vector of G, or case B holds for every Fielder vector of G.)

Fiedler [7] further extends these partition properties and shows that if v_k is an articulation point of G, with components of $G \setminus \{v_k\}$ given by $C_0, \ldots C_r$, then the following hold.

(a) If $y_k > 0$ then exactly one of the components $C_0, \ldots, C_r$ has some vertices valuated negatively in y. For all other vertices v_i in the remaining components, $y_i > y_k$.
(b) If $y_k = 0$ and there is a component C_i with both negatively and positively valuated vertices, then it is the only component with that property; all other vertices v_i have $y_i = 0$.
(c) If $y_k = 0$ and there is no component C_i with both negatively and positively valuated vertices, then each component C_i is either positively valuated or negatively valuated or zero valuated in y.

Some 20 years after Fiedler's work appeared, Kirkland, Neumann, Shader and Fallat used Fiedler's theory and the Perron values of matrices associated to the components $G \setminus \{v\}$ (v an articulation point) in order to characterize the algebraic connectivity of trees [9] and of graphs with articulation points [10]. A further extension of this work appears in [4], which deals with irreducible matrices all of whose off–diagonal entries are nonpositive.

Observe that the results above give no information about the characteristic valuation inside a block of G. Theorem 2.4 of [2], gives some information about the behavior of the characteristic valuations on vertices of the non characteristic blocks. Nevertheless, if G has no articulation point, i.e. if it is k-connected, with $k > 1$, then G is composed of a single block containing both nonpositive and nonnegative vertices. The main purpose of this paper is to extend the kinds of results discussed above, to graphs without articulation points, describing the structure of the partition arising from a characteristic valuation. In particular, we show how a set of $k > 1$ vertices that disconnect G may induce connected subgraphs having vertex valuations of the same sign, introducing a natural partition of vertices in G. The technique used is similar to Fiedler's, but relies on a generalized notion of articulation point. This is discussed in section 3. We also extend the result in [9] and [10] to graphs with no articulation points. This is the topic of section 4.

The remainder of the paper is organized as follows. We start in section 2 with the definitions and known lemmata which will be used throughout this note. Section 5 discusses an application to bounding the algebraic connectivity based on a parameter derived from the number of connecting edges between components.

2. Definitions and Lemmata

First, we generalize the idea of an articulation point. Given a k-connected graph (with $k \geq 1$), a *hinge* is a set of k vertices $\widehat{H}$ such that $G \setminus \widehat{H}$ is disconnected. It is easy to see that for any connected graph with an articulation point, the hinges are precisely the articulation points of that graph. Hinges will be crucial for the results obtained in this paper.

Now we will get into the definitions that distinguish hinges and components by means of their characteristic valuations. Let G be a graph with a characteristic valuation $y = [y_i]$ and a hinge $\widehat{H}$. Label the vertices of $\widehat{H}$ as v_{l_1}, v_{l_2},…,v_{l_k}, such that $y_{l_1} \leq y_{l_2} \leq \cdots \leq y_{l_k}$.

- We say that a *hinge* $\widehat{H}$ *is null* if $y_{l_1} = y_{l_2} = \cdots = y_{l_k} = 0$.
- We say that a *hinge* $\widehat{H}$ *is positive* if $y_{l_1} > 0$, that is, all vertices at $\widehat{H}$ have positive valuation.
- We say that a *hinge* $\widehat{H}$ *is nonnegative* if $y_{l_1} \geq 0$ and $y_{l_k} > 0$.
- We say that a *hinge* $\widehat{H}$ *is mixed* whenever $y_{l_1} < 0 < y_{l_k}$, that is, $\widehat{H}$ has simultaneously positive and negative valuation.

Similarly, we define *negative hinge* and *nonpositive hinge*.

Now, consider a connected component C of the graph $G \setminus \widehat{H}$.

- We say that C is a *null component* if the valuation of each of its vertices is zero.
- We say that C is a *positive component* if the valuation of each of its vertices is positive.
- We say that C is a *negative component* if the valuation of each of its vertices is negative.
- We say that C is a *mixed component* if it contains vertices with both positive and negative valuations.

We emphasize here that the identification of C as a null (respectively positive, negative, mixed) component depends fundamentally upon the particular Fiedler vector under consideration.

For a symmetric matrix M of order n we denote its eigenvalues by $\lambda_1(M) \leq \lambda_2(M) \leq \ldots \leq \lambda_n(M)$.

The following results were proved by Fiedler [7].

Lemma 2.1. *Consider the symmetric matrix A with nonpositive off diagonal entries and with nonnegative eigenvalues. If A is irreducible and singular, then there exists a unique vector $y > 0$ (up to a scalar multiple), such that $Ay = 0$.*

Lemma 2.2. *Let A be a real square matrix with nonpositive off diagonal entries. If all eigenvalues of A are positive, then $A^{-1} \geq 0$.*

Lemma 2.3. *Consider a connected graph G with vertex set $V = \{v_1, v_2, \ldots, v_n\}$. Let $y = [y_i]$ be a characteristic valuation of G. For $r \geq 0$, let*

$$V_1(r) = \{v_i \in V | y_i \geq -r\}, \; V_2(r) = \{v_i \in V | y_i \leq r\}.$$

Then both the subgraphs induced by $V_1(r)$ and $V_2(r)$ are connected.

Lemma 2.4. *Let $y = [y_i]$ be a characteristic valuation of the connected graph $G = (V, E)$. If $y_i > 0$, then there exists a vertex v_j such that $v_i, v_j \in E$ and $y_i > y_j$.*

In the paper [4], the perturbed Laplacian matrix of a graph is defined as $L(D) = D - A$, where D is any given diagonal matrix and A is the weighted adjacency matrix. The next lemma follows from [4], where the authors studied the Fiedler vector of the perturbed Laplacian matrix and we rewrite it in the context of Laplacian matrix to fit our framework.

Lemma 2.5. *Let G be a connected graph. Let y be a Fiedler vector of L. Let W be a nonempty set of vertices of G such that $y(u) = 0$, for all $u \in W$ and suppose $G \setminus W$ is disconnected with $t \geq 2$ components $C_1, C_2, \ldots, C_t$, such that $y(C_i) \neq 0$, $i = 1, \ldots, t$. Then each $y(C_i)$ is either all positive or all negative.*

3. Structural Results

We begin with a convenient expression for the Laplacian matrix of the connected graph G. Let $\widehat{H}$ be a hinge of G and let $C_0, C_1, \ldots, C_r$ be the components of $G \setminus \widehat{H}$. For convenience, we assume that the last rows and columns of the Laplacian matrix represent the vertices of $\widehat{H}$. Therefore, the Laplacian matrix has the following format.

$$L = \begin{bmatrix} A_0 & 0 & \cdots & 0 & c_0^1 & \cdots & c_0^k \\ 0 & A_1 & & 0 & c_1^1 & \cdots & c_1^k \\ \vdots & & \ddots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & A_r & c_r^1 & \cdots & c_r^k \\ (c_0^1)^T & (c_1^1)^T & \cdots & (c_r^1)^T & d_1 & & \\ \vdots & \vdots & & \vdots & & \ddots & \\ (c_0^k)^T & (c_1^k)^T & \cdots & (c_r^k)^T & & & d_k \end{bmatrix}, \tag{3.1}$$

where A_i corresponds to vertices of the connected component C_i, for $i = 0, 1, \ldots, r$ and c_i^j is a $(0, -1)$-vector that accounts for the edges between the vertex v_j of $\widehat{H}$ and the connected component C_i. We note that it is possible for a hinge to have edges between its vertices. From now on, we shall use the format (3.1).

Theorem 3.1. *Let G be connected graph and $y = [y_i]$ a characteristic valuation of G. Let $\widehat{H}$ be a hinge of G and let $C_0, C_1, \ldots, C_r$ be the connected components of $G \setminus \widehat{H}$. Label the vertices of $\widehat{H}$ as $l_1, \ldots, l_k$, where $y_{l_1} \leq y_{l_2} \leq \cdots \leq y_{l_k}$.*

(i) If $\widehat{H}$ is null and there exists a mixed component C_i, then it is the only mixed component and all the other components are null.

(ii) If $\widehat{H}$ is null and there is no mixed component, then each component is either null, positive or negative.

(iii) If $\widehat{H}$ is non-negative, then only one component has vertices with negative valuation. All the remaining vertices v_s outside the hinge and the component with negative vertices satisfies $y_{l_1} < y_s$.

Proof. We first apply Lemma 2.5 with $W = \widehat{H}$. To prove (i) suppose that there exists another component that is not null. Then by Lemma 2.5 each component is positive, negative or null. This is a contradiction since there exists a mixed component. Hence the only non-null component is the mixed one. This shows part (i).

To prove (ii), since $\mathbf{1}$ is an eigenvector of $L(G)$, then $\sum y_i = 0$. Since there are no mixed components there are at least two nonzero components, say C_0 and C_1, such that $y(C_0) \neq 0$ and $y(C_1) \neq 0$. Therefore, Lemma 2.5 ensures that each component $C_0, C_1, \ldots, C_r$ is positive, negative or null and it proves part (ii).

To prove (iii), we split the proof in two cases

Case a: $y_{l_1} = 0$.

Conforming to the structure of L we can partition y as

$$y = [y^{(0)}, y^{(1)}, \ldots, y^{(r)}, y_{l_1}, \ldots, y_{l_k}]^T,$$

where $y_{l_k} > 0$.

From equation (3.1) and since $Ly = ay$, we have $A_i y^{(i)} + \sum_{j=1}^{k} c_i^j y_{l_j} = a y^{(i)}$ for $i = 0, \ldots, r$, so that we can write

$$(A_i - aI)y^{(i)} = -\sum_{j=1}^{k} c_i^j y_{l_j} \tag{3.2}$$

for $i = 0, \ldots, r$. As before, we see that B has at most one negative eigenvalue. Hence, we assume that for each $i = 1, \ldots, r$ $A_i - aI$ is positive semidefinite.

Since G is a k-connected graph the hinge has k vertices. Now, for each component C_j and each vertex u of $\hat{H}$, there is at least one vertex of C_j that is adjacent to u, otherwise we could disconnect the graph by removing a proper subset of the hinge, i.e. those vertices of $\hat{H}$ that vertices that are adjacent to some vertex of C_j. We thus deduce that that $c_i^j \neq 0$.

If $A_i - aI$ were singular, then by means of Lemma 2.1, there exists a vector $u > 0$ such that $u^T(A_i - aI) = 0$. Hence $u^T(A_i - aI)y^{(j)} = 0$ implying that $\sum_{j=1}^{k} c_i^j y_{l_j} = 0$. Now, since $c_i^j \leq 0$ and $y_{l_k} > 0$, we have that $c_i^j = 0$, a contradiction. Thus, $A_i - aI$ is invertible. Hence, by applying Lemma 2.2, its inverse is positive. Using equation (3.2), we obtain

$$y^{(i)} = -(A_i - aI)^{-1} \sum_{j=1}^{k} c_i^j y_{l_j}$$

for $i = 1, \ldots, r$. Since $c_i^j \leq 0$, $y_{l_j} \geq 0$ and $(A_i - aI)^{-1} \geq 0$, we have that $y^{(i)} \geq 0$ for $i = 1, \ldots, r$.

It remains to show that $y^{(i)} > 0$ for $i = 1, \ldots, r$. We have $A_i y^{(i)} + \sum_{j=1}^k c_i^j y_{l_j} = a y^{(i)}$ for $i = 0, \ldots, r$, which can be rewritten as

$$\frac{1}{a(G)} y^{(i)} = A_i^{-1} y^{(i)} - \frac{1}{a(G)} A_i^{-1} \left(\sum_{j=1}^k c_i^j y_{l_j} \right).$$

Let $\mathbf{1}$ denote an all ones vector of the appropriate order; multiplying and dividing the rightmost term by $\mathbf{1}^T y^{(i)}$, we obtain

$$\begin{aligned} \frac{1}{a(G)} y^{(i)} &= A_i^{-1} y^{(i)} - \frac{1}{a(G)\mathbf{1}^T y^{(i)}} A_i^{-1} \left(\sum_{j=1}^k c_i^j y_{l_j} \right) \mathbf{1}^T y^{(i)} \\ &= \left[A_i^{-1} - \frac{1}{a(G)\mathbf{1}^T y^{(i)}} A_i^{-1} \left(\sum_{j=1}^k c_i^j y_{l_j} \right) \mathbf{1}^T \right] y^{(i)}. \end{aligned}$$

Since A_i is an irreducible and nonsingular M-matrix, $(A_i)^{-1}$ is a positive matrix, and it now follows readily that the matrix $M = A_i^{-1} - \frac{1}{a(G)\mathbf{1}^T y^{(i)}} A_i^{-1} \left(\sum_{j=1}^k c_i^j y_{l_j} \right) \mathbf{1}^T$ is positive. Since $y^{(i)}$ is a nonnegative eigenvector of the positive matrix M, it follows from Perron-Frobenius theory that in fact $y^{(i)}$ is a positive vector. Hence the vertices of G with negative valuation are in the same component C_0.This completes the proof in case a.

Case b: $y_{l_1} > 0$.

Since $\sum y_i = 0$, there is a vertex in G with negative valuation. Suppose that C_0 contains such vertex. To prove case b, it is sufficient to show that each vertex v_t with $y_t \leq y_{l_1}$ is in C_0 or in $\widehat{H}$.

First, suppose that $y_t < y_{l_1}$ for some v_t. Then there exists $\epsilon > 0$ such that $y_{l_1} - \epsilon > 0$. By means of Lemma 2.3, the subgraph G' induced by the set of vertices $M = \{v_s \in V | y_s \leq y_{l_1} - \epsilon\}$ is connected. Since G' contains at least one negative vertex and $H \not\subset G'$, then $G' \subset C_0$ and, therefore, $v_t \in C_0$. On the other hand, suppose that $y_t = y_{l_1}$. By applying Lemma 2.4, there is a vertex $v_s \in G$ adjacent to v_t with $y_s < y_t$. Since $y_s \neq y_{l_1}$, by the previous argument, we have $v_s \in C_0$ and since C_0 is a connected component, then $v_t \in C_0$ or $\widehat{H}$, which shows case b. $\square$

In the cases of negative or nonpositive hinges, the results above may also be applied, using a negative multiple of the Fiedler vector. For the case of mixed hinge, the example below shows that there could be more than one mixed component or even no mixed component.

Example 1. *The 10-vertex cycle of Figure 1 has algebraic connectivity* $2-2\cos(\frac{\pi}{5})$ *(with multiplicity two), and a Fiedler vector given approximately by*

$$y = [-0.26287, -0.42533, -0.42533, -0.26287, 0, 0.26287, 0.42533, 0.42533, 0.26287, 0]^T.$$

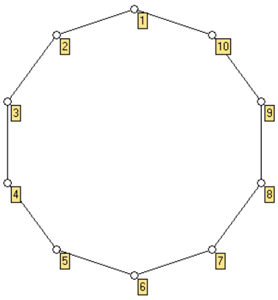

FIGURE 1. The 10-vertex cycle.

Considering the hinges $\widehat{H} = \{v_1, v_6\}$ *and* $\widehat{I} = \{v_3, v_8\}$*, we see that at* $\widehat{H}$ *there are two components and none is mixed, whereas at* $\widehat{I}$ *there are two mixed components.*

The graph of the next example illustrates the three cases of Theorem 3.1.

Example 2. *The graph in the Figure 2 has algebraic connectivity* $2-\sqrt{2}$ *(with multiplicity one), and a corresponding Fiedler vector is given approximately by*

$$y = [-0.35355, -0.5, -0.35355, 0.35355, 0.5, 0.35355, 0, 0, 0, 0]^T.$$

This graph is 2-connected and we can choose $\{v_7, v_9\}$ *as a null hinge. It has one*

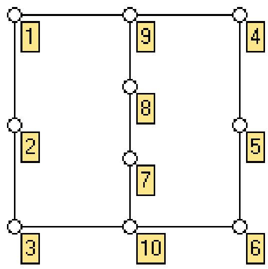

FIGURE 2. A 2-connected graph.

mixed component and one null component in accordance with case (i) of Theorem 3.1.

Clearly, $\{v_9, v_{10}\}$ *is a null hinge that fulfils case (ii) of Theorem 3.1 with only positive, negative and null components.*

Finally, the set $\{v_4, v_6\}$ *forms a non-negative hinge (in fact positive) and case (iii) of Theorem 3.1 ensures that it has only one component with negative valuation. Besides, the remaining component formed by the singleton* $\{v_5\}$ *has larger characteristic valuation than* v_4*.*

We can also provide graphs that have only one hinge that is a mixed hinge. The next example illustrates this.

Example 3. *The graph of Figure 3 has algebraic connectivity equal to the smallest root of the cubic* $z^3-15z^2+66z-74$*, which is approximately* 1.7101*, and the algebraic connectivity is algebraically simple. A Fiedler vector is (approximately) equal to*

$$y \quad = \quad [-0.37945, -0.37945, -0.13567, 0.02565, 0.55636, 0.34601, -0.37945, 0.34601]^T.$$

Therefore, this characteristic valuation gives a mixed hinge $\{v_3, v_4\}$*. Further, this*

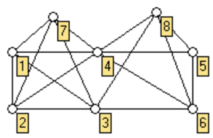

FIGURE 3. A graph with a unique mixed-hinge

is the only hinge for this graph.

4. CHARACTERIZING THE ALGEBRAIC CONNECTIVITY

In this section, we use our structure theorem to compute the algebraic connectivity for graphs with nonnegative or null hinges. The results are natural extensions of those in [10], where graphs with articulation points were considered.

Let G be a graph and L its Laplacian matrix. For a hinge $\widehat{H}$ of G, denote the connected components of $G \setminus \widehat{H}$ by $C_0, C_1, \ldots, C_r$. For each component, let $L(C_i)$ be the principal submatrix of L, corresponding to the vertices of C_i. The *Perron value* of C_i is the Perron value of the positive matrix $L^{-1}(C_i)$ and we say C_j is a *Perron component at* $\widehat{H}$ if its Perron values is the largest among all components $C_0, C_1, \ldots, C_r$.

Theorem 4.1. *Let G be a connected graph and $y = [y_i]$ a characteristic valuation of G. If there exists a null cut set W, such that $G \setminus W$ has no mixed component, then there are two or more Perron components at W. In this case, $a(G) = \dfrac{1}{\rho(L(C)^{-1})}$ for each Perron component C at W.*

Proof. Let $C_0, C_1, \dots, C_r$ be the components of $G \setminus W$ and assume the Laplacian matrix is in the form (3.1). Partitioning the eigenvector y according to the characteristic valuation in each component C_i as $y = [y^{(0)}, y^{(1)}, \dots, y^{(r)}, 0, \dots, 0]^T$, and from the relation $Ly = a(G)y$, we have

$$L(C_i)y^{(i)} = a(G)y^{(i)},$$

where $A_i = L(C_i)$. From the fact that $\sum y_i = 0$, we know there exist positive and negative entries in y. Since there are no mixed components there are at least two nonzero components, say C_r and C_s, such that $y^{(r)} \neq 0$ and $y^{(s)} \neq 0$. Therefore, Lemma 2.5 ensures that each component $C_0, C_1, \dots, C_r$ is positive, negative or null.

Using the Perron-Frobenius theorem, the only eigenvectors with entries of the same sign are Perron vectors. Hence, $y^{(r)}$ and $y^{(s)}$, are Perron vectors for $L(C_r)^{-1}$ and $L(C_s)^{-1}$, respectively. Therefore, for each non-null component we have the relation for the Perron vector $y^{(i)}$

$$L(C_i)^{-1}y^{(i)} = \frac{1}{a(G)}y^{(i)}.$$

It remains to show that C_r and C_s are Perron components at W. Suppose to the contrary, that this is not the case. Then there would exist another component, say C_t, whose Perron value is larger than $1/a(G)$. Let x be the Perron vector of $L(C_t)^{-1}$, normalized so that $\mathbf{1}^T x = 1/\sqrt{2}$, and define $u = y^{(r)}/(\sqrt{2}\mathbf{1}^T \|y^{(r)}\|)$. Consider the vector

$$w = [u, 0, \cdots 0, -x, 0, \dots, 0]^T,$$

which is obviously orthogonal to $\mathbf{1}$.
Since

$$w^T L w = a u^T u + \frac{1}{\rho(L(C_t)^{-1})} x^T x < a u^T u + a x^T x = a w w^T \tag{4.1}$$

we obtain a contradiction with the fact that the algebraic connectivity can be characterized as

$$a(G) = \min_{\substack{\|x\|=1 \\ x \perp \mathbf{1}}} x^T L x. \tag{4.2}$$

Thus, we obtain that C_r and C_s are indeed the Perron components at W. $\square$

One question we want to address is whether or not the set of Fiedler vectors identifies the same null hinge. One can describe other Fiedler vectors identifying the same null hinge, as long as there is some information about a Fiedler vector. More precisely, the following result constructs the set of all Fiedler vectors that identify $\widehat{H}$ as a null hinge.

Theorem 4.2. *Let G be a graph and $y = [y_i]$ a characteristic valuation of G. Suppose there exists a null hinge $\widehat{H}$, such that $G \setminus \widehat{H}$ has no mixed component, and for $t \geq 2$, let $C_1, C_2, \ldots, C_t$ be the set of Perron components of $G \setminus \widehat{H}$. Assume the Laplacian matrix is in the form (3.1). Let $y^{(1)}, y^{(2)}, \ldots, y^{(t)}$ be the set of Perron vectors for the set of matrices $L(C_1)^{-1}, L(C_2)^{-1}, \ldots, L(C_t)^{-1}$ such that $\mathbf{1}^T y^{(i)} = 1$. Define, for $i = 2, .., t$, the vector*

$$f_i = \begin{cases} y^{(1)}(v) & v \in C_1, \\ -y^{(i)}(v) & v \in C_i, \\ 0 & \textit{otherwise}. \end{cases} \tag{4.3}$$

Then $f_2, f_3, \ldots, f_t$ is a set of linearly independent eigenvectors associated with $a(G)$ and each Fiedler vector that identifies $\widehat{H}$ as a null hinge is a linear combination of f_i, therefore it has no mixed component.

Proof. It is easy to see that $\mathbf{1}^T f_i = 0$ and that $f_2, f_3, \ldots, f_t$ is a linearly independent set of vectors. Further, $\frac{1}{\rho(L(C_1)^{-1})}$ is the eigenvalue of $L(C_1)$ associated with $y^{(i)}$, for $i = 2, \ldots, t$. By the construction of f_i, we conclude that $\frac{1}{\rho(L(C_1)^{-1})}$ is the eigenvalue associated with f_i. Then, we have

$$\frac{f_i^T L f_i}{f_i^T f_i} = \frac{1}{\rho(L(C_1)^{-1})}.$$

Using Theorem 4.1, we obtain

$$\frac{f_i^T L f_i}{f_i^T f_i} = a(G),$$

for $i = 2, \ldots, t$, therefore $f_2, f_3, \ldots, f_t$ is a set of linearly independent eigenvectors associated with $a(G)$.

Now let z be a Fiedler vector that identifies $\widehat{H}$ as a null hinge. From the relation $Lz = a(G)z$, it follows for each component C_i at $\widehat{H}$ that $L(C_i)z(C_i) = a(G)z(C_i)$. Since $L(C_i)y^{(i)} = a(G)y^{(i)}$, by the Perron-Frobenius theorem $\frac{1}{\rho(L(C_1)^{-1})}$ is a simple eigenvalue of $L(C_i)^{-1}$. Hence, it follows that $z(C_i)$ is a scalar multiple of $y^{(i)}$. That implies that z is a linear combination of the f_is. $\square$

We note that the proof of the last theorem can also be obtained using results of [5]. Since we construct the whole set of Fiedler vectors that identify the same null hinge, it is natural to ask whether or not it is possible to have other Fiedler vectors that do not identify the same hinge as null. The answer is positive, and the following example shows that.

Example 4. *Consider the complete bipartite graph $K_{3,3}$. It is readily determined that the eigenvalues of the corresponding Laplacian matrix are $0, 6$ and 3, the latter with multiplicity four. In particular, $K_{3,3}$ has algebraic connectivity 3. If we label the vertices in one partition as v_1, v_2 and v_3, then the set $\widehat{H} = \{v_4, v_5, v_6\}$ is a hinge. At $\widehat{H}$, we have three Perron components, namely $C_i = \{v_i\}, i = 1, 2, 3,$ with $\rho(L(C_i)^{-1}) = \frac{1}{3}, i = 1, 2, 3$. It is easy to see that a set of eigenvectors which spans the Fiedler eigenspace is given by $w = [1, -1, 0, 0, 0, 0]^T$, $x = [1, 0, -1, 0, 0, 0]^T$, $y = [0, 0, 0, 0, -1, 1]^T$ and $z = [0, 0, 0, -1, 0, 1]^T$. Now, we can see that w and x identify $\widehat{H}$ as null, but that $\widehat{H}$ is not null for y and z.*

For the case where null hinges have mixed components we can also describe the algebraic connectivity in terms of its components.

Theorem 4.3. *Let G be graph and $y = [y_i]$ a characteristic valuation of G. Suppose that there exists a null cut set W, such that $G \setminus W$ has a mixed component C, then $a(G) = \lambda_2(L(C))$.*

Proof. By part (i) of Theorem 3.1 all components are null, except for the mixed one. If we use the Laplacian matrix in the form (3.1) and set $A_0 = L(C)$, we can write y such that

$$y = [y^{(0)}, y^{(1)}, \ldots, y^{(r)}, y_{l_1}, \ldots, y_{l_k}]^T.$$

By the fact that $\widehat{H}$ is null, we have $y_{l_i} = 0,\ i = 1, \ldots, k$. From the relation $Ly = a(G)y$, we obtain

$$A_0 y^{(0)} = a(G) y^{(0)} \tag{4.4}$$

And hence, $a(G)$ is an eigenvalue of A_0. By the eigenvalue interlacing theorem, we have $a(G) \leq \lambda_2(A_0)$. Since the smallest eigenvalue of A_0, say $\lambda_1(A_0)$, satisfies $\lambda_1(A_0) = \dfrac{1}{\rho(L(C)^{-1})}$, we find that the eigenvector of A_0 associated with $\lambda_1(A_0)$ has all entries with the same sign. Therefore, by applying equation (4.4), $a(G)$ can not be the smallest eigenvalue of A_0. Therefore we have $a(G) = \lambda_2(A)$, as desired. □

Corollary 4.4. *Under the hypotheses of Theorem 4.3, C is the only Perron component at W.*

Proof. Consider the submatrix of L

$$L^* = \begin{bmatrix} A_0 & & & \\ & A_1 & & \\ & & \ddots & \\ & & & A_r \end{bmatrix}.$$

By the eigenvalue interlacing theorem, we have $a(G) \leq \lambda_2(L^*)$. Since $\lambda_1(L(C)) = \dfrac{1}{\rho(L(C)^{-1})}$ is a simple eigenvalue of A_0, by Theorem 4.3 we have $\lambda_1(L(C)) < \lambda_2(L(C)) \leq \lambda_2(L^*)$. Assume to the contrary that there exists another component at W, say D, that is a Perron component. We get $\rho(L(C)^{-1}) \leq \rho(L(D)^{-1})$ and, furthermore, it is easy to see that $\lambda_1(L(C)) = \dfrac{1}{\rho(L(C)^{-1})}$ and $\lambda_1(L(D)) = \dfrac{1}{\rho(L(D)^{-1})}$ are eigenvalues of L^*. Therefore, we obtain

$$\lambda_1(L(D)) \leq \lambda_1(L(C)) < \lambda_2(L(C)) \leq \lambda_2(L^*).$$

This is a contradiction, because we have two eigenvalues of L^* smaller than $\lambda_2(L^*)$. □

Next example provides a concrete case where Theorem 4.3 applies.

Example 5. *Recalling graph G of Figure 2, it has algebraic connectivity approximately 0.58579. Also, $\{v_7, v_9\}$ is a null hinge with one mixed component $C = \{v_1, v_2, v_3, v_4, v_5, v_6, v_8, v_{10}, \}$. Its corresponding Laplacian submatrix is given by*

$$L(C) = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 0 & 3 \end{bmatrix}$$

and $L(C)$ has eigenvalues (approximately) equal to

$$0.32487, 0.58579, 1.46081, 2, 2, 3, 3.41421, 4.21431.$$

Therefore, $a(G) = \lambda_2(L(C))$ in accordance with Theorem 4.3.

For nonnegative or nonpositive hinges, it is also possible to describe the algebraic connectivity by using the Perron value of some matrices. The following theorem describes how it can be done.

Theorem 4.5. *Let G be a graph and $y = [y_i]$ a characteristic valuation of G. Let $\widehat{H}$ be a nonnegative (nonpositive) or null hinge of G. For each positive (or negative) component C at $\widehat{H}$, there is a nonnegative matrix M of rank at most 1, and a scalar $\gamma > 0$, such that*

$$\rho(L(C)^{-1} + \gamma M) = \frac{1}{a(G)}.$$

Furthermore, $M = 0$ if and only if $\widehat{H}$ is null.

Proof. If we use the Laplacian matrix in the form (3.1) and set $A_0 = L(C)$, we can write y such that

$$y = [y^{(0)}, y^{(1)}, \ldots, y^{(r)}, y_{l_1}, \ldots, y_{l_k}]^T.$$

Since C is positive, then $y^{(0)} \geq 0$ (when C is negative the proof is similar). By the fact that $\widehat{H}$ is nonnegative, we have $y_{l_i} \geq 0$ and from the relation $Ly = a(G)y$, we obtain

$$A_0 y^{(0)} + \sum_{j=1}^{k} c_0^j y_{l_j} = a(G) y^{(0)}$$

or, equivalently,

$$\frac{1}{a(G)} y^{(0)} = A_0^{-1} y^{(0)} - \frac{1}{a(G)} A_0^{-1} \left(\sum_{j=1}^{k} c_0^j y_{l_j} \right).$$

Now, multiplying and dividing the rightmost term by $\mathbf{1}^T y_0^A$, we obtain

$$\begin{aligned} \frac{1}{a(G)} y^{(0)} &= A_0^{-1} y^{(0)} - \frac{1}{a(G)\mathbf{1}^T y^{(0)}} A_0^{-1} \left(\sum_{j=1}^{k} c_0^j y_{l_j} \right) \mathbf{1}^T y^{(0)} \\ &= (A_0^{-1} + \gamma M) y^{(0)}, \end{aligned}$$

where $\gamma = \frac{1}{a(G)\mathbf{1}^T y^{(0)}}$ and $M = -A_0^{-1} \left(\sum_{j=1}^{k} c_0^j y_{l_j} \right) \mathbf{1}^T$. Since C is a positive component, then $y^{(0)} > 0$ and, hence, $\gamma > 0$. By applying Lemma 2.2, we have $A_0^{-1} \geq 0$ and by the fact that $\sum_{j=1}^{k} c_0^j y_{l_j} \leq 0$, we obtain $M \geq 0$. Since $c_0^j \leq 0$, $M = 0$ if, and only if, $y_{l_j} = 0$ for each $j = 1, \ldots, k$, that is, whenever $\widehat{H}$ is null. $\square$

Example 6. *Figure 2, provides a graph G where the set $\{v_4, v_6\}$ forms a nonnegative hinge with one positive component formed by the singleton $C = \{v_5\}$. In order to characterize the algebraic connectivity using Theorem 4.5, the proof gives a description of γ and M.*

The graph G has Fiedler vector (approximately)

$$y = [-0.35355, -0.5, -0.35355, 0.35355, 0.5, 0.35355, 0, 0, 0, 0]^T$$

and $a(G) = 0.58579$ *(also approximately). Using the notation of the proof of Theorem 4.5, we have* $L(C) = [2]$ *and* $y^{(0)} = [0.5]$. *Therefore, again using approximate values, we have* $\gamma = \frac{1}{a(G)y^0} = 3.41419$ *and*

$$M = -L(C)^{-1}\left(\sum_{j=1}^{2} -y_{l_j}\right) = -[0.5]\left([-0.35355] + [-0.35355]\right) = [0.35355].$$

Hence, $L(C)^{-1} + \gamma M = [1.707087864]$ *which has Perron vector* $[0.5]$ *and satisfies* $\rho(L(C)^{-1} + \gamma M) = \frac{1}{a(G)}$.

5. Bounding the algebraic connectivity

In this section we introduce some concepts which will help us to better understand how the algebraic connectivity is bounded. More specifically, we want to investigate the algebraic connectivity as a function of the number of edges between a hinge and its components, as an attempt to generalize the well known fact (see [6]) that, for a k-connected graph, $a(G) \leq k$.

Let $\widehat{H}$ be a hinge of the graph G and let C be a component at $\widehat{H}$. Let $v_1, v_2, \ldots, v_t$ be the vertices in the component C. We shall denote by $d_{\widehat{H}}(v_i)$ the number of edges connecting v_i to the vertices of $\widehat{H}$. Similarly, for each vertex u of $\hat{H}$, we let $d_C(u)$ denote the number of vertices in C that are adjacent to u.

Further, we define the quantity

$$\mathcal{S}_C = \max_{v_i \in C} \left\{ d_{\widehat{H}}(v_i) \right\} \tag{5.1}$$

which shall be named *strength* of the component C.

Denoting the set of components at $\widehat{H}$ by H, we define the quantity

$$\mathcal{S}_{\widehat{H}} = \max_{C \in H} \{\mathcal{S}_C\} \tag{5.2}$$

which shall be named *strength* of the hinge $\widehat{H}$.

Theorem 5.1. *Let G be a graph and let $\widehat{H}$ be a hinge of G. For each $j = 0, \ldots, r$, let the component C_j have p_j vertices. Then we have the following conclusions.*
a) $a(G) \leq \mathcal{S}_{\widehat{H}}$.
b) If $a(G) = \mathcal{S}_{\widehat{H}}$, *then each vertex of* $G \setminus \hat{H}$ *is adjacent to* $\mathcal{S}_{\widehat{H}}$ *vertices in the hinge*

$\hat{H}$ *and for each* $i, j = 0, \ldots, r$ *and each* $u \in \hat{H}$ *we have* $p_i d_{C_j}(u) = p_j d_{C_i}(u)$.
c) If each vertex of $G \setminus \hat{H}$ *is adjacent to* $\mathcal{S}_{\widehat{H}}$ *vertices in the hinge* $\hat{H}$ *and for each* $i, j = 0, \ldots, r$ *and each* $u \in \hat{H}$ *we have* $p_i d_{C_j}(u) = p_j d_{C_i}(u)$, *then* $\mathcal{S}_{\widehat{H}}$ *is a Laplacian eigenvalue of* G. *In this case, the multiplicity of* $\mathcal{S}_{\widehat{H}}$ *as an eigenvalue is at least* r.

Proof. We write the Laplacian matrix L in the form (3.1). Let x^T be the vector given by

$$\left[\begin{array}{ccccc} p_1\mathbf{1}^T & -p_0\mathbf{1}^T & 0^T & \ldots & 0^T \end{array}\right],$$

where the partitioning of x conforms with that of L in (3.1). Clearly $\mathbf{1}^T x = 0$. Since $\mathbf{1}^T A_0 \mathbf{1} \le p_0 \mathcal{S}_{\widehat{H}}$ and $\mathbf{1}^T A_1 \mathbf{1} \le p_1 \mathcal{S}_{\widehat{H}}$, it is readily determined that $x^T L x \le x^T x \mathcal{S}_{\widehat{H}}$. Conclusion a) now follows readily.

To establish conclusion b), suppose that $a(G) = \mathcal{S}_{\widehat{H}}$. Then necessarily the vector x above must be an eigenvector of G. In that case, we find that each vertex of $C_0 \cup C_1$ is adjacent to $\mathcal{S}_{\widehat{H}}$ vertices in the hinge $\hat{H}$, and in addition, for each vertex $u \in \hat{H}$ we have $p_1 d_{C_0}(u) = p_0 d_{C_1}(u)$. Evidently the argument above applies to any pair of components C_i, C_j, and b) now follows.

To establish c), suppose that each vertex of $G \setminus \hat{H}$ is adjacent to $\mathcal{S}_{\widehat{H}}$ vertices in the hinge $\hat{H}$, and further if for each $i, j = 0, \ldots, r$ and each $u \in \hat{H}$ we have $p_i d_{C_j}(u) = p_j d_{C_i}(u)$. Then $\mathbf{1}^T A_0 \mathbf{1} = p_0 \mathcal{S}_{\widehat{H}}$ and $\mathbf{1}^T A_1 \mathbf{1} = p_1 \mathcal{S}_{\widehat{H}}$, which implies x is an eigenvector for the eigenvalue $\mathcal{S}_{\widehat{H}}$. Since we can associate a suitable vector x to the pair of components $C_0, C_j, j = 1, \ldots, r$, we can thereby construct r linearly independent eigenvectors for the eigenvalue $\mathcal{S}_{\widehat{H}}$. □

Example 7. *Consider the graph* G_1 *of the Figure 4. The hinge* $\widehat{H} = \{9, 10\}$ *has two Perron components* C *and* D, *each one with* $\mathcal{S}_C = \mathcal{S}_D = 1$. *Therefore,* $\mathcal{S}_{\widehat{H}} = 1$ *and Theorem 5.1 a) ensures* $a(G_1) \le 1$. *It turns out that* $a(G_1) = 1$ *in this example; the conditions in Theorem 5.1 b) may be verified by inspecting Figure 4.*

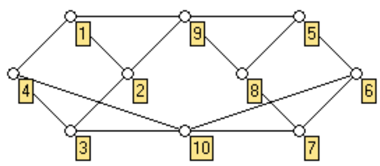

FIGURE 4. Illustration of Theorem 5.1

On the other hand, consider the graph G_2 *of the Figure 5, which also has* $\{9, 10\}$ *as a hinge. By a similar application of Theorem 5.1 a) it follows that* $a(G_2) \le 1$. *In fact,* $a(G_2) = 3 - \sqrt{6} < 1$, *while Theorem 5.1 c) ensures that* 1 *is an eigenvalue of* G_2.

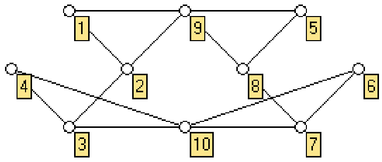

FIGURE 5. Illustration of Theorem 5.1

These examples show that conditions in Theorem 5.1 c) are not enough to have $a(G) = \mathcal{S}_{\widehat{H}}$ *and that it depends not only on the strength of a hinge, but also on the structure of the components as well. Thus, finding necessary and sufficient conditions to ensure* $a(G) = \mathcal{S}_{\widehat{H}}$ *remains an open problem.*

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