Stochastic modelling of epidemic spread

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1. Introduction

2. Stochastic processes

3. The SIS model used as an example

4. DTMC

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6. ODE to CTMC
1 Introduction

2 Stochastic processes

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Deterministic SIR

Use the standard KMK

\[ S' = -\beta SI \]
\[ I' = \beta SI - \alpha I \]
\[ R' = \alpha I \]

Basic reproduction number

\[ R_0 = \frac{\beta}{\alpha} S_0 \]

- If \( R_0 < 1 \), no outbreak
- If \( R_0 > 1 \), outbreak

Use \( \alpha = 1/4 \), \( N = S + I + R = 100 \), \( I(0) = 1 \), \( R_0 = \{0.5, 1.5\} \) and \( \beta \) so that this is true
$R_0 = 0.5$

$R_0 = 1.5$
Consider \( S(t) \) and \( I(t) \) \((R(t) = N - S(t) - I(t))\). Define transition probabilities

\[
p_{(s',i'),(s,i)}(\Delta t) = \begin{cases} 
\beta s i \Delta t & (s', i') = (-1, 1) \\
\alpha i \Delta t & (s', i') = (0, -1) \\
1 - [\beta s i + \alpha i] \Delta t & (s', i') = (0, 0) \\
0 & \text{otherwise}
\end{cases}
\]

Parameters as previously and \( \Delta t = 0.01 \)
\[ R_0 = 0.5 \]

\[ R_0 = 1.5 \]
Number of outbreaks

outbreak: $\exists t \text{ s.t. } I(t) > I(0)$
Why stochastic models?

Important to choose right type of model for given application

**Deterministic models**
- Mathematically easier
- Perfect reproducibility
- For given parameter set, one sim suffices

**Stochastic models**
- Often harder
- Each realization different
- Needs many simulations

In early stage of epidemic, very few infective individuals:
- Deterministic systems: behaviour entirely governed by parameters
- Stochastic systems: allows “chance” to play a role
In the context of the SIR model

**Deterministic**
- \( R_0 \) strict threshold
- \( R_0 < 1 \) \( \Rightarrow \) \( I(t) \rightarrow 0 \) monotonically
- \( R_0 > 1 \) \( \Rightarrow \) \( I(t) \rightarrow 0 \) after a bump (outbreak)

**Stochastic**
- \( R_0 \) threshold for mean
- \( R_0 < 1 \) \( \Rightarrow \) \( I(t) \rightarrow 0 \) monotonically (roughly) on average but some realizations have outbreaks
- \( R_0 > 1 \) \( \Rightarrow \) \( I(t) \rightarrow 0 \) after a bump on average (outbreak) but some realizations have no outbreaks
Introduction

Stochastic processes

The SIS model used as an example

DTMC

CTMC

ODE to CTMC
Theoretical setting of stochastic processes

Stochastic processes form a well defined area of probability, a subset of measure theory, itself a well established area of mathematics

Strong theoretical content

Very briefly presented here for completeness, will not be used in what follows


DON’T PANIC!!!
Definition 1 (σ-algebra)

Let $\Omega \neq \emptyset$ be a set and $\mathcal{A} \subset 2^\Omega$, the set of all subsets of $\Omega$, be a class of subsets of $\Omega$. $\mathcal{A}$ is called a $\sigma$-algebra if:

1. $\Omega \in \mathcal{A}$
2. $\mathcal{A}$ is closed under complements, i.e., $A^c := \Omega \setminus A \in \mathcal{A}$ for any $A \in \mathcal{A}$
3. $\mathcal{A}$ is closed under countable unions, i.e., $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$ for any choice of countably many sets $A_1, A_2, \ldots \in \mathcal{A}$

$\Omega$ is the space of all elementary events and $\mathcal{A}$ the system of observable events
Probability space

Definition 2

1. A pair \((\Omega, \mathcal{A})\), with \(\Omega\) a nonempty set and \(\mathcal{A} \subset 2^\Omega\) a \(\sigma\)-algebra is a **measurable space**, with sets \(A \in \mathcal{A}\) measurable sets. If \(\Omega\) is at most countably infinite and if \(\mathcal{A} = 2^\Omega\), then the measurable space \((\Omega, 2^\Omega)\) is **discrete**.

2. A triple \((\Omega, \mathcal{A}, \mu)\) is a **measure space** if \((\Omega, \mathcal{A})\) is a measurable space and \(\mu\) is a measure on \(\mathcal{A}\).

3. If in addition, \(\mu(\Omega) = 1\), then \((\Omega, \mathcal{A}, \mu)\) is a **probability space** and the sets \(A \in \mathcal{A}\) are **events**. \(\mu\) is then usually denoted \(\mathbb{P}\) (or \text{Prob}).
Topological spaces and Borel $\sigma$-algebras

Definition 3 (Topology)

Let $\Omega \neq \emptyset$ be an arbitrary set. A class of sets $\tau \subset \Omega$ is a topology on $\Omega$ if:

1. $\emptyset, \Omega \in \tau$
2. $A \cap B \in \tau$ for any $A, B \in \tau$
3. $(\bigcup_{A \in F} A) \in \tau$ for any $F \in \tau$

$(\Omega, \tau)$ is a topological space, sets $A \in \tau$ are open and $A \in \Omega$ with $A^c \in \tau$ are closed

Definition 4 (Borel $\sigma$-algebra)

Let $(\Omega, \tau)$ be a topological space. The $\sigma$-algebra $\mathcal{B}(\Omega, \tau)$ generated by the open sets is the Borel $\sigma$-algebra on $\Omega$, with elements $A \in \mathcal{B}(\Omega, \tau)$ the Borel sets
Polish spaces

Definition 5

A separable topological space whose topology is induced by a complete metric is a **Polish space**

\( \mathbb{R}^d, \mathbb{Z}^d, \mathbb{R}^\mathbb{N}, (C([0, 1]), \| \cdot \|_\infty) \) are Polish spaces
Stochastic process

Definition 6 (Stochastic process)

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space, \((E, \tau)\) be a Polish space with Borel \(\sigma\)-algebra \(\mathcal{E}\) and \(T \subset \mathbb{R}\) be arbitrary (typically, \(T = \mathbb{N}\), \(T = \mathbb{Z}\), \(T = \mathbb{R}_+\) or \(T\) an interval).

A family of random variables \(X = (X_t, t \in T)\) on \((\Omega, \mathcal{F}, \mathbb{P})\) with values in \((E, \mathcal{E})\) is called a \textbf{stochastic process} with index set (or time set) \(T\) and range \(E\).
Types of processes

So a stochastic process is a collection of random variables (r.v.)

\[ \{X(t, \omega), t \in T, \omega \in \Omega\} \]

The index set \( T \) and r.v. can be **discrete** or **continuous**

- **Discrete time Markov chain** (DTMC) has \( T \) and \( X \) discrete (e.g., \( T = \mathbb{N} \) and \( X(t) \in \mathbb{N} \))

- **Continuous time Markov chain** (CTMC) has \( T \) continuous and \( X \) discrete (e.g., \( T = \mathbb{R}_+ \) and \( X(t) \in \mathbb{N} \))

In this course, we focus on these two. Also exist Stochastic differential equations (SDE), with both \( T \) and \( X(t) \) continuous
Markov property

Definition 7 (Markov property)
Let $E$ be a Polish space with Borel $\sigma$-algebra $\mathcal{B}(E)$, $T \subset \mathbb{R}$ and $(X_t)_{t \in T}$ an $E$-valued stochastic process. Assume that $(\mathcal{F}_t)_{t \in T} = \sigma(X)$ is the filtration generated by $X$. $X$ has the Markov property if for every $A \in \mathcal{B}(E)$ and all $s, t \in T$ with $s \leq t$,

$$\mathbb{P}[X_t \in A|\mathcal{F}_s] = \mathbb{P}[X_t \in A|X_s]$$

Theorem 8 (The usable one)
If $E$ is countable then $X$ has the Markov property if for all $n \in \mathbb{N}$, all $s_1 < \cdots < s_n < t$ and all $i_1, \ldots, i_n, i \in E$ with

$$\mathbb{P}[X_{s_1} = i_1, \ldots, X_{s_n} = i_n] > 0,$$ there holds

$$\mathbb{P}[X_t = i|X_{s_1} = i_1, \ldots, X_{s_n} = i_n] = \mathbb{P}[X_t = i|X_{s_n} = i_n]$$
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We consider an SIS model with demography. In ODE,

\[ S' = b - dS - \beta SI + \gamma I \]
\[ I' = \beta SI - (\gamma + d)I \]

Total population

\[ N' = b - dN \]

asymptotically constant \((N(t) \to b/d =: N^* \text{ as } t \to \infty)\). Can work with asymptotically autonomous system where \(N = N^*\) (and \(S = N^* - I\))
\[ S' = b - \beta SI - dS + \gamma I \]
\[ I' = \beta SI - (\gamma + d)I \]

DFE: \((S, I) = (b/d, 0)\). EEP: \((S, I) = \left(\frac{\gamma+d}{\beta}, \frac{b}{d} - \frac{\gamma+d}{\beta}\right)\)

VdD&W: \(F = D_I[\beta S]\) \(\mid_{DFE} = \beta N^*\), \(V = D_I[(\gamma + d)I]\) \(\mid_{DFE} = \gamma + d\)

\[ \Rightarrow R_0 = \rho(FV^{-1}) = \frac{\beta}{\gamma + d}N^* \]

- If \(R_0 < 1\), \(\lim_{t \to \infty}(S(t), I(t)) = (N^*, 0)\)
- If \(R_0 > 1\), \(\lim_{t \to \infty}(S(t), I(t)) = \left(\frac{N^*}{R_0}, N^* - \frac{N^*}{R_0}\right)\)
Reduction to one equation

As $N$ is asymptotically constant, we can write $S = N^* - I$ and so only need

$$I' = \beta(N^* - I)I - (\gamma + d)I$$

We can then reconstruct dynamics of $S$ from that of $I$
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Definition 9 (DTMC – Simple definition)

An experiment with finite number of possible outcomes $S_1, \ldots, S_N$ is repeated. The sequence of outcomes is a **discrete time Markov chain** if there is a set of $N^2$ numbers $\{p_{ij}\}$ such that the conditional probability of outcome $S_j$ on any experiment given outcome $S_i$ on the previous experiment is $p_{ij}$, i.e., for $1 \leq i, j \leq N$, $t = 1, \ldots,$

$$p_{ji} = \Pr(S_j \text{ on experiment } t + 1|S_i \text{ on experiment } t).$$

The outcomes $S_1, \ldots, S_N$ are the **states**, and the $p_{ij}$ are the **transition probabilities**. The matrix $P = [p_{ij}]$ is the **transition matrix**.
Homogeneity

One major distinction:

- If $p_{ij}$ does not depend on $t$, the chain is **homogeneous**
- If $p_{ij}(t)$, i.e., transition probabilities depend on $t$, the chain is **nonhomogeneous**
Markov chains operate on two level:

- Description of probability that the system is in a given state
- Description of individual realizations of the process

Because we assume the Markov property, we need only describe how system switches (transitions) from one state to the next.

We say that the system has no memory, the next state depends only on the current one.
The states are here the different values of $I$. $I$ can take integer values from 0 to $N$.

The chain links these states.
Probability vector

Let

\[ p_i(t) = \mathbb{P}(I(t) = i) \]

\[ p(t) = \begin{pmatrix} p_0(t) \\ p_1(t) \\ \vdots \\ p_N(t) \end{pmatrix} \]

is the probability distribution

We must have \( \sum_i p_i(t) = 1 \)

Initial condition at time \( t = 0 \): \( p(0) \)
Transition probabilities

We then need description of probability of transition

In general, suppose that $S_i$ is the current state, then one of $S_1, \ldots, S_N$ must be the next state. If

$$p_{ji} = \mathbb{P}\{X(t+1) = S_j|X(t) = S_i\}$$

then we must have

$$p_{i1} + p_{i2} + \cdots + p_{iN} = 1, \quad 1 \leq i \leq N$$

Some of the $p_{ij}$ can be zero
Transition matrix

The matrix

$$P = \begin{pmatrix}
p_{11} & p_{12} & \cdots & p_{1N} \\
p_{21} & p_{22} & \cdots & p_{2N} \\
p_{N1} & p_{N2} & \cdots & p_{NN}
\end{pmatrix}$$

has

- nonnegative entries, $p_{ij} \geq 0$
- entries less than 1, $p_{ij} \leq 1$
- row sum 1, which we write

$$\sum_{j=1}^{N} p_{ij} = 1, \quad i = 1, \ldots, N$$

or, using the notation $\mathbb{1} = (1, \ldots, 1)^T$,

$$P \mathbb{1} = \mathbb{1}$$
In matrix form

\[ p(t + 1) = Pp(t), \quad t = 1, 2, 3, \ldots \]

where \( p(t) = (p_1(t), p_2(t), \ldots, p_N(t))^T \) is a (column) probability vector and \( P = (p_{ij}) \) is a \( N \times N \) transition matrix,

\[
P = \begin{pmatrix}
p_{11} & p_{12} & \cdots & p_{1N} \\
p_{21} & p_{22} & \cdots & p_{2N} \\
p_{N1} & p_{N2} & \cdots & p_{NN}
\end{pmatrix}
\]
Stochastic matrices

Definition 10 (Stochastic matrix)

The nonnegative $N \times N$ matrix $M$ is **stochastic** if $\sum_{i=1}^{N} a_{ij} = 1$ for all $j = 1, 2, \ldots, N$. In other words, $\mathbb{1}^T M = \mathbb{1}^T$

Theorem 11

Let $M$ be a stochastic matrix $M$. Then the spectral radius $\rho(M) \leq 1$, i.e., all eigenvalues $\lambda$ of $M$ are such that $|\lambda| \leq 1$. Furthermore, $\lambda = 1$ is an eigenvalue of $M$.

$M$ has all column sums 1 $\Leftrightarrow \mathbb{1}^T M = \mathbb{1}^T$

$\lambda$ e.value of $M$ with associated left e.vector $\nu$ $\Leftrightarrow \nu^T M = \lambda \nu^T$

$\Rightarrow \mathbb{1}^T M = \mathbb{1}^T$: left e.vector $\mathbb{1}^T$ and e.value 1.
Long “time” behavior

Let $p(0)$ be the initial distribution (column) vector. Then

$$
p(1) = Pp(0)$$

$$
p(2) = Pp(1)
\quad = P(Pp(0))
\quad = P^2p(0)
$$

Iterating, we get that for any $t \in \mathbb{N}$,

$$\quad p(t) = P^t p(0)$$

Therefore,

$$\lim_{t \to +\infty} p(t) = \lim_{t \to +\infty} P^t p(0) = \left(\lim_{t \to +\infty} P^t\right) p(0)$$

if this limit exists (it does if $P$ is constant and chain is regular)
Regular Markov chain

Regular Markov chains not covered here, as most chains in the demographic context are absorbing. For information, though..

Definition 12 (Regular Markov chain)

A regular Markov chain is one in which $P^k$ is positive for some integer $k > 0$, i.e., $P^k$ has only positive entries, no zero entries.

Theorem 13

If $P$ is the transition matrix of a regular Markov chain, then

1. the powers $P^t$ approach a stochastic matrix $W$,
2. each column of $W$ is the same (column) vector $w = (w_1, \ldots, w_N)^T$,
3. the components of $w$ are positive.

So if the Markov chain is regular,

$$\lim_{t \to +\infty} p(t) = Wp(0)$$
Interesting properties of matrices

Definition 14 (Irreducibility)
The square nonnegative matrix $M$ is **reducible** if there exists a permutation matrix $P$ such that $P^T M P$ is block triangular. If $M$ is not reducible, then it is **irreducible**.

Definition 15 (Primitivity)
A nonnegative matrix $M$ is **primitive** if and only if there is an integer $k > 0$ such that $M^k$ is positive.
Directed graphs (digraphs)

Definition 16 (Digraph)

A directed graph (or digraph) $G = (V, A)$ consists in a set $V$ of vertices and a set $A$ of arcs linking these vertices.
Connecting graphs and matrices

**Adjacency matrix** $M$ has $m_{ij} = 1$ if arc from vertex $j$ to vertex $i$, 0 otherwise

$M = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}$

Column of zeros (except maybe diagonal entry): vertex is a **sink** (nothing leaves)
Paths of length exactly 2

Given adjacency matrix $M$, $M^2$ has number of paths of length exactly two between vertices

$$M^2 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 2 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}$$

Write $m^k_{ij}$ the $(i, j)$ entry in $M^k$

$m^2_{32} = 2$ since $2 \rightarrow 1 \rightarrow 3$ and $2 \rightarrow 4 \rightarrow 3$. ..
Connecting matrices, graphs and regular chains

Theorem 17
A matrix $M$ is irreducible iff the associated connection graph is strongly connected, i.e., there is a path between any pair $(i, j)$ of vertices

Irreducibility: $\forall i, j, \exists k, m_{ij}^k > 0$

Theorem 18
A matrix $M$ is primitive if it is irreducible and there is at least one positive entry on the diagonal of $M$

Primitivity: $\exists k, \forall i, j, m_{ij}^k > 0$

Theorem 19
Markov chain regular $\iff$ transition matrix $P$ primitive
Strong connectedness (1)

Not strongly connected
**Strong connectedness (2)**

![Graph](image_url)

*Strongly connected*
Strong components

Strong component 1

Strong component 2
Absorbing states, absorbing chains

Definition 20
A state $S_i$ in a Markov chain is **absorbing** if whenever it occurs on the $n^{th}$ generation of the experiment, it then occurs on every subsequent step. In other words, $S_i$ is absorbing if $p_{ii} = 1$ and $p_{ij} = 0$ for $i \neq j$.

Definition 21
A Markov chain is said to be absorbing if it has at least one absorbing state, and if from every state it is possible to go to an absorbing state.

In an absorbing Markov chain, a state that is not absorbing is called **transient**.
Some questions on absorbing chains

1. Does the process eventually reach an absorbing state?

2. Average number of times spent in a transient state, if starting in a transient state?

3. Average number of steps before entering an absorbing state?

4. Probability of being absorbed by a given absorbing state, when there are more than one, when starting in a given transient state?
Answer to question 1:

**Theorem 22**

*In an absorbing Markov chain, the probability of reaching an absorbing state is 1*
Standard form of the transition matrix

For an absorbing chain with $k$ absorbing states and $N - k$ transient states, the transition matrix can be written as

$$P = \begin{pmatrix} \mathbb{I}_k & 0 \\ R & Q \end{pmatrix}$$

with following meaning,

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<td>Absorbing states</td>
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with $\mathbb{I}_k$ the $k \times k$ identity matrix, 0 an $k \times (N - k)$ matrix of zeros, $R$ an $(N - k) \times k$ matrix and $Q$ an $(N - k) \times (N - k)$ matrix.
The matrix $\mathbb{I}_{N-k} - Q$ is invertible. Let

- $F = (\mathbb{I}_{N-k} - Q)^{-1}$ be the fundamental matrix of the Markov chain
- $T_i$ be the sum of the entries on row $i$ of $F$
- $B = FR$

Answers to our remaining questions:

1. $F_{ij}$ is the average number of times the process is in the $j$th transient state if it starts in the $i$th transient state
2. $T_i$ is the average number of steps before the process enters an absorbing state if it starts in the $i$th transient state
3. $B_{ij}$ is the probability of eventually entering the $j$th absorbing state if the process starts in the $i$th transient state
Back to the SIS model

ODE:
\[ I' = \beta(N^* - I)I - (\gamma + d)I \]

This equation has 3 components:
1. \( \beta(N^* - I)I \) new infection
2. \( \gamma I \) recovery
3. \( dI \) death

In the DTMC, want to know what transitions mean and what their probabilities are
Consider small amount of time $\Delta t$, small enough that only one event occurs: new infection, loss of infectious individual, nothing.

Let

$$B(i) := \beta(N^* - i)i \quad D(i) := (\gamma + d)i$$

be rates of “birth” and “death” of infectious. Then

$$p_{ji}(\Delta t) = \mathbb{P}\{I(t + \Delta) = j|I(t) = i\}$$

$$= \begin{cases} 
B(i)\Delta t & j = i + 1 \\
D(i)\Delta t & j = i - 1 \\
1 - [B(i) + D(i)]\Delta t & j = i \\
0 & \text{otherwise}
\end{cases}$$
Transition matrix

\[
\begin{pmatrix}
1 & D(1)\Delta t \\
0 & 1 - (B(1) + D(1))\Delta t \\
0 & B(1)\Delta t \\
\vdots & \ddots \\
0 & 0 & 0 & \cdots & B(N-1)\Delta t & 1 - D(N)\Delta t \\
\end{pmatrix}
\]

Note that indices in the matrix range from 0 to \(N\) here
Properties

$p_0$ is an absorbing state

For any initial distribution $p(0) = (p_0(t), p_1(0), \ldots, p_N(0))$,

$$\lim_{t \to \infty} p(t) = (1, 0, \ldots, 0)^T \quad \lim_{t \to \infty} p_0(t) = 1$$

Exists quasi-stationary distribution conditioned on non-extinction

$$q_i(t) = \frac{p_i(t)}{1 - p_0(t)}, \quad i = 1, \ldots, N$$
Consequence of absorption

We have

\[ \lim_{t \to \infty} p(t) = (1, 0, \ldots, 0)^T \quad \lim_{t \to \infty} p_0(t) = 1 \]

but time to absorption can be exponential
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Several ways to formulate CTMC’s

A continuous time Markov chain can be formulated in terms of

• infinitesimal transition probabilities
• branching process
• time to next event

Here, time is in $\mathbb{R}_+$
For small $\Delta t$,

$$p_{ji}(\Delta t) = \mathbb{P}\{I(t+\Delta) = j | I(t) = i\}$$

$$= \begin{cases} 
B(i)\Delta t + o(\Delta t) & j = i + 1 \\
D(i)\Delta t + o(\Delta t) & j = i - 1 \\
1 - [B(i) + D(i)]\Delta t + o(\Delta t) & j = i \\
o(\Delta t) & \text{otherwise}
\end{cases}$$

with $o(\Delta t) \to 0$ as $\Delta t \to 0$
Forward Kolmogorov equations

Assume we know $I(0) = k$. Then

$$p_i(t + \Delta t) = p_{i-1}(t)B(i - 1)\Delta t + p_{i+1}(t)D(i + 1)\Delta t$$

$$+ p_i(t)[1 - (B(i) + D(i))\Delta t] + o(\Delta t)$$

Compute $(p_i(t + \Delta t) - p_i(t))/\Delta t$ and take $\lim_{\Delta t \to 0}$, giving

$$\frac{d}{dt}p_0 = p_1D(1)$$

$$\frac{d}{dt}p_i = p_{i-1}B(i - 1) + p_{i+1}D(i + 1) - p_i[B(i) + D(i)] \quad i = 1, \ldots, N$$

**Forward Kolmogorov equations** associated to the CTMC
In vector form

Write previous system as

\[ p' = Qp \]

with

\[
Q = \begin{pmatrix}
0 & D(1) & 0 & \cdots & 0 \\
0 & -(B(1) + D(1)) & D(2) & \cdots & 0 \\
0 & B(1) & -(B(2) + D(2)) & \cdots & 0 \\
& & & \ddots & \vdots \\
& & & & D(N) \\
& & & & -D(N)
\end{pmatrix}
\]

\textit{Q generator matrix}. Of course,

\[ p(t) = e^{Qt} p(0) \]
Linking DTMC and CTMC for small $\Delta t$

DTMC:

\[ p(t + \Delta t) = P(\Delta t)p(t) \]

for transition matrix $P(\Delta t)$. Let $\Delta t \to 0$, obtain Kolmogorov equations for CTMC

\[ \frac{d}{dt}p = Qp \]

where

\[ Q = \lim_{\Delta t \to 0} \frac{P(\Delta t) - I}{\Delta t} = P'(0) \]
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Converting ODE to CTMC

Extremely easy to convert compartmental ODE model to CTMC!!
The general setting: flow diagram

$\frac{dS}{dt} = -\beta SI + bS$

$\frac{dI}{dt} = \beta SI - \gamma I$
ODE approach

Focus on compartments
CTMC approach

Focus on arrows (transitions)
Focus on transitions

- \( b \): \( S \rightarrow S + 1 \)  
  - birth

- \( dS \): \( S \rightarrow S - 1 \)  
  - death of susceptible

- \( \beta SI \): \( S \rightarrow S - 1, I \rightarrow I + 1 \)  
  - new infection

- \( \gamma I \): \( S \rightarrow S + 1, I \rightarrow I - 1 \)  
  - recovery

- \( dI \): \( I \rightarrow I - 1 \)  
  - death of infectious

Will use \( S = N^* - I \) and omit \( S \) dynamics
Gillespie’s algorithm (SIS model with only I equation)

1: Set $t \leftarrow t_0$ and $I(t) \leftarrow I(t_0)$
2: while $t \leq t_f$ do
3: \[ \xi_t \leftarrow \beta(N^* - i)i + \gamma i + di \]
4: Draw $\tau_t$ from $T \sim \mathcal{E}(\xi_t)$
5: $v \leftarrow [\beta(N^* - i)i/\xi_t, \beta(N^* - i)i/\xi_t + \gamma i/\xi_t, 1]$ \[ \]
6: Draw $\zeta_t$ from $\mathcal{U}([0, 1])$
7: Find $pos$ such that $\zeta_t \geq v(pos)$
8: switch ($pos$)
9: case 1: New infection, $I(t + \tau_t) = I(t) + 1$
10: case 2: End of infectious period, $I(t + \tau_t) = I(t) - 1$
11: case 3: Death, $I(t + \tau_t) = I(t) - 1$
12: end switch
13: $t \leftarrow t + \tau_t$
14: end while
IMPORTANT: in Gillespie’s algorithm, we do not consider $p_{ii}$, the event “nothing happens”
Drawing at random from an exponential distribution

Want $\tau_t$ from $T \sim \mathcal{E}(\xi_t)$, i.e., $T$ has probability density function

$$f(x, \xi_t) = \xi_t e^{-\xi_t x} \mathbf{1}_{x \geq 0}$$

Use cumulative distribution function $F(x, \xi_t) = \int_{-\infty}^{x} f(s, \xi_t) \, ds$

$$F(x, \xi_t) = (1 - e^{-\xi_t x}) \mathbf{1}_{x \geq 0}$$

which has values in $[0, 1]$. So draw $\zeta$ from $\mathcal{U}([0, 1])$ and solve $F(x, \xi_t) = \zeta$ for $x$:

$$F(x, \xi_t) = \zeta \iff 1 - e^{-\xi_t x} = \zeta$$

$$\iff e^{-\xi_t x} = 1 - \zeta$$

$$\iff \xi_t x = - \ln(1 - \zeta)$$

$$\iff x = \frac{- \ln(1 - \zeta)}{\xi_t}$$