NOTES ON GRASSMANNIANS AND SCHUBERT VARIETIES

JAYDEEP V. CHIPALKATTI

These notes form an enlarged version of a series of four lectures, given in the Curves Seminar at Queen's in the winter of 2000. This is a purely expository account and except possibly in the commission of errors, no originality is in evidence. Since the general theorems of the subject are very well-documented, there are few complete proofs to be found here. Instead, I have consistently preferred to work out a specific low dimensional example to illustrate a theorem, and to let the reader do the general case. My intent has been to stay away from some of the notational clutter which must be suffered by anyone studying the Grassmann varieties.

I thank Brian Coolen, Susan Cooper and Sindi Sabourin for typesetting the original lecture notes and the audience for helping me sustain my illusions. I have used Macaulay-2 and the Maple package SF for some computations and it is my pleasure to thank their creators.

1. Introduction

Consider a line in the projective space $\mathbb{P}^3$. It is of course given to us as a set of points, but there are occassions when it is necessary to consider it atomically, i.e. as an indivisible geometric unit. For instance, we would like to make sense of the claim that as a point moves continuously on a space curve, the tangent at that point moves continuously as well. What we are looking for is a 'space' whose 'points' naturally correspond to lines in $\mathbb{P}^3$. This by stipulation is a Grassmann variety G. Now given a space curve C we can define a function $C \longrightarrow G$ which sends $x \in C$ to the 'point' corresponding to the tangent to C at x. Once this is done, we are then free to worry about whether it is continuous.

These Grassmann varieties will be discussed at length in the sequel. Our notation will follow the dictates of algebra rather than geometry, so a line as above will be thought of as a 2-dimensional vector subspace of a 4-dimensional ambient vector space; the Grassmannian will then appear as $G(2,4)$. Throughout, we will work over complex numbers, except in §8.

2. The Plücker relations

Let V be $\mathbb{C}$–vector space of dimension n; and k an integer, $0 < k < n$. The Grassmannian $G(k,V)$ or $G(k,n)$ is defined to be the set

$$\{W : W \text{ is a } k\text{-dimensional subspace of } V\}.$$

Alternately, it is the set of $(k-1)$-dimensional linear subspaces of $\mathbb{P}^{n-1}$. For $W \subseteq V$, we write $\langle W \rangle$ for the corresponding element of G.

This set has the structure of a smooth projective variety (and also that of a compact complex manifold). This we proceed to describe.

Once and for all, fix an ordered basis $\{v_1, v_2, \dots, v_n\}$ for V. For any k-dimensional $W \subseteq V$, choose an ordered basis $\mathcal{B} = \{w_1, \dots, w_k\}$. Say $w_i = \sum_{j=1}^n z_{ij} v_j$, then W is completely described by the $k \times n$ matrix

$$M = \begin{bmatrix} z_{11} \dots z_{1n} \\ \vdots \ddots \vdots \\ z_{k1} \dots z_{kn} \end{bmatrix}$$

For a sequence $I : 1 \leq i_1 < \dots < i_k \leq n$, the determinant of the maximal minor corresponding to columns in I is called the Plücker coordinate P_I. Since M has maximal rank, at least one of the coordinates is nonzero. Moreover, changing the basis of W has the effect of multiplying M on the left by a $k \times k$ nonsingular matrix, say N. Then each P_I gets multiplied by $\det(N)$, a fact which will be repeatedly used below.

Hence it makes sense to define a map $G(k,n) \xrightarrow{\pi} \mathbb{P}^{\binom{n}{k}-1} = \mathbb{P}^N$ by sending $\langle W \rangle$ to its collection of Plücker coordinates:

$$\langle W \rangle \xrightarrow{\pi} [P_{12\dots k}, \dots, P_I, \dots].$$

This of course presupposes an ordering of the indices I, which may be fixed in any way you like. By what we have said, the image is well-defined and does not depend upon $\mathcal{B}$. We will see later that this is a set-theoretic injection, but it is far from being a surjection. There are algebraic relations a point must satisfy to lie in the image, these are the famous Plücker relations. We will illustrate with the case of $G(3,6)$.

Take M to be the matrix shown above, where the entries are now thought of as *indeterminates*, and let P_I be its minors. In practice it will be convenient not to require that I be increasing. Then we have obvious relations such as $P_{213} = -P_{123}, P_{133} = 0$ etc.

Now the leftmost 3×3 minor is nonsingular (thought of as a matrix over the coefficient field $\mathbb{C}(\{z_{ij}\})$), so we multiply M on the left by the inverse of this minor and get a matrix

$$M' = \begin{bmatrix} 1 & 0 & 0 & (423) & (523) & (623) \\ 0 & 1 & 0 & (143) & (153) & (163) \\ 0 & 0 & 1 & (124) & (125) & (126) \end{bmatrix} \quad \text{where } (abc) \text{ stands for } \frac{P_{abc}}{P_{123}}. \tag{1}$$

(To see this, say the $(1,5)$ entry of M' is y. Then its $[523]$ minor is y, but it must also be $\frac{P_{523}}{P_{123}}$.)

Now the $[453]$ minor of M' equals

$$(423)(153) - (523)(143) = \frac{P_{453}}{P_{123}}$$

Clearing denominators, we get

$$\begin{aligned} P_{423}P_{153} - P_{523}P_{143} &= P_{453}P_{123}, \quad \text{i.e.} \\ P_{\underline{1}23}P_{453} &= P_{\underline{4}23}P_{\underline{1}53} + P_{\underline{5}23}P_{4\underline{1}3} \end{aligned}$$

The $[456]$ minor of M' is

$$\begin{aligned} \frac{P_{456}}{P_{123}} &= \det \begin{vmatrix} (423) & (523) & (623) \\ (143) & (153) & (163) \\ (124) & (125) & (126) \end{vmatrix} \\ &= (124) \begin{vmatrix} (523) & (623) \\ (153) & (163) \end{vmatrix} - \dots \text{etc.} \quad \text{(expanding by the third row)} \\ &= \frac{P_{124}}{P_{123}}\frac{P_{563}}{P_{123}} - \frac{P_{125}}{P_{123}}\frac{P_{463}}{P_{123}} + \frac{P_{126}}{P_{123}}\frac{P_{453}}{P_{123}} \end{aligned}$$

which amounts to

$$P_{12\underline{3}}P_{456} = P_{12\underline{4}}P_{\underline{3}56} + P_{12\underline{5}}P_{4\underline{3}6} + P_{12\underline{6}}P_{45\underline{3}} \tag{2}$$

The formulae (2) and (2) are instances of a general pattern, which we proceed to describe.

Let M be a $k \times n$ matrix of indeterminates with minors denoted P_I; and q an integer, $1 \le q \le k$. Fix two length k sequences $I, J \subseteq \{1, \dots, n\}$ and a length q sequence $R \subseteq \{1, \dots, k\}$;

$$\begin{aligned} &I : i_1 < \dots < i_k, \quad J : j_1 < \dots < j_k, \\ &R : r_1 < \dots < r_q. \end{aligned}$$

If $S \subseteq \{1, \dots, k\}$ is a length q sequence, say $S : s_1 < \dots < s_q$; let I' be obtained from I by replacing $(i_{r_1}, \dots, i_{r_q})$ with $(j_{s_1}, \dots, j_{s_q})$, and likewise J' from J by replacing $(j_{s_1}, \dots, j_{s_q})$ with $(i_{r_1}, \dots, i_{r_q})$.

Then we have the general Plücker relation

$$P_I P_J = \sum_S P_{I'} P_{J'} \tag{3}$$

where the sum is quantified over all increasing length q sequences $S \subseteq \{1, \dots, k\}$.

The replacements above are point to point, i.e. j_{s_1} replacing i_{r_1} etc. The resulting I', J' may not be increasing sequences. Now the advantage of allowing such Plücker coordinates becomes clear: there are no signs in (3).

To recapitulate, fix two sequences I, J and a subsequence of I. Then successively exchange this subsequence with each subsequence of the same length in J. E.g., in formula (2), the 3 in $I = (1\,2\,3)$ is being exchanged with each entry in $J = (4\,5\,6)$. As a more elaborate example, say $I = (2\,4\,5\,8), J = (1\,3\,6\,7)$ and $R = (1\,3)$. Then

$$P_{\underline{2}4\underline{5}8} P_{1367} = P_{\underline{1}4\underline{3}8} P_{\underline{25}67} + P_{\underline{1}4\underline{6}8} P_{\underline{2}3\underline{5}7} + \cdots + P_{\underline{6}4\underline{7}8} P_{13\underline{25}}$$

Remark 2.1. Although the details are notationally laborious, the central idea behind the proof of (3) is rather simple. Let M' be obtained by multiplying M on the left by the inverse of the $(1, \dots, k) \times I$-minor. The determinant of the $(1, \dots, k) \times J$-minor of M' is $\frac{P_I}{P_J}$. Now recalculate this determinant by Laplace expansion along the rows in R, equate the two and clear denominators. See [SS51, Ch. II, §9] for the precise statement of the Laplace expansion theorem.

A point $[P_I] \in \mathbb{P}^{\binom{n}{k}-1}$ belongs to the image of the map π iff the P_I satisfy the relations (3). The 'only if' part is clear. To see the 'if' part (for the 3×6 case !), notice that one of the P_I must be nonzero. By reordering the basis of V if necessary, we can assume it to be P_{123}. But then the P_I are Plücker coordinates of the rowspace of the matrix (1), hence the point lies in the image of π.

The same argument shows that W can be completely recovered from the Plücker coordinates, thereby proving that π is injective. This exhibits G as a projective subvariety of $\mathbb{P}^N$. Indeed much more is true, the relations in (3) generate the ideal of G in $\mathbb{P}^N$. This fact is sometimes called the 'second fundamental theorem of invariant theory.' (See [ACGH85, Ch. 2] for a proof.) This ideal will be denoted I_G.

For later use, let R be the polynomial ring $\mathbb{C}[\{P_I\}]$, i.e. the coordinate ring of $\mathbb{P}^{\binom{n}{k}-1}$. Then $S = R/I_G$ is the coordinate ring of the Plücker embedding. In more fancy terms $G = \text{Proj } S$.

Here is a quick and dirty way to calculate the dimension of G. A $(k-1)$-dimensional subspace Λ in $\mathbb{P}^{n-1}$ is specified by k points, this altogether gives $k(n-1)$ parameters. But any k general points on Λ suffice to determine it. Hence $k(k-1)$ of the parameters are superfluous, so Λ depends on $k(n-1) - k(k-1) = k(n-k)$ parameters. This is $\dim G$.

Exercise 2.2. Make this precise.

The Plücker embedding can be defined without any reference to coordinates (which is preferable on occassion). In these terms we have

$$\begin{aligned} \pi : G &\longrightarrow \mathbb{P}(\wedge^k V) \\ \langle W \rangle &\longrightarrow [\wedge^k W] \end{aligned}$$

The exterior vectors $v_I = v_{i_1} \wedge \cdots \wedge v_{i_k}$ give a basis of $\wedge^k V$, hence an arbitrary vector $\tau \in \wedge^k V$ may be written as $\tau = \sum P_I v_I$. We say that τ is *decomposable* if it lies in the image of π, i.e. if it can written as $w_1 \wedge \cdots \wedge w_k$ for some $w_j \in V$. To rephrase what we have said before, τ is decomposable iff the P_I satisfy the relations (3).

The preceding discussion brings out one fact. The group $GL(V)$ naturally acts on $\wedge^k V$, and any element of GL will carry a decomposable vector to another decomposable vector. Hence GL acts compatibly on the embedding $G \hookrightarrow \mathbb{P}(\wedge^k V)$, i.e. for any $g \in GL$ we have a commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{\pi} & \mathbb{P}(\wedge^k V) \\ {\scriptstyle g}\downarrow & & {\scriptstyle g}\downarrow \\ G & \xrightarrow{\pi} & \mathbb{P}(\wedge^k V) \end{array}$$

Hence all vector spaces such as $H^0(\mathbb{P}, \mathcal{I}_G(m))$ or $H^0(G, \mathcal{O}_G(m))$ which naturally arise in the geometry of the Plücker embedding, are in fact $GL(V)$-representations. We will attempt to describe some of these representations in §5.

Let us note that we have a natural isomorphism

$$\begin{aligned} \delta : G(k, V) &\longrightarrow G(n-k, V^*) \\ \langle W \rangle &\longrightarrow \langle \mathrm{ann}(W) \rangle \end{aligned} \tag{4}$$

This fact has several manifestations in the geometry of G, as will be amply clear in the sequel.

3. Singular homology

In this section we consider G in its classical topology unless stated otherwise. We will describe a cellular decomposition for G; along the way, we will encounter some naturally defined closed subvarieties of G which play a key role in its intersection theory.

3.1. **The Bruhat decomposition.** Fix a basis of V as before and let V_i be the subspace generated by $\{v_1, \dots, v_i\}$. This determines a *complete flag*

$$V_\bullet : 0 \subsetneq V_1 \subsetneq V_2 \subsetneq \dots \subsetneq V_n = V.$$

By way of illustration, we will settle on $G = G(3,6)$. For $\langle W \rangle \in G$, consider the following string of subspaces:

$$0 \subseteq V_1 \cap W \subseteq V_2 \cap W \subseteq \dots \subseteq V_6 \cap W (= W).$$

At any step in the ladder, the dimension of $V_j \cap W$ is at most one larger than that of $V_{j-1} \cap W$. Hence there must be exactly 3 values of j (between 1 and 6), where the inclusion is strict. Let $(\gamma_1, \gamma_2, \gamma_3)$ be this sequence; it will be called the *gap sequence*[1] Γ_W.

Let us suppose that $\Gamma_W = (2,4,5)$; we will try to construct a canonical ordered basis $\{w_1, w_2, w_3\}$ for W. Let w_1 be a nonzero vector in $V_2 \cap W$, say $w_1 = \alpha_{11} v_1 + \alpha_{12} v_2$. We can assume $\alpha_{12} \neq 0$ (since otherwise $\gamma_1 = 1$), and by dividing we set $\alpha_{12} = 1$. These requirements determine w_1 uniquely. Now choose $w_2 \in V_4 \cap W \setminus V_3 \cap W$, say $w_2 = \alpha_{21} v_1 + \alpha_{22} v_2 + \alpha_{23} v_3 + \alpha_{24} v_4$. Since $\alpha_{24} \neq 0$ (otherwise $\gamma_2 < 4$) we can assume it to be 1. Moreover, by subtracting a multiple of w_1, we can assume $\alpha_{22} = 0$. This completely determines w_2. Similarly, there is a unique $w_3 \in V_5 \cap W \setminus V_4 \cap W$ of the form $w_3 = \alpha_{31} v_1 + \alpha_{33} v_3 + v_5$. Altogether W can be uniquely represented as the rowspace of a matrix of the form

$$B_W = \begin{bmatrix} \alpha_{11} & 1 & 0 & 0 & 0 & 0 \\ \alpha_{21} & 0 & \alpha_{23} & 1 & 0 & 0 \\ \alpha_{31} & 0 & \alpha_{33} & 0 & 1 & 0 \end{bmatrix}, \tag{5}$$

henceforth called the Bruhat matrix of W. Conversely, for any choice of values α_{11}, α_{21} etc., the rowspace of this matrix will have gap sequence $(2,4,5)$. Hence the α's define a homeomorphism

$$\{\langle W \rangle : \Gamma_W = (2,4,5)\} \longleftrightarrow \mathbb{C}^5$$

[1] The terminology is not standard.

The former set, called the 'open Bruhat cell', is denoted $\Omega^o_{(245)}$. The set

$$\Omega_{(2,4,5)} := \{\langle W\rangle : \gamma_1 \leq 2, \gamma_2 \leq 4, \gamma_3 \leq 5\}$$

is the closure of $\Omega^o_{(245)}$ in G.

Exercise 3.1. For an arbitrary sequence $\underline{a} : 1 \leq a_1 < \cdots < a_k \leq n$, establish a homeomorphism

$$\Omega^o_{\underline{a}} := \{\langle W\rangle : \Gamma_W = \underline{a}\} \longleftrightarrow \mathbb{C}^{(\sum a_j - \frac{k(k+1)}{2})}$$

and show that the closure of $\Omega^o_{\underline{a}}$ (in classical or Zariski topology) equals

$$\Omega_{\underline{a}} := \{\langle W\rangle : \gamma_i \leq a_i \text{ for all } i\}.$$

The closed subvariety $\Omega_{\underline{a}}$ is called a Schubert variety.

This shows that the Grassmann variety can be written as a disjoint union of affine spaces

$$G = \coprod_{\underline{a}} \Omega^o_{\underline{a}}$$

quantified over all possible gap sequences $\underline{a}$.

We are well on our way to defining the structure of a CW complex on G. We will use the following subsets of $\mathbb{R}^q$:

$$E_q = [-1,1]^q, \;\; U_q = (-1,1)^q, \;\; B_q = E_q \setminus U_q.$$

For q between 0 and $2k(n-k)$ define the q-skeleton as the union

$$K_q = \cup_{\underline{a}} \Omega_{\underline{a}}$$

over all cells having *real* dimension $\leq q$. For each such cell, we define a characteristic map $\chi_{\underline{a}} : E_q \longrightarrow \Omega_{\underline{a}}$ such that

- $\chi_{\underline{a}}$ maps U_q homeomorphically onto $\Omega^o_{\underline{a}}$,
- the image of B_q lands in K_{q-1}.

We will describe $\chi_{(245)}$ and the general pattern will then be clear. The numbers of α's in the successive rows of B_W is $1, 2$ and 2, so we make the identification $E_{10} = E_2 \times E_4 \times E_4$. Write a typical element $z \in E_{10}$ as

$$z = (x_1^{(1)}, y_1^{(1)}; x_1^{(2)}, y_1^{(2)}, x_2^{(2)}, y_2^{(2)}; x_1^{(3)}, \dots, y_2^{(3)}).$$

For $r = 1,2,3$, let $d^{(r)}$ be the maximum of the absolute values of all $x_j^{(r)}, y_j^{(r)}$. So z lies in U_{10} iff all $d^{(r)} < 1$. Now define $\chi(z)$ to be the rowspace of

$$\begin{bmatrix} x_1^{(1)} + i\,y_1^{(1)} & 1-d^{(1)} & 0 & 0 & 0 & 0 \\ x_1^{(2)} + i\,y_1^{(2)} & 0 & x_2^{(2)} + i\,y_2^{(2)} & 1-d^{(2)} & 0 & 0 \\ x_1^{(3)} + i\,y_1^{(3)} & 0 & x_2^{(3)} + i\,y_2^{(3)} & 0 & 1-d^{(3)} & 0 \end{bmatrix}.$$

The reader may verify that it has the correct properties and write a general expression for $\chi_{\underline{a}}$, if he so desires. (See [LW69] in case of difficulty.) This defines the structure of a CW complex on G.

The term 'open cell' follows the usage in the theory of CW complexes, but it is almost unnecessary to remark that $\Omega^o_{\underline{a}}$ is in general not open in G.

Exercise 3.2. Show that the sequences $\underline{a}$ such that $\dim \Omega_{\underline{a}} = r$ (as a complex variety) are in bijection with the set of partitions of r into at most k parts such that no part exceeds $n-k$.

(Hint: Pass to the sequence $(a_k - k, \dots, a_i - i, \dots, a_1 - 1)$.) The number of such partitions will be denoted $\text{part}(r; k, n-k)$.

Since the CW complex has cells only in even dimension, all chain maps are zero. Hence for $0 \le q \le 2k(n-k)$,

$$H_q(G, \mathbb{Z}) = H_q(K_q, K_{q-1}) = \begin{cases} 0 & \text{for } q \text{ odd,} \\ \mathbb{Z}^{\text{part}(q/2;\, k, n-k)} & \text{for } q \text{ even.} \end{cases} \tag{6}$$

Then $H^q(G, \mathbb{Z}) \simeq \text{Hom}(H_q(G, \mathbb{Z}), \mathbb{Z})$. There is an alternate indexing convention for Schubert varieties. We have

$$\text{codim}\,(\Omega_{\underline{a}}, G) = k(n-k) - (\sum_{j=1}^{k} a_j - j) = \sum_{j=1}^{k} (n-k+j-a_j).$$

Hence define a sequence λ by letting $\lambda_j = n-k+j-a_j$ for $1 \le j \le k$. Then we write X_λ for $\Omega_{\underline{a}}$, and $|\lambda| = \sum \lambda_j$. Hence

$$X_\lambda = \{W : \dim\,(W \cap V_{n-k+j-\lambda_j}) \ge j \;\text{ for } 1 \le j \le k.\} \tag{7}$$

The admissible sequences λ are those with

$$n-k \ge \lambda_1 \ge \cdots \ge \lambda_k \ge 0. \tag{8}$$

3.2. **The dual Bruhat cells.** It is interesting to trace this cell decomposition through the duality map δ (on page 5).

Let $U_i = \text{ann}(V_{n-i})$, then we have the dual complete flag

$$U_\bullet : 0 \subsetneq U_1 \subsetneq U_2 \subsetneq \ldots \subsetneq U_n = V^*.$$

Now

$$\begin{aligned} &W \cap V_{r-1} \subsetneq W \cap V_r \Rightarrow \operatorname{ann}(W \cap V_{r-1}) \supsetneq \operatorname{ann}(W \cap V_r) \\ &\Rightarrow \operatorname{ann}(W) + \operatorname{ann}(V_{r-1}) \supsetneq \operatorname{ann}(W) + \operatorname{ann}(V_r). \end{aligned}$$

Taking dimensions, this implies

$$\begin{aligned} &\dim(\operatorname{ann}(W) + \operatorname{ann}(V_{r-1})) > \dim(\operatorname{ann}(W) + \operatorname{ann}(V_r)) \\ \Rightarrow &\dim\operatorname{ann}(W) + \dim\operatorname{ann}(V_{r-1}) - \dim(\operatorname{ann}(W) \cap \operatorname{ann}(V_{r-1})) \\ > &\dim\operatorname{ann}(W) + \dim\operatorname{ann}(V_r) - \dim(\operatorname{ann}(W) \cap \operatorname{ann}(V_r)), \\ \Rightarrow &\dim(\operatorname{ann}(W) \cap \operatorname{ann}(V_r)) + 1 > \dim(\operatorname{ann}(W) \cap \operatorname{ann}(V_{r-1})). \end{aligned}$$

Now the dimension of $\operatorname{ann}(W) \cap \operatorname{ann}(V_r)$ could be at most one less than that of $\operatorname{ann}(W) \cap \operatorname{ann}(V_{r-1})$, so this relation forces them to be equal. We have proved that if $\gamma \in \Gamma_W$, then $n-\gamma+1 \notin \Gamma_{\operatorname{ann}(W)}$. Since $\Gamma_{\operatorname{ann}(W)}$ contains $n-k$ elements, it now follows that if $\gamma \notin \Gamma_W$ then $n-\gamma+1 \in \Gamma_{\operatorname{ann}(W)}$. Hence the sequences Γ_W and $\Gamma_{\operatorname{ann}(W)}$ completely determine each other, and the map δ takes an open Bruhat cell isomorphically to an open Bruhat cell.

Exercise 3.3. Let X_μ denote the image $\delta(X_\lambda) \subseteq G(n-k, V^*)$. Determine the relationship between the sequences λ and μ.

(Answer: $\mu_i = \#\{\lambda_j : \lambda_j \geq i\}$.) One says that μ is the conjugate of λ, written $\mu = \lambda'$.

4. Standard Monomials

In this section we will write $[I]$ for P_I. In contradistinction to §2, all such I will be assumed to be increasing sequences.

Let $S = R/I_G$ denote the coordinate ring of $G(k,n)$ in the Plücker embedding. The standard monomials (first introduced by W.V.D. Hodge) are a natural vector space basis for S. The definition is as follows:

Put a partial order on the Plücker coordinates by declaring

$$[i_1, \ldots, i_k] \prec [j_1, \ldots, j_k] \quad \text{if } i_r \leq j_r \text{ for all } r.$$

A monomial $[I_1] \ldots [I_m]$ is called standard if any two I's are comparable in the partial order. (In which case we may assume $[I_1] \prec \cdots \prec [I_m]$.) We have made two claims:

- The standard monomials span S as a vector space, and
- they are linearly independent modulo I_G.

The second claim is the more serious one, and we say nothing about it here; except referring the reader to [ACGH85, Ch.2].

To establish the first claim is to show that a nonstandard monomial can be rewritten as a sum of standard ones using the Plücker relations (3). We will put a lexicographic order on the coordinates by declaring that

$$[i_1, \dots, i_k] \leq [j_1, \dots, j_k] \quad \text{if we have}$$
$$i_1 = j_1, \dots, i_{r-1} = j_{r-1} \text{ and } i_r \leq j_r \text{ for some } r.$$

Note that $I \prec J$ implies $I \leq J$. We extend this to a 'deglex' order on all monomials in the variables $[I]$.

Now we claim that a nonstandard monomial can be rewritten (modulo I_G) as a sum of monomials that strictly precede it in this order. Since this cannot continue indefinitely, we will eventually end up with a sum of standard monomials. Clearly it is enough to show this for a degree 2 monomial. We give an example of this reduction leaving the general proof to the reader.

Example 4.1. The monomial $[1\,3\,4\,7\,8][2\,3\,5\,6\,9]$ is nonstandard, and the violation occurs at $7 > 6$. Now set

$$I : (1\,3\,4\,7\,8), J : (2\,3\,5\,6\,9) \text{ and } R = (4\,5)$$

and use the relation (3) to rewrite the monomial as $\sum \pm[I'][J']$. (We agree to rewrite I', J' in increasing order, this would introduce signs.) We are successively exchanging the pair $(7\,8) \subseteq I$ with every pair in J, hence $I' \leq I$ holds for every summand.

Exercise 4.2. Let $[i_1 \dots i_k][j_1 \dots j_k]$ be a nonstandard monomial. Assume without loss of generality that $i_1 \leq j_1$. Let l be the least integer such that $i_l > j_l$. Show that if

$$I : (i_1 \dots i_k), J : (j_1 \dots j_k) \text{ and } R = (l, \dots, k),$$

then (3) rewrites the monomial in terms of smaller ones.

This is an instance of a 'straightening law', a theme which is ubiquitous in group representation theory.

4.1. **The ideal of the Schubert variety.** A slight refinement of this idea can be used to give an additive basis for the coordinate ring of the Schubert variety. We begin by describing the ideal $I_{\Omega_{\underline{a}}} < S$. The key point is that if $\langle W \rangle$ lies in $\Omega^o_{\underline{a}}$, then some of its Plücker coordinates are forced to be zero. These can be read off from the structure of the Bruhat matrix.

So let B_W be the Bruhat matrix of $\langle W\rangle \in \Omega^o_{\underline{a}}$. From the picture (5) on page 6 or otherwise, it is clear that for any column in B_W strictly after the a_i th, the rows $1, \dots, i$ are zero. Now consider a coordinate $[I] = [i_1, \dots, i_k]$.

Suppose $i_s > a_s$ for some $1 \le s \le n$, and consider the submatrix $[i_s, i_{s+1}, \dots, i_k]$. It has $k-s+1$ columns and at most $k-s$ nonzero rows. Hence the columns have to be linearly dependent, and the coordinate itself has to vanish.

Exercise 4.3. Show that if $i_s \le a_s$ for all s, then $[i_1, \dots, i_k]$ does not vanish at a general point of $\Omega^o_{\underline{a}}$.

Now we declare a monomial $[I_1] \dots [I_m]$ to be $\underline{a}$–standard if

- any two I's are comparable in the partial order '$\prec$', and
- each coordinate satisfies $[I_r] \prec [a_1, \dots, a_k]$.

If $\underline{a} = (n-k+1, \dots, n)$ (so $\Omega_{\underline{a}} = G$), then the latter condition is vacuous. Hence a standard monomial is $(n-k+1, \dots, n)$–standard.

Now we have the following big theorem:

Theorem 4.4. *The ideal $I(\Omega_{\underline{a}}) < S$ is generated by the $\underline{a}$-nonstandard Plücker coordinates. The classes of the $\underline{a}$-standard monomials make an additive basis for the coordinate ring $S/I(\Omega_{\underline{a}})$.*

See [Mus72] for the theory.

5. Sheaf cohomology

We start by defining some natural vector bundles on G. The tautological bundle S has rank k, its fibre over a point $\langle W\rangle$ is the vector space W (hence the name). The universal quotient bundle Q has fibre V/W over $\langle W\rangle$. The short exact sequence

$$0 \longrightarrow W \longrightarrow V \longrightarrow V/W \longrightarrow 0 \tag{9}$$

globalizes to

$$0 \longrightarrow S \longrightarrow V \otimes \mathcal{O}_G \longrightarrow Q \longrightarrow 0. \tag{10}$$

The last is called the Euler sequence, although Euler would have had some difficulty recognizing it.

Now the dual of (9) is

$$\begin{array}{ccccccccc} 0 \longrightarrow & (V/W)^* & \longrightarrow & V^* & \longrightarrow & W^* & \longrightarrow 0 \\ & || & & || & & || & \\ 0 \longrightarrow & \operatorname{ann}(W) & \longrightarrow & V^* & \longrightarrow & V^*/\operatorname{ann}(W) & \longrightarrow 0 \end{array}$$

so under the duality (4), S^* and Q correspond.

The hyperplane bundle $\mathcal{O}_G(1)$ is by definition $\wedge^{n-k}Q(=\wedge^k S^*)$. It equals the restriction of $\mathcal{O}_{\mathbb{P}^N}(1)$ to G.

We are interested in the cohomology of these bundles and some of their concoctions. It is clear that any such cohomology group will carry the structure of a $GL(V)$–representation. These will turn out to be the Schur-Weyl modules. Although we will not describe their construction here, we will state enough properties to get things off the ground.

5.1. **The Schur-Weyl modules.** The group $GL(V)$ is reductive, so any finite dimensional complex representation can be uniquely written as a direct sum of irreducible representations (exactly as in the case of a finite group). Hence we would like to know the irreducible representations.

By an index λ we will mean a nonincreasing sequence of integers $\lambda : \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_q$. We will not distinguish between indices if they differ only by trailing zeros, e.g. $(6,3,1,0,0)$ and $(6,3,1)$. We write $|\lambda| = \sum \lambda_i$ and $\lambda \geq 0$ if all λ_i are nonnegative. If $\lambda \geq 0$ holds, then the conjugate index λ' is determined by the recipe of exercise 3.3.

The Schur-Weyl module $\mathbb{S}_\lambda V$ (which we will not define) is a certain $\mathbb{C}$-vector space functorially associated to V. The pleasant fact is that $\mathbb{S}_\lambda V$ is an irreducible $GL(V)$-representation, and every irreducible representation arises from precisely one index λ. We list some properties of this construction to orient the reader.

(1) If λ has more nonzero entries than $\dim V(=n)$, then (and only then) $\mathbb{S}_\lambda V = 0$.

(2) If $\lambda = (r)$ with $r \geq 0$ (resp. $(\overbrace{1,\dots,1}^{r \text{ times}},0,\dots,0)$), then $\mathbb{S}_\lambda V = \operatorname{Sym}^r V$ (resp. $\wedge^r V$). If $\lambda = (\underbrace{m,\dots,m}_{n \text{ times}})$, then $\mathbb{S}_\lambda V = (\wedge^n V)^{\otimes m}$.

These are some of the cases where the Weyl module is easily described, in general it is a mysterious amalgam of symmetric and exterior powers.

(3) If we let $\lambda^* = (-\lambda_q,\dots,-\lambda_1)$, then there are isomorphisms of GL-representations

$$(\mathbb{S}_\lambda V)^* \simeq_{GL} \mathbb{S}_\lambda(V^*) \simeq_{GL} \mathbb{S}_{\lambda^*} V$$

(4) We have the dimension formula

$$\dim \mathbb{S}_\lambda V = \prod_{1 \leq i < j \leq n} \frac{\lambda_i - \lambda_j + j - i}{j - i} \tag{11}$$

(5) If we let $\gamma = (\lambda_1+m, \lambda_2+m, \ldots, \lambda_k+m)$ for an integer m, then $\mathbb{S}_\gamma V \simeq_{GL} \mathbb{S}_\lambda V \otimes (\wedge^n V)^m$. Hence $\mathbb{S}_\gamma V$ and $\mathbb{S}_\lambda V$ are isomorphic qua $SL(V)$-representations.
(6) The compound representation $\mathbb{S}_\lambda(\mathbb{S}_\mu V)$ can be decomposed as a direct sum $\oplus S_\nu V$. The *plethysm problem* consists in determining which ν's appear in the sum and with what multiplicity. We will come across one such explicit formula later in §5.4.
(7) For two vector spaces V and W, we have isomorphisms of $GL(V) \times GL(W)$ representations

$$\wedge^r(V\otimes W) \simeq \oplus\, \mathbb{S}_\alpha V \otimes \mathbb{S}_{\alpha'} W, \quad \mathrm{Sym}^r\,(V\otimes W) \simeq \oplus\, \mathbb{S}_\alpha V \otimes \mathbb{S}_\alpha W \quad (12)$$

both sums over $\alpha \geq 0, |\alpha| = r$. (By property 1, the sum is finite.)

The construction of Schur-Weyl modules is functorial in V, hence given a vector bundle E there is an associated bundle $\mathbb{S}_\lambda E$.

Exercise 5.1. Verify that (11) gives the correct formula for the dimension of a symmetric or exterior power.

5.2. **The vector bundles $\mathbb{S}_\alpha Q \otimes \mathbb{S}_\beta S$.** Let $\langle W \rangle \in G(k, V)$. The tangent space to G at $\langle W \rangle$ is of course a $k(n-k)$ dimensional space. We would like to describe it intrinsically. The group $GL(V)$ acts transitively on G. Let H_W denote the stabilizer of the point $\langle W \rangle$. The tangent space to $GL(V)$ at the identity is $\mathrm{End}(V)$ and that of H_W is the subspace

$$U_W = \{\varphi \in \mathrm{End}(V) : \varphi(W) \subseteq W\}.$$

The tangent space to G at $\langle W \rangle$ is given by the quotient $\mathrm{End}(V)/U_W$. Define a map $\mathrm{End}(V)/U_W \longrightarrow \mathrm{Hom}(W, V/W)$ by sending $\varphi + U_W$ to the composite

$$W \xrightarrow{\varphi|_W} V \longrightarrow V/W.$$

It is immediate that this is well-defined and injective. Then it is surjective for dimensional reasons.

Hence we have a canonical identification $T_{G,\langle W \rangle} = \mathrm{Hom}(W, V/W)$. Globally we have an isomorphism of vector bundles $T_G = \mathrm{Hom}(S, Q) = S^* \otimes Q$. Then the cotangent bundle $\Omega^1_G = S \otimes Q^*$. By (12) above,

$$\Omega^r = \bigoplus_{\substack{|\alpha|=r \\ \alpha \geq 0}} \mathbb{S}_\alpha S \otimes \mathbb{S}_{\alpha'} Q^* \quad (13)$$

If $r = k(n-k)$, then the only α making a contribution would be $(n-k, \dots, n-k)$. Hence the canonical bundle $\omega_G = (\wedge^k S)^{n-k} \otimes (\wedge^{n-k} Q^*)^k = \mathcal{O}_G(-n)$.

If we set $\lambda = (\overbrace{m, \dots, m}^{n-k \text{ times}})$, then $\mathbb{S}_\lambda Q = \mathcal{O}_G(m)$. The global sections of this line bundle are the equations of degree m hypersurfaces in $\mathbb{P}^N$ modulo those vanishing on G.

It is clear at this point, that many geometrically natural vector bundles on G are of the form $E_{\alpha,\beta} = \mathbb{S}_\alpha Q \otimes \mathbb{S}_\beta S$, and a general mechanism to calculate their cohomology would be quite valuable. This is given by the Borel-Weil-Bott theorem. In order to state it, some auxiliary definitions will be needed.

For an arbitrary sequence of integers $\lambda := (\lambda_1, \dots, \lambda_n)$, define

$$\lambda^{(1)} := (\lambda_1 + n, \dots, \lambda_j + n - j + 1, \dots, \lambda_n + 1),$$

and let l_λ be the cardinality of the set

$$\{(i,j) : 1 \leq i < j \leq n \text{ and } \lambda_i^{(1)} < \lambda_j^{(1)}\}.$$

This is also the number of *adjacent* transpositions necessary to rearrange the sequence $\lambda^{(1)}$ into nonincreasing order. Let $\lambda^{(2)}$ be the rearranged sequence and finally let

$$\lambda^\sharp = (\lambda_1^{(2)} - n, \dots, \lambda_j^{(2)} - (n-j+1), \dots, \lambda_n^{(2)} - 1).$$

Theorem 5.2 (Borel–Weil–Bott). *Let $\alpha : \alpha_1 \geq \cdots \geq \alpha_{n-k}$, $\beta : \beta_1 \geq \cdots \geq \beta_k$ be indices, and set $E_{\alpha,\beta} = \mathbb{S}_\alpha Q \otimes \mathbb{S}_\beta S$. Define λ to be $(\alpha_1, \dots, \alpha_{n-k}; \beta_1, \dots, \beta_k)$. Then*

$$H^q(G, E_{\alpha,\beta}) \simeq_{SL} \begin{cases} \mathbb{S}_{\lambda^\sharp} V & \text{if } q = l_\lambda \text{ and } \lambda^\sharp \text{ is nonincreasing,} \\ 0 & \text{otherwise.} \end{cases}$$

Examples 5.3.
(1) Let $G = G(2,5), \alpha = (3,0,-2)$ and $\beta = (5,1)$. Then $\lambda = (3,0,-2,5,1), \lambda^{(1)} = (8,4,1,7,2)$, so $l_\lambda = 3$. After the transpositions

$$(8,4,1,7,2) \to (8,4,7,1,2) \to (8,7,4,1,2) \to (8,7,4,2,1) (= \lambda^{(2)}),$$

we have $\lambda^\sharp = (3,3,1,0,0)$. Hence $H^3(G, E_{\alpha,\beta}) = \mathbb{S}_{(3,3,1)} V$ is the only nonzero cohomology group.

(2) Let $G = G(2,5), \alpha = (3,2,1)$ and $\beta = (4,0)$. Then $\lambda^{(1)} = (8,6,4,6,1) \to (8,6,6,4,1) = \lambda^{(2)}$. But now $\lambda^\sharp = (3,2,3,2,0)$ which is not nonincreasing. So $E_{\alpha,\beta}$ has no cohomology in any dimension.

Exercise 5.4. Show that $E_{\alpha,\beta}$ has no cohomology in any dimension iff $\lambda_i - i = \lambda_j - j$ for some $i \neq j$ iff $\lambda^{(1)}$ contains two equal entries. If this holds, we will say that the bundle is nilcyclic.

Remark 5.5. We stress that the isomorphism in the theorem is that of SL-representations. E.g. consider the trivial bundle $\mathcal{O}_G$ written as $\wedge^k S \otimes \wedge^{n-k} Q$. The theorem gives $H^0(G, \mathcal{O}_G) \simeq \wedge^n V$, which is indeed trivial as an $SL(V)$-module.

5.3. **The Hilbert polynomial of G.** Now it is immediate that if $m > 0$, then

$$H^0(G, \mathcal{O}_G(m)) \simeq_{SL} \mathbb{S}_{\lambda_m} V^*$$

where $\lambda_m = (\underbrace{m, \dots, m}_{k \text{ times}}, 0, \dots, 0)$. By formula (11), the Hilbert polynomial of G in the Plücker embedding is given by

$$P(G, m) = \dim H^0(G, \mathcal{O}_G(m)) = \prod \frac{m + j - i}{j - i} \tag{14}$$

the product over all $1 \leq i \leq k$ and $k + 1 \leq j \leq n$. Its degree in m is $k(n-k)$, as it should be.

Exercise 5.6. Calculate the leading term and prove the formula

$$\deg G = \dim G\,! \prod_{i=1}^{k} \frac{(k-i)!}{(n-i)!}$$

We have a short exact sequence

$$0 \longrightarrow \mathcal{I}_G(m) \longrightarrow \mathcal{O}_{\mathbb{P}^N}(m) \longrightarrow \mathcal{O}_G(m) \longrightarrow 0.$$

Take cohomology and consider the piece

$$\begin{array}{ccc} H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(m)) & \xrightarrow{\rho_m} & H^0(G, \mathcal{O}_G(m)) \\ \| & & \| \\ \mathrm{Sym}^m(\wedge^k V^*) & \longrightarrow & \mathbb{S}_{\lambda_m} V^* \end{array}$$

The point is that ρ_m is $SL(V)$-equivariant and its target is an irreducible representation. So by Schur's lemma, the map is either zero or surjective. Since not every degree m hypersurface can contain G, it cannot be zero. We have proved that G is projectively normal in the Plücker embedding.

It is true more generally that $\Omega_{\underline{a}}$ is projectively normal. In fact we have an isomorphism

$$\begin{aligned} H^0(G, \Omega_{\underline{a}}(m)) &\simeq (S/I_{\Omega_{\underline{a}}})_m \\ &\simeq \text{vector space generated by } \underline{a}\text{-standard monomials of degree } m. \end{aligned} \tag{15}$$

Hence the computation of the Hilbert polynomial of $\Omega_{\underline{a}}$ reduces to the combinatorial problem of counting such monomials. This is done by a very delicate descending induction on $\underline{a}$, and the answer can be represented as the determinant of the matrix

$$(i,j) \longrightarrow \binom{m+i-j+a_{k-i+1}-1}{m+i-j}, \quad 1 \leq i,j \leq k. \tag{16}$$

See [HP47, vol. II].

From this expression it is possible to extract a horrifyingly complicated expression for the degree of $\Omega_{\underline{a}}$. I do not know if it can be simplified any further.

5.4. **The Plücker relations (bis).** The relations (3) on page 4 span $H^0(\mathbb{P}^N, \mathcal{I}_G(2))$ and we can ask for a representation theoretic description of this vector space.

Now by the plethysm formula on [Mac95, Ch I,§8, Example 9]

$$\mathrm{Sym}^2(\wedge^k V^*) = \oplus\, \mathbb{S}_{\lambda'} V^*, \tag{17}$$

the sum over all $\lambda = (k+r, k-r)$, with r even and $0 \leq r \leq k$. The image of ρ_2 is given by the summand corresponding to $r=0$, and $\ker \rho_2$ by the remaining summands. For instance, say for $G(5,11)$ we have

$$\ker \rho_2 = H^0(\mathbb{P}^N, \mathcal{I}_G(2)) = \mathbb{S}_{(7,3)'} V^* \oplus \mathbb{S}_{(9,1)'} V^*.$$

See [FH91, Exercise 15.44] for a geometric interpretation of this decomposition.

We will calculate the Hodge numbers of the projective space using the B-W-B theorem. With the identification $G(1,V) = \mathbb{P}V$, we have $S = \mathcal{O}_{\mathbb{P}}(-1)$ and $Q^* = \Omega^1_{\mathbb{P}}(1)$. Then $\Omega^p_{\mathbb{P}} = S^{\otimes p} \otimes \wedge^p Q^*$. By the recipe of the theorem, we set

$$\lambda = (\underbrace{0, \dots, 0}_{n-p-1 \text{ times}}, \underbrace{-1, \dots, -1}_{p \text{ times}}, p).$$

Then

$$\begin{aligned} &\lambda^{(1)} = (n, n-1, \dots, p+2, p, p-1, \dots, 1, p+1) \rightsquigarrow \\ &\lambda^{(2)} = (n, n-1, \dots, 1) \quad \text{after } p \text{ adjacent transpositions.} \end{aligned}$$

Hence $l_\lambda = p$ and $\lambda_\sharp = 0$. Thus $H^q(\mathbb{P}, \Omega^p) = \mathbb{C}$ if $p = q$ and zero otherwise.

Exercise 5.7. Prove that

$$\dim H^q(G, \Omega^p_G) = \begin{cases} \operatorname{part}(q; k, n-k) & \text{if } p = q, \\ 0 & \text{otherwise.} \end{cases}$$

You will need formula (13) on page 13 and the following lemma:
Given indices

$$\alpha : k \geq \alpha_1 \geq \cdots \geq \alpha_{n-k} \geq 0 \quad \text{and} \quad \beta : n-k \geq \beta_1 \geq \cdots \geq \beta_k \geq 0,$$

$$H^q(G, E_{\alpha,\beta}) \simeq_{SL} \begin{cases} \mathbb{C} & \text{if } \alpha = \beta' \text{ and } q = |\alpha|, \\ 0 & \text{otherwise.} \end{cases}$$

Exercise 5.8. Prove the lemma.

As another application of the theorem, we will prove Serre duality for the bundle $E_{\alpha,\beta}$ on $G(4,9)$. So set $\lambda = (\alpha_1, \dots, \alpha_5; \beta_1, \dots, \beta_4)$, the bundle $E^* \otimes \omega_G$ then corresponds to the sequence

$$\mu = (-\alpha_5, \dots, -\alpha_1; 9 - \beta_4, \dots, 9 - \beta_1).$$

We need to compare l_λ and l_μ.

For a pair of indices (i,j), we define a related pair (r,s) by $r = 6 - i, s = 15 - j$. Then as i (resp. j) takes values in $\{1, \dots, 5\}$ (resp. $\{6, \dots, 9\}$), so does r (resp. s).

Note that in calculating l_λ or l_μ, we need only consider pairs in $\{1, \dots, 5\} \times \{6, \dots, 9\}$. But for any such pair (i,j),

$$\alpha_i - i \leq \beta_{j-5} - j \iff -\alpha_i + i \geq -\beta_{j-5} + j \iff$$
$$-\alpha_{6-r} + 6 - r \geq -\beta_{10-s} + 15 - s \iff -\alpha_{6-r} - r \geq 9 - \beta_{10-s} - s.$$

This proves several things:

- Equality holds in the first expression iff it holds in the last. So E is nilcyclic iff $E^* \otimes \omega_G$ is.
- If neither is nilcyclic, then the pair (i,j) contributes to l_λ if and only if the pair (r,s) does not contribute to l_μ. Since there are 20 such pairs, $l_\lambda = 20 - l_\mu$. Moreover as *sets*

$$\lambda^{(1)} = \{\alpha_i + 10 - i : 1 \leq i \leq 5\} \cup \{\beta_{j-5} + 10 - j : 6 \leq j \leq 9\}$$
$$\mu^{(1)} = \{-\alpha_i + i + 4 : 1 \leq i \leq 5\} \cup \{-\beta_{j-5} + j + 4 : 6 \leq j \leq 9\}$$

So $\mu^{(1)}$ is obtained by negating each entry of $\lambda^{(1)}$ and adding 14. Hence $\lambda_\sharp^* = \mu_\sharp + (14, \dots, 14)$ and thus

$$H^{l_\lambda}(G, E)^* \simeq H^{20-l_\lambda}(G, E^* \otimes \omega).$$

Exercise 5.9. Obvious.

Recall that a vector bundle E on a variety X is ample if the following holds: For any coherent $\mathcal{O}_X$-module $\mathcal{F}$, the module $\mathcal{F} \otimes \operatorname{Sym}^m E$ is generated by global sections for sufficiently large m.

You may find formula (12) on page 13 useful for the next exercise.

Exercise 5.10.

1. Prove that the tangent bundle of a projective space is ample. Since $\mathcal{F}$ can be written as a quotient of a direct sum of line bundles, it is enough to prove the claim for $\mathcal{F} = \mathcal{O}_{\mathbb{P}}(r)$. Do this by a direct calculation and interpreting the map $H^0(\mathbb{P}, \operatorname{Sym}^m T_{\mathbb{P}} \otimes \mathcal{O}_{\mathbb{P}}(r)) \longrightarrow \operatorname{Sym}^m T_{\mathbb{P}} \otimes \mathcal{O}_{\mathbb{P}}(r)$ at the level of stalks.
2. Now assume $1 < k < n-1$ and exhibit an integer r such that $\operatorname{Sym}^m Q \otimes \operatorname{Sym}^m S^* \otimes \mathcal{O}_G(r)$ has no global sections even for large m. Conclude that the tangent bundle of $G(k, n)$ is not ample.

Also see [Mor79].

6. Intersection Theory

We have seen that $H_q(G, \mathbb{Z})$ is a free abelian group on the Schubert cycles of real dimension q. Let $[G]$ denote the canonical generator of $H_{2k(n-k)}(G, \mathbb{Z}) \simeq \mathbb{Z}$. (Recall that a complex manifold has a canonical orientation.) Then Poincaré duality gives an isomorphism

$$H^q(G, \mathbb{Z}) \longrightarrow H_{2k(n-k)-q}(G, \mathbb{Z}), \quad x \longrightarrow x \cap [G].$$

A cycle X_λ having *complex* codimension r corresponds to a class $[X_\lambda] \in H_{2k(n-k)-2r}$, and we let s_λ denote its preimage in $H^{2r}(G, \mathbb{Z})$ under this isomorphism. Then $H^{2r}(G, \mathbb{Z})$ is a free abelian group on the s_λ with $|\lambda| = r$, which satisfy the inequalities of (8) on page 8. The classical 'Schubert Calculus [2]' is a complete description of the multiplicative structure of the cohomology ring $H^\bullet(G, \mathbb{Z})$.

[2] According to De Morgan, 'A *calculus*, or *science of calculation*, in the modern sense, is one which has organized processes by which passage is made, or may be made, mechanically, from one result to another. A *calculus* always contains something which it would be *possible* to do by machinery.'

Remark 6.1. We have a natural map

$$H^\bullet(G, \mathbb{Z}) \longrightarrow A^\bullet(G)$$

defined by sending s_λ to the rational equivalence class of X_λ. (See [Har77, Appendix A] for the definition and first properties of $A^\bullet$.) This is an isomorphism of graded rings. This entails that

- the rational equivalence class of X_λ is independent of the flag chosen,
- all singular cohomology of G comes from algebraic cycles.

Thus topological and algebraic intersection theories on G are the 'same'. For an algebraic geometer, Schubert calculus is perhaps more naturally thought of as calculation in the ring $A^\bullet(G)$.

In any event, given two cycles s_λ, s_μ, their product must be a formal sum of cycles of codimension $|\lambda| + |\mu|$. Hence

$$s_\lambda s_\mu = \sum_\nu N_{\lambda\mu\nu}\, s_\nu$$

for some integers $N_{\lambda\mu\nu}$. Here the sum is quantified over ν satisfying

$$|\nu| = |\lambda| + |\mu|, \quad n-k \geq \nu_1 \geq \cdots \geq \nu_k \geq 0.$$

The coefficients $N_{\lambda\mu\nu}$ all turn out to be nonnegative. They are governed by a rather strange combinatorial rule, which we proceed to describe.

To forestall any confusion, note that for *any* three partitions λ, μ, ν satisfying $|\nu| = |\lambda| + |\mu|$, the Littlewood–Richardson coefficient $N_{\lambda\mu\nu}$ is defined. The numerical restrictions in (8) on 8 become relevant only while doing Schubert Calculus. In order to state the rule, some preliminaries will be needed.

6.1. **The Littlewood–Richardson Rule.** A *lattice word* (say on the alphabet $\mathbb{N}_+$) is a finite sequence of positive integers $w_1 w_2 \ldots w_m$ such that in any subsequence $w_1 w_2 \ldots w_p$ $(p \leq m)$, the number of occurrences of 1 is at least the number of occurrences of 2, which in turn is at least that of 3 and so on. E.g., (1 1 2 3 2 1 1 2 4 3) is a lattice word, but (1 2 1 3 4 2 2) is not.

Recall the notion of the Young diagram of a partition, e.g. the diagram of $(4, 3, 1)$ is

$$\begin{matrix} * & * & * & * \\ * & * & * & \\ * & & & \end{matrix}$$

Represent λ by its Young diagram and replace all the stars in the jth row of (the Young diagram of) μ with the number j. Now enlarge the diagram of λ by the following procedure, starting from the first row of μ, then moving on to the second etc.

Attach all the entries in the j-th row of μ in such a way that

(1) the result retains the shape of a Young diagram,
(2) no number appears twice in the same column.

The final result is called a μ-expansion of λ.

We have shown a possible $(3,1,1)$-expansion of $(2,1)$.

$$\lambda : \begin{array}{cc} * & * \\ * & \end{array} \quad \mu : \begin{array}{ccc} 1 & 1 & 1 \\ 2 & & \\ 3 & & \end{array} \quad \Rightarrow \begin{array}{cccc} * & * & 1 & 1 \\ * & & & \\ 1 & & & \end{array} \quad \Rightarrow \begin{array}{cccc} * & * & 1 & 1 \\ * & 2 & & \\ 1 & & & \end{array} \quad \Rightarrow \begin{array}{cccc} * & * & 1 & 1 \\ * & 2 & & \\ 1 & 3 & & \end{array}$$

$$\not\Rightarrow \begin{array}{cccc} * & * & 1 & 1 \\ * & 1 & & \\ 3 & 2 & & \end{array} \quad \Bigg| \quad \not\Rightarrow \begin{array}{ccc} * & * & 1 \\ * & 1 & 1 \\ 2 & 3 & \end{array}$$

These are not possible expansions. In the first, the result was no longer a Young diagram when the 2 was attached. In the second, two 1's appear in the same column.

Now construct the *word* of the expansion by listing all the newly added entries in the right to left, top to bottom order. The expansion is called *strict* if this a lattice word. The expansion in the top diagram above is strict, since its word is $(1\,1\,2\,3\,1)$.

Finally $N_{\lambda\mu\nu}$ is the number of strict μ-expansions of λ which give the diagram ν.

Example 6.2. Let $\lambda = (2,1), \mu = (3,1,1)$ and $\nu = (4,2,2)$. We know that that $N_{\lambda\mu\nu}$ is at least one. The reader should check that there is no other strict expansion which would give $(4,2,2)$, hence $N_{\lambda\mu\nu} = 1$.

Example 6.3. Say $\lambda = (3,2,1), \mu = (2,1,1)$ and $\nu = (4,3,2,1)$. We will calculate $N_{\lambda\mu\nu}$ and $N_{\mu\lambda\nu}$.

$$\lambda : \begin{array}{ccc} * & * & * \\ * & * & \\ * & & \end{array} \qquad \mu : \begin{array}{cc} 1 & 1 \\ 2 & \\ 3 & \end{array} \qquad \Rightarrow \quad \begin{array}{cccc} * & * & * & 1 \\ * & * & x & \\ * & y & & \\ z & & & \end{array}$$

The topmost 1 is forced and now x, y, z are the numbers $1, 2, 3$ in some order. There are two constraints: firstly the expansion must be legal and secondly $(1\,x\,y\,z)$ must be a lattice word. The only possibilities for (x, y, z) are $(1, 2, 3), (2, 1, 3)$ or $(2, 3, 1)$, so $N_{\lambda\mu\nu} = 3$.

To calculate $N_{\mu\lambda\nu}$,

$$\mu : \begin{matrix} * & * \\ * & \\ * & \end{matrix} \qquad \lambda : \begin{matrix} 1 & 1 & 1 \\ 2 & 2 & \\ 3 & & \end{matrix} \qquad \Rightarrow \qquad \begin{matrix} * & * & 1 & 1 \\ * & x & 2 & \\ * & y & & \\ z & & & \end{matrix}$$

Necessarily (x, y, z) is one of $(1, 2, 3), (1, 3, 2)$ or $(2, 3, 1)$, so $N_{\mu\lambda\nu} = 3$.

In fact, $N_{\lambda\mu\nu} = N_{\mu\lambda\nu}$ is true generally, although this is hardly obvious to mere mortals.

Example 6.4. Let $\lambda = (4, 3, 1), \mu = (5)$ and consider a possible μ-expansion of λ. Any such expansion is trivially strict. For $1 \le i \le 4$, let x_i be the number of new entries in the i-th row of the expansion. By rule 2, the inequalities $x_2 \le 1, x_3 \le 2, x_4 \le 1$ are forced. Hence $N_{\lambda\mu\nu}$ equals 1 for each nonincreasing ν of the form $(4+x_1, 3+x_2, 1+x_3, x_4)$, with

$$x_1 + \cdots + x_4 = 5, \quad x_2 \le 1, x_3 \le 2, x_4 \le 1, \quad x_i \ge 0;$$

and 0 otherwise. This illustrates Pieri's rule – see part 4 of the next exercise.

Exercise 6.5. (1) Let

$$\lambda = (\underbrace{1, \ldots, 1}_{r \text{ times}}) \quad \text{and} \quad \mu = (\underbrace{1, \ldots, 1}_{s \text{ times}}) \text{ with } r \ge s,$$

then $N_{\lambda\mu\nu} = 1$ when

$$\nu = (\underbrace{2, \ldots, 2}_{t \text{ times}}, \underbrace{1, \ldots, 1}_{r+s-2t \text{ times}}) \text{ for some } 0 \le t \le s,$$

and 0 otherwise.

(2) Let

$$\lambda = (r), \quad \text{and} \quad \mu = (s) \text{ with } r \ge s,$$

then $N_{\lambda\mu\nu} = 1$ when

$$\nu = (r + t, s - t) \text{ for some } 0 \le t \le s,$$

and 0 otherwise.

(3) Let

$$\lambda = (r), \quad \text{and} \quad \mu = (\underbrace{1, \ldots, 1}_{s \text{ times}}),$$

then $N_{\lambda\mu\nu} = 1$ when

$$\nu = (r+1, \underbrace{1, \dots, 1}_{s-1 \text{ times}}) \text{ or } (r, \underbrace{1, \dots, 1}_{s \text{ times}}),$$

and 0 otherwise.

(4) Pieri's Rule: Show that $s_\lambda s_{(r)} = \sum_\nu s_\nu$ quantified over ν satisfying

$$|\nu| = |\lambda| + r, \quad \nu_1 \geq \lambda_1 \geq \nu_2 \geq \lambda_2 \geq \dots \geq \nu_k \geq \lambda_k.$$

(Hint: Do not write anything down.)

Exercise 6.6. Let $\lambda = (13, 11, 10, 7), \mu = (9, 6, 5, 3)$. Prove that on $G(4, 20)$ we have $s_\lambda s_\mu = s_{(16,16,16,16)}$.

The degree of $G(k, n)$ in the Plücker embedding is of course the $\dim G$-th power of the hyperplane section, i.e. $s_1^{k(n-k)}$. We will try to verify this using the L-R rule when $k = 2$. By way of illustration, say $n = 5$ and consider a possible sequence of steps for building the diagram $(3, 3)$.

$$\begin{array}{c} * \\ \end{array} \to \begin{array}{c} * \\ * \end{array} \to \begin{array}{cc} * & * \\ * & \end{array} \to \begin{array}{ccc} * & * & * \\ * & & \end{array} \to \begin{array}{ccc} * & * & * \\ * & * & \end{array} \to \begin{array}{ccc} * & * & * \\ * & * & * \end{array}$$

At each step, we add a star to either of the rows in such a way that the top row has at least as many of them as the bottom one. We want to calculate the number of such sequences, when there are $2(n-2)$ stars in all.

This problem is solved in most elementary books on combinatorics. (It is equivalent to counting the number of expressions with matching parentheses. See e.g. [Bru92, Theorem 8.1.1].) The answer is the Catalan number $C_{n-2} = \frac{1}{n-1}\binom{2n-4}{n-2} = \frac{(2n-4)!}{(n-1)!(n-2)!}$, which is in agreement with Exercise 5.6. I do not know if this argument can be stretched to cover the general case.

Example 6.7. We will calculate the number of lines intersecting five general 3-planes in $\mathbb{P}^6$. We are in $G(2, 7)$ and the lines in $\mathbb{P}^6$ intersecting a general 3-plane define a cycle $s_{(2,0)}$. (This follows from formula (7) on page 8.) The reader is invited to check that $s_{(2,0)}^5 = 6 s_{(5,5)}$, so there are 6 such lines.

Exercise 6.8. Prove that the cup product form

$$H^q(G, \mathbb{Z}) \cup H^{2k(n-k)-q}(G, \mathbb{Z}) \longrightarrow \mathbb{Z}$$

is unimodular. (Hint: Exercise 6.6.)

Exercise 6.9. Calculate the degree of the subvariety $X_{(322)} \subseteq G(3,7)$ in the Plücker embedding. (Answer: 5)

6.2. **The characteristic classes of Q.** We will write s_r for $s_{(r)}$. Then it is the case that $c_r(Q) = s_r$ for $0 \le r \le n-k$. These are called special Schubert cycles.

Exercise 6.10. Prove this as follows.

Let $G' = G(k, V_{n-1})$ and $G' \overset{i}{\hookrightarrow} G$ be the obvious inclusion.

(1) Show that $H_q(G', \mathbb{Z}) \xrightarrow{H_q(i)} H_q(G, \mathbb{Z})$ is an injection for all q.
(2) Show that $H^q(G, \mathbb{Z}) \xrightarrow{H^q(i)} H^q(G', \mathbb{Z})$ is a surjection for all q and s_λ belongs to its kernel iff $\lambda_1 = n - k$.
(3) Show that $c_{n-k}(Q) = s_{n-k}$ directly as follows. Topologically $c_{n-k}(Q)$ is represented by the zero locus of a global section of Q. By the BWB theorem, $H^0(Q) = V$. Interpret what it means to say that a section $v \in V$ vanishes at $\langle W \rangle$.
(4) We have a short exact sequence

$$0 \longrightarrow Q' \longrightarrow Q|_{G'} \longrightarrow V/V_{n-1} \otimes \mathcal{O}_{G'} \longrightarrow 0,$$

so by Whitney product formula, $c_{\mathrm{tot}}(Q|_{G'}) = c_{\mathrm{tot}}(Q')$. Now finish the proof using induction and the fact that Chern classes behave well under pullbacks.

Dually, $c_r(S^*) = s_{\lambda_r}$ where $\lambda_r = (\underbrace{1, \ldots, 1}_{r \text{ times}}, 0, \ldots, 0)$.

The special cycles generate $A^\bullet(G)$ as an algebra over $\mathbb{Z}$. Giambelli's formula directly expresses a general cycle in terms of the special cycles. For instance,

$$s_{4421} = \det \begin{bmatrix} \underline{s_4} & s_5 & s_6 & s_7 \\ s_3 & \underline{s_4} & s_5 & s_6 \\ s_0 & s_1 & \underline{s_2} & s_3 \\ 0 & 0 & s_0 & \underline{s_1} \end{bmatrix}$$

The general formula follows the expected pattern,

$$s_\lambda = \det(s_{\lambda_i + j - i}), \quad \text{with the conventions } s_0 = 1, s_i = 0 \text{ if } i < 0.$$

In fact, Giambelli's and Pieri's rule together suffice to determine all products; hence the L-R rule is technically superfluous.

7. The intersection theory on $G(2,4)$

The variety of lines in $\mathbb{P}^3$ is the simplest Grassmannian which is not a projective space, and also perhaps the most vivid. It is a quadric hypersurface in $\mathbb{P}^5$ defined by a single equation

$$\Psi : P_{12}P_{34} - P_{13}P_{24} + P_{14}P_{23} = 0.$$

The cycles are

$$\begin{array}{llll} \text{codim } 0: & s_{(0,0)} & \text{codim } 1: & s_{(1,0)} \\ \text{codim } 2: & s_{(2,0)} \text{ and } s_{(1,1)} & & \\ \text{codim } 3: & s_{(2,1)} & \text{codim } 4: & s_{(2,2)} \end{array}$$

Our flag $V_\bullet$ corresponds to the plane $\Pi = \mathbb{P}V_3$ containing the line $L = \mathbb{P}V_2$, which in turn contains the point $P = \mathbb{P}V_1$.

We will rename our cycles more suggestively. Let

$$\sigma_L := s_{(1,0)}, \quad \sigma_P := s_{(2,0)}, \quad \sigma_\Pi := s_{(1,1)},$$
$$\sigma_{\mathfrak{P}} := s_{(2,1)}, \quad \sigma_0 := s_{(2,2)}.$$

In the sequel, l denotes a typical point of G, i.e. a line in $\mathbb{P}^3$.

Now

$$\begin{aligned} \sigma_L &= \{l \in G : l \text{ intersects } L\} \\ \sigma_P &= \{l \in G : l \text{ passes through } P\} \\ \sigma_\Pi &= \{l \in G : l \text{ lies in } \Pi\} \\ \sigma_{\mathfrak{P}} &= \{l \in G : l \text{ belongs to the pencil contained in } \Pi \text{ with vertex } P\} \\ \sigma_0 &= \{L\} \end{aligned}$$

and the products are

$$\begin{array}{lll} \sigma_L^2 = \sigma_P + \sigma_\Pi & \sigma_P^2 = \sigma_\Pi^2 = \sigma_0 & \sigma_P\sigma_\Pi = 0 \\ \sigma_P\sigma_L = \sigma_\Pi\sigma_L = \sigma_{\mathfrak{P}} & \sigma_{\mathfrak{P}}\sigma_L = \sigma_0 & \end{array}$$

The oldest problem of Schubert calculus is this: Given four general lines in $\mathbb{P}^3$, how many lines intersect them all?

If $L_1, \dots, L_4$ be the lines, the intersection $\cap_{i=1}^4 \sigma_{L_i}$ has class

$$\sigma_L^4 = (\sigma_P + \sigma_\Pi)^2 = 2\sigma_0,$$

so there are two such lines.

Exercise 7.1. Let the lines L_i correspond to subspaces spanned by

$$\{v_1, v_2\},\ \{v_3, v_4\},\ \{v_1 + v_3, v_2 + v_4\},\ \text{and } \{v_1 + v_4, v_2 + v_3\}.$$

Find the two lines intersecting them all.

A space curve carries some naturally associated families of lines with it, and the latter define subvarieties of $G(2,4)$. Here we look at a few of the possibilities. In the sequel, X is a nondegenerate smooth projective curve of degree d and genus g in $\mathbb{P}^3$.

7.1. **The Cayley Form.** Let

$$C_X = \{l \in G : l \text{ intersects } X\}.$$

This is a three dimensional closed subvariety of G, so its class $[C_X] \in A^1(G)$ must equal $\alpha\sigma_L$ for some integer α. Now $[C_X]\sigma_{\mathfrak{P}} = \alpha\sigma_0$, hence α is the number of lines in a general planar pencil which intersect X. Now X will intersect the plane of the pencil in d points, and each point will give one line intersecting X. So $\alpha = d$, and the divisor C_X belongs to the linear series $|d\sigma_L| = \mathbb{P}H^0(G, \mathcal{O}_G(d))$.

Since the map $H^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}}(d)) \xrightarrow{\rho_d} H^0(G, \mathcal{O}_G(d))$ is surjective, there must exist a degree d homogeneous polynomial f_X in $P_{12}, \dots, P_{34}$ such that C_X equals the set $\{f_X = 0\} \cap G$. This is the 'Cayley (or Chow) form' of the curve X. The Cayley form is not unique, indeed for an arbitrary nonzero constant c and an arbitrary degree $d-2$ form h_{d-2}, the expression $cf_X + h_{d-2}\Psi$ is also a Cayley form of X. The reader should check that all possible Cayley forms of X are obtainable this way.

For the next two examples, we will set

$$\begin{aligned} a &= P_{12}, \quad b = P_{13}, \quad c = P_{14}, \\ d &= P_{23}, \quad e = P_{24}, \quad f = P_{34}. \end{aligned}$$

Example 7.2. We will determine the Cayley form of the twisted cubic curve realized as the image of the map

$$\mathbb{P}^1 \longrightarrow \mathbb{P}^3; \quad t \longrightarrow [1, t, t^2, t^3]$$

If a line l contains such an image point, then it has a matrix representation

$$\begin{bmatrix} 1 & t & t^2 & t^3 \\ \zeta_0 & \zeta_1 & \zeta_2 & \zeta_3 \end{bmatrix}$$

for some ζ_i, not all zero. So write down relations

$$a = \zeta_1 - t\zeta_0, \, b = \zeta_2 - t^2\zeta_0, \dots, f = t^2\zeta_3 - t^3\zeta_2;$$

and eliminate the ζ_i and t. Then

$$f_X = d^3 - bde + ae^2 + b^2 f - acf - 2adf.$$

Example 7.3. Let X be the elliptic quartic curve given as the intersection of two space quadrics

$$Y_1^2 + Y_2^2 + Y_3^2 + Y_4^2 = 0, \quad Y_1^2 + 2Y_2^2 + 3Y_3^2 + 4Y_4^2 = 0. \tag{18}$$

We will regard $a, \dots, f$ as indeterminates. Then a line l with Plücker coordinates $[a, \dots, f]$ has a matrix representation

$$\begin{bmatrix} 1 & 0 & -d/a & -e/a \\ 0 & 1 & b/a & c/a \end{bmatrix},$$

so l is the set of points with coordinates $[1, \lambda, -d/a + \lambda\, b/a, -e/a + \lambda\, c/a]$ as λ varies over $\mathbb{P}^1$. If l is to intersect X, then this point must lie on each of the quadrics for some λ. Hence substitute these coordinates in the equations (18) and take the Sylvester resultant with respect to λ. Setting this to zero, our condition becomes

$$\frac{1}{a^2}\begin{vmatrix} (1) & (2) & (3) & 0 \\ 0 & (1) & (2) & (3) \\ (4) & (5) & (6) & 0 \\ 0 & (4) & (5) & (6) \end{vmatrix} = 0, \quad \text{where}$$

$$\begin{aligned} &(1) = b^2 + c^2 + a^2, && (2) = -2(ce + bd), && (3) = e^2 + d^2 + a^2, \\ &(4) = 3b^2 + 4c^2 + 2a^2, && (5) = -2(3bd + 4ce), && (6) = 3d^2 + 4e^2 + a^2. \end{aligned}$$

A straightforward expansion of the determinant would give an octavic equation in the variables $b/a, c/a$ etc. But we can reduce the degree using the relation $f/a = 1/a^2(be - cd)$, and finally multiply by a^4 to get

$$\begin{aligned} f_X =& f^4 + f^2(-4a^2 - 4b^2 + 6c^2 - 2d^2 + 4e^2) - 20abef \\ &+ 4b^4 + 12b^2c^2 + 9c^4 + 4b^2d^2 + d^4 + 14b^2e^2 + 12c^2e^2 + 4d^2e^2 + 4e^4 \\ &+ a^2(a^2 + 4b^2 + 6c^2 - 2d^2 - 4e^2). \end{aligned}$$

To recapitulate, a line with Plücker coordinates $[a, \dots, f]$ intersects the elliptic quartic iff $f_X(a, \dots, f) = 0$.

Needless to say, the procedure applies in principle to any complete intersection curve. I do not know if the answer can be given in a more elegant form.

7.2. **The Chordal Surface.** Let

$$S_X = \{l \in G : l \text{ is a chord of } X\}.$$

A chord is a line intersecting X in at least two (possibly coincident) points. From a Scholastic viewpoint, l is a chord iff the scheme-theoretic intersection $l \cap X$ has length at least 2.

In any event, S_X is a surface in G with

$$[S_X] = \alpha\sigma_P + \beta\sigma_\Pi \quad \text{for some integers } \alpha, \beta.$$

We will calculate α and β. Now $\alpha = [S_X].\sigma_P$, the number of chords of X passing through a general point P. Fix a general plane Λ in $\mathbb{P}^3$ and consider the projection of X on Λ with centre P.

$$\text{proj}_P : X \longrightarrow \mathbb{P}^2, \quad x \longrightarrow \overline{Px} \cap \Lambda.$$

Now X has only finitely many trichords, and they all avoid a general point P. Hence the image of the projection is a plane curve of degree d with only ordinary double points. Each chord through P contributes exactly one double point, so by the genus formula $\alpha = \frac{(d-1)(d-2)}{2} - g$.

We have $\beta = [S_X].\sigma_\Pi$, i.e. β is the number of chords of X lying in a general plane Π. There are d points in $X \cap \Pi$, and all chords lying in Π come from their pairwise intersections. Hence $\beta = \binom{d}{2}$.

If in particular X is a twisted cubic, then $[S_X] = \sigma_P + 3\sigma_\Pi$. Hence two general twisted cubics in space have $[S_X]^2 = 10$ common chords.

7.3. **The First Associated Curve.** Consider the curve

$$T_X = \{l \in G : l \text{ is tangent to } X\}$$

and say $[T_X] = \mu\sigma_{\mathfrak{P}}$. Then $[T_X].\sigma_L = \mu.\sigma_0$, so μ is the number of tangents of X intersecting a general line L. To determine this number, choose another general line M and consider the projection

$$\text{proj}_P : X \longrightarrow \mathbb{P}^1, \quad x \longrightarrow \overline{Lx} \cap M.$$

This is a d-fold cover of $\mathbb{P}^1$, and x is a simple ramification point iff the tangent to X at x intersects L. (Since L is general, we can assume that no inflexional tangent of X intersects L.) By Riemann-Hurwitz,

$$2g - 2 = -2d + \mu, \quad \text{hence } \mu = 2(g + d - 1).$$

Exercise 7.4. Let $\mathfrak{T}_X$ be the tangential surface of X, i.e. the union of tangent lines to X. Convince yourself that as a hypersurface in $\mathbb{P}^3$, it has degree μ.

7.4. **The Number of Lines on a Cubic Surface in $\mathbb{P}^3$.** While discussing Schubert calculus, this computation is obligatory.

Let $Y : \{f = 0\}$ be a smooth cubic surface in $\mathbb{P}V$, where $\dim V = 4$. In fancier terms, f is an element of $\text{Sym}^3 V^* = H^0(G, \text{Sym}^3 S^*)$. An arbitrary line $l \in G(2,4)$ corresponds to a two dimensional subspace $W \subseteq V$. The fibre of the vector bundle $E = \text{Sym}^3 S^*$ over l is the space $\text{Sym}^3 W^*$. The global section f gives an element of this space, which is nothing but the composite functional

$$\text{Sym}^3 W \longrightarrow \text{Sym}^3 V \xrightarrow{f} \mathbb{C}.$$

This map vanishes iff W lies entirely in Y. Hence the lines on Y precisely correspond to zeros of the section f in G. Note that E is a rank 4 bundle on a 4 dimensional space, so we expect a global section to vanish at a finite number of points. If it does so, then this number is given by the top Chern class $c_4(E)$.

This is mindless hack-work. Let ξ_1, ξ_2 be the Chern roots of S^*, that is to say, $\xi_1 + \xi_2 = \sigma_L$, $\xi_1\xi_2 = \sigma_\Pi$. Hence the Chern roots of E are $3\xi_1, 2\xi_1 + \xi_2, \xi_1 + 2\xi_2, 3\xi_2$. Thus

$$\begin{aligned} c_4(E) &= 3\xi_1.(2\xi_1 + \xi_2).(\xi_1 + 2\xi_2).3\xi_2 = 9\xi_1\xi_2(2(\xi_1 + \xi_2)^2 + \xi_1\xi_2) \\ &= 9\sigma_\Pi(2\sigma_L^2 + \sigma_\Pi) = 9\sigma_\Pi(3\sigma_\Pi + 2\sigma_P) = 27\sigma_0. \end{aligned}$$

Exercise 7.5. Calculate the number of lines on a smooth quintic threefold in $\mathbb{P}^4$. (Answer: 2875)

7.5. **The self-duality of $G(2,4)$.** We have a natural isomorphism $G(2,V) \overset{\delta}{\simeq} G(2,V^*)$. Now if we *choose* an isomorphism $V \xrightarrow{\phi} V^*$, then δ corresponds to an involution (i.e. a degree 2 automorphism) of $G(2,V)$. Its geometry is very interesting.

Define $\Phi = \{[v] \in \mathbb{P}V : \phi(v) \text{ annihilates } v\}$. This is a quadric in $\mathbb{P}^3$ (this is immediate if you write ϕ as a matrix), and the injectivity of ϕ implies that it is smooth.

If $x = [v]$ is a point of $\mathbb{P}^3$, its polar is defined to be the plane corresponding to the three dimensional space

$$\{u \in V : \phi(v) \text{ annihilates } u\} \subseteq V.$$

Geometrically, all points $y \in \Phi$ such that the line $\overline{xy}$ is tangent to Φ lie in this plane. Then Φ is precisely the set of points which lie on their own polars.

For any linear subspace $\Lambda \subseteq \mathbb{P}V$, define $\text{polar}(\Lambda) = \cap_{x\in\Lambda}\text{polar}(x)$. Then obviously $\Lambda_1 \subseteq \Lambda_2 \Rightarrow \text{polar}(\Lambda_1) \supseteq \text{polar}(\Lambda_2)$.

Now it is clear how the involution operates on Schubert cycles. The cycles $\sigma_L, \sigma_{\mathfrak{P}}$ are self-dual and σ_P, σ_Π get interchanged.

8. The zeta function

In this section we will calculate an example of the zeta function of the Grassmann variety. The notation will follow that of [Har77, Appendix C], where excellent reasons are given why one should bother about the zeta function in the first place.

We fix a finite field $\mathbb{F}_q$, then $G = G(3,6)$ denotes the variety $G(3, \overline{\mathbb{F}_q}^6)$. We will try to determine N_r, the number of points of G over $\mathbb{F}_{q^r}$.

The zeta function of G is then defined to be

$$Z(G,t) = \exp(\sum_{r=1}^{\infty} \frac{N_r}{r} t^r)$$

A three dimensional subspace of $(\mathbb{F}_{q^r})^6$ with an ordered basis is given by a rank 3 matrix of size 3×6, consisting of elements in $\mathbb{F}_{q^r}$. We will count the number of such matrices. The first row is nonzero, but otherwise arbitrary; hence there are $q^{6r} - 1$ choices for it. For the second row there are $q^{6r} - q^r$ choices, since it cannot be a multiple of the first row. There are $q^{6r} - q^{2r}$ choices for the third row, since it cannot be a linear combination of the first two rows. Hence in all there are $(q^{6r} - 1)(q^{6r} - q^r)(q^{6r} - q^{2r})$ such matrices. Since there are $|GL(3, \mathbb{F}_{q^r})|$ choices for an ordered basis of this subspace, we must divide this number by

$$|GL(3, \mathbb{F}_{q^r})| = (q^{3r} - 1)(q^{3r} - q^r)(q^{3r} - q^{2r}).$$

Hence

$$\begin{aligned} N_r =& \frac{(q^{6r} - 1)(q^{6r} - q^r)(q^{6r} - q^{2r})}{(q^{3r} - 1)(q^{3r} - q^r)(q^{3r} - q^{2r})} \\ =& q^{9r} + q^{8r} + 2q^{7r} + 3q^{6r} + 3q^{5r} + 3q^{4r} + 3q^{3r} + 2q^{2r} + q^r + 1. \end{aligned}$$

Now note the simple identity

$$\exp(\sum_{r=1}^{\infty} q^{sr} \frac{t^r}{r}) = \exp(-\log(1 - q^s t)) = \frac{1}{(1 - q^s t)}$$

In sum, then

$$\begin{aligned} & Z(G,t) \\ =& \{(1 - q^9 t)(1 - q^8 t)(1 - q^7 t)^2 (1 - q^6 t)^3 (1 - q^5 t)^3 (1 - q^4 t)^3 \\ & (1 - q^3 t)^3 (1 - q^2 t)^2 (1 - qt)(1 - t)\}^{-1}. \end{aligned}$$

So we see that the Betti numbers are $(1, 1, 2, 3, 3, 3, 3, 2, 1, 1)$, in agreement with formula (6) on page 8.

The functional equation is now straightforward. We have

$$Z(\frac{1}{q^9 t}) = q^{90} t^{20} Z(t)$$

as expected; since $E = c_9(G)$ is just the sum of the Betti numbers, which is 20.

Exercise 8.1. Carry this out for $G(k, n)$. You may wish to look up what q–binomial coefficients are.

9. Bibliographical remarks

General references for Grassmann and Schubert varieties are [GH78, Ch.1, §5], [Har92, Lectures 6, 16] and the article [KL72]. A brief account of Grassmann's original work (which I don't find altogether easy to understand) may be found in [Kle79, pages 160 ff.].

There are several ways to derive the Plücker relations. I have given here the only one which I understand and which in my opinion does not anticipate the answer.

For singular (co)homology and CW complexes I have used [Mas80] and [LW69] respectively. The theory of standard monomials may be found in [ACGH85]. For a generalization to all reductive groups (and hence further instances of 'straightening laws'), see [MS81].

The account of Schur–Weyl modules follows [FH91], also see [Ful97]. A proof of the Borel-Weil-Bott theorem may be found in [Jan87]. For the theory of ample vector bundles, see [Har66]. The Littlewood–Richardson rule is treated in detail in [Ful97]. The SF (for Symmetric Functions) package by John Stembridge can calculate these numbers, besides doing much else. (It may be found on his website: www.math.lsa.umich.edu/~jrs/) The definitive reference for intersection theory is of course [Ful98].

De Morgan's quotation appears in his 'Trigonometry and Double Algebra, Book II'. I have in turn borrowed it from [Ewa96, vol. I] where some sections of the former are reprinted.

Cayley considered the more difficult problem of determining which homogeneous forms in Plücker coordinates can occur as Cayley forms of curves. See [GM86] and [GKZ94] for a modern account.

References

[ACGH85] E. Arbarello, M. Cornalba, P.A. Griffiths och J. Harris. *Geometry of Algebraic Curves, Volume I.* Grundlehren der mathematischen Wissenschaften, No. 267. Springer–Verlag, New York, 1985.

[Bru92] R. Brualdi. *Introductory Combinatorics.* North–Holland, 2nd utgva, 1992.

[Ewa96] W. Ewald. *From Kant to Hilbert, A Source Book in the Foundations of Mathematics, 2 vols.* Oxford University Press, 1996.

[FH91] W. Fulton och J. Harris. *Representation Theory, A First Course.* Graduate Texts in Mathematics. Springer–Verlag, New York, 1991.

[Ful97] W. Fulton. *Young Tableaux.* London Mathematical Society Student Texts. Cambridge University Press, 1997.

[Ful98] W. Fulton. *Intersection Theory.* Ergebnisse der Mathematik und ihrer Grenzgebiete, 3 Folge. Springer–Verlag, Berlin, 2nd utgva, 1998.

[GH78] P.A. Griffiths och J. Harris. *Principles of Algebraic Geometry.* Wiley Interscience, New York, 1978.

[GKZ94] I.M. Gelfand, M.M. Kapranov och A.V. Zelevinsky. *Discriminants, Resultants and Multidimensional Determinants.* Birkhäuser, Boston, 1994.

[GM86] M. Green och I. Morrison. Equations defining Chow varieties. *Duke Math J.*, vol. 53, pp.733–747, 1986.

[Har66] R. Hartshorne. Ample vector bundles. *I.H.E.S. Publ. Math.*, vol. 29, pp. 63–94, 1966.

[Har77] R. Hartshorne. *Algebraic Geometry.* Graduate Texts in Mathematics. Springer–Verlag, New York, 1977.

[Har92] J. Harris. *Algebraic Geometry, A First Course.* Graduate Texts in Mathematics. Springer–Verlag, New York, 1992.

[HP47] W.V.D. Hodge och D. Pedoe. *Methods of Algebraic Geometry, 3 vols.* Cambridge Mathematical Library. Cambridge University Press, 1947.

[Jan87] J. Jantzen. *Representations of algebraic groups.* Pure and Applied Mathematics. Academic Press, Boston, 1987.

[KL72] S. Kleiman och D. Laksov. Schubert calculus. *American Math. Monthly*, vol. 79, pp. 1061–1082, 1972.

[Kle79] F. Klein. *Development of Mathematics in the 19th century.* Lie Groups, vol. IX. Math Sci Press, 1979.

[LW69] A. Lundell och S. Weingram. *The Topology of CW Complexes.* Van Nostrand Reinhold Co., 1969.

[Mac95] I.G. MacDonald. *Symmetric Functions and Hall Polynomials.* Oxford University Press, 2nd utgva, 1995.

[Mas80] W. Massey. *Singular Homology Theory.* Graduate Texts in Mathematics. Springer–Verlag, New York, 1980.

[Mor79] S. Mori. Projective manifolds with ample tangent bundles. *Annals of Math.*, vol. 110, pp.593–606, 1979.

[MS81] C. Musili och C.S. Seshadri. Standard monomial theory. I *Lecture Notes in Math, No. 867*, sida 441–476. Springer–Verlag, 1981.

[Mus72] C. Musili. Postulation formula for Schubert varieties. *J. Indian Math. Soc.*, vol. 36, pp. 143–171, 1972.

[SS51] O. Schreier och E. Sperner. *Introduction to Modern Algebra and Matrix Theory.* Chelsea Publishing Company, New York, 2nd utgva, 1951.