2013 Manitoba Mathematical Competition Solutions

Thursday, February 21, 2013

1. (a) The angles of a triangle are in the ratio $2:3:4$. Find the largest of the three angles.

   (b) The sides of a triangle are integers. It has positive area and its perimeter is 12. How many such triangles exist?

Solution:

(a) $\frac{4}{2+3+4} \cdot 180 = 80°$.

(b) The two shorter sides must have total length greater than the longest side. The longest side is therefore < 6. If the longest side is 5 the others total 7, yielding cases 3-4-5 or 2-5-5; if the longest side is 4 the others must also be 4. Therefore three such triangles are possible.

Comments:

If mirror reflections are counted as distinct (i.e., 5-4-3 is not the same triangle 3-4-5) and this is clear in the solution then "four" is an acceptable answer.

2. (a) Solve $|x^2 - 6| = 2$.

(b) The roots of the quadratic equation

$$3x^2 + bx + c = 0$$

are $2 \pm \sqrt{3}$. Find the values of b and c.

Solution:

(a) (Solution 1)

$$\begin{aligned} x^2 - 6 &= \pm 2 \\ x^2 &= 6 \pm 2 = 8 \text{ or } 4 \\ x &= \pm 2\sqrt{2} \text{ or } \pm 2 \end{aligned}$$

(Solution 2)

$$\begin{aligned} (x^2 - 6)^2 - 2^2 &= 0 \\ (x^2 - 8)(x^2 - 4) &= 0 \\ x &= \pm 2\sqrt{2} \text{ or } \pm 2 \end{aligned}$$

(b) (Solution 1)
Let $x = 2 \pm \sqrt{3}$. Then $x - 2 = \pm\sqrt{3}$ so $x^2 - 4x + 4 = 3$, which simplifies to $x^2 - 4x + 1 = 0$. Multiplying by 3 puts the quadratic into the given form, so $b = 3(-4) = -12$ and $c = 3(1) = 3$.

(Solution 2)
$3x^2 + bx + c = 3(x - 2 + \sqrt{3})(x - 2 - \sqrt{3}) = 3((x-2)^2 - \sqrt{3}^2) = 3(x^2 - 4x + 1)$.
Equating coefficients, we have $b = -12$, $c = 3$.

(Solution 3)
Substitute and subtract:

$$\begin{aligned} 3(2+\sqrt{3})^2 + b(2+\sqrt{3}) + c &= 21 + 12\sqrt{3} + 2b + b\sqrt{3} + c = 0 \\ 3(2-\sqrt{3})^2 + b(2-\sqrt{3}) + c &= 21 - 12\sqrt{3} + 2b - b\sqrt{3} + c = 0 \\ 24\sqrt{3} + 2b\sqrt{3} &= 0 \end{aligned}$$

Sio $b = -12$. Substituting into the equation gives $-3 + c = 0$, so $c = 3$.

Comments:

Almost anything involving equating "rational" and "irrational" parts is incorrect because it is not known initially whether b and c are rational. Many students made this mistake.

3. In each part, solve for x.

(a) $4x = x^3 + 3x^2$.

(b) $\dfrac{3}{x^2-1} = \dfrac{1}{x-1}$.

Solution:

(a)
$$\begin{aligned} x^3 + 3x^2 - 4x &= 0 \\ x(x+4)(x-1) &= 0 \\ x &= 0, 1 \text{ or } -4. \end{aligned}$$

(b) (Solution 1)
$$\begin{aligned} 3(x-1) &= x^2 - 1 \\ 3x - 3 &= x^2 - 1 \\ x^2 - 3x + 2 &= 0 \\ (x-1)(x-2) &= 0 \end{aligned}$$

$x \in \{1, 2\}$ but $x \neq 1$ (division by 0). Therefore $x = 2$.

(Solution 2)
Since $x \neq 1$, multiplying by $x - 1$ gives $\frac{3}{x+1} = 1$, so $x + 1 = 3$, so $x = 2$.

(Solution 3)
Neither side is 0. Reciprocate both sides to get $\frac{x^2-1}{3} = x - 1$.
Reject $x = 1$ (not a solution) and cancel $x - 1$ to get $\frac{x+1}{3} = 1$, so $x = 2$.

4. (a) A historian divides the past 2013 years into 63 time intervals (each consisting of a positive integer number of years). Explain why two of these intervals must have the same length.

   (b) Consider the following game of dice: Player A rolls a standard die with faces numbered 1,2,3,4,5,6. Player B rolls a modified die with faces numbered 3,4,5,6,7,8. With what exact probability will player B roll the higher value?

Solution:

(a) The smallest sum of 63 distinct positive integers is $1 + 2 + 3 + \cdots + 63 = \frac{63 \cdot 64}{2} = 2016 > 2013$. It follows that 2013 years cannot be divided into 63 distinct positive integer multiples of one year. The result follows.

(b) Face 3 of B's die will beat 2 faces (1 and 2) of A's die. The others will beat 3, 4, 5, 6 and 6 faces respectively. So $2 + 3 + 4 + 5 + 6 + 6 = 26$ of the 36 possible combinations will give B the higher value. Since all combinations are equally likely the required probability is $\frac{26}{36} = \frac{13}{18}$.

5. (a) Given that $ab = 10$, $bc = 12$ and $ad = 5$, evaluate $(a + c)(b + d)$.

(b) A rectangle is cut into four smaller rectangles by two perpendicular lines. Four of the five numbers 6, 9, 10, 12 and 15 are the areas of the four smaller rectangles. Which one is not, and why not?

Solution:

(a) $cd = \frac{(bc)\cdot(ad)}{ab} = \frac{12\cdot 5}{10} = 6$. Therefore, $(a + c)(b + d) = ab + ad + bc + cd = 10 + 5 + 12 + 6 = 33$.

(b)

A	B
C	D

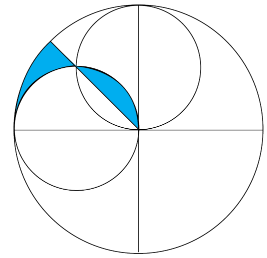
We must have $A : B = C : D$ since in both cases this is the proportion into which the rectangle's horizontal side is cut by the vertical line. The only pairs among of the numbers 6, 9, 10, 12, 15 having common ratios are $6 : 9 = 10 : 15$. Therefore 12 is not the area of any of the four rectangles.

Comments:

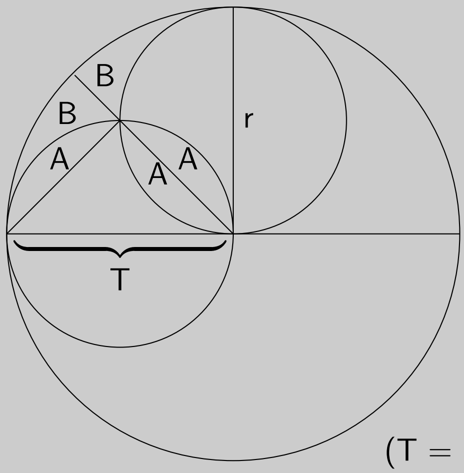
Many students lost marks in (a) for work that explicitly assumed that a, b, c, d are integers.

Similarly students lost marks in (b) for answers that relied on the assumption that the sides of the four regions are of integer length.

6. The sketch shown is from the files of Leonardo da Vinci. Two perpendicular diameters divide a circle into four parts. On each of these diameters a circle of half the diameter is drawn, tangent to the original circle and meeting at its centre. A radius to the large circle is drawn through the intersection points of these smaller circles. Show that the two shaded regions are of the same area.



Solution:



(T = right isosceles triangle of hypotenuse r)

(Solution 1)

$T + A + B = \pi\frac{r^2}{8}$, a one-eigth portion of the large circle, while $T + 2A = \frac{1}{2}\pi\left(\frac{r}{2}\right)^2$, half the small circle. So $T + A + B = T + 2A$. It follows that $A = B$.

(Solution 2)

A quarter of the large circle is equal to the small circle (doubling the radius multiplies the area by 4). Thus the two overlapping small semicircles have the same total area as the large quarter-circle in which they are located. They overlap in a region of area $2A$ and do not cover a region of area $2B$. It follows that $2A = 2B$, so $A = B$.

(Or numerous explicit calculations of area values with or without assigning a value to r)

7. Given that $x^2 + y^2 = 4$ and $x + y = 1$, find all possible values of $x^3 + y^3$.

Solution: $2xy = (x+y)^2 - (x^2+y^2) = 1 - 4 = -3$, so $xy = -\frac{3}{2}$. Now,

$$x^3 + y^3 = (x+y)(x^2 - xy + y^2) = 4 + \frac{3}{2} = \frac{11}{2}.$$

(Alternative second step)

$$x^3 + y^3 = (x+y)^3 - 3xy(x+y) = 1^3 - 3(-\frac{3}{2}) \cdot 1 = 1 + \frac{9}{2} = \frac{11}{2}.$$

Comments:

It is possible to determine the values $\{x, y\}$ and work explicitly from there. This requires a lot of error-prone work.

8. Let A, B, C, X, Y represent distinct, non-zero digits. Consider the following subtraction (and specific example, taking $(A, B, C) = (4, 5, 2)$):

$$\begin{array}{rccc} & A & B & C \\ - & C & B & A \\ \hline & 1 & X & Y \end{array} \qquad \text{Example:} \quad \begin{array}{rccc} & 4 & 5 & 2 \\ - & 2 & 5 & 4 \\ \hline & 1 & 9 & 8 \end{array}$$

There are many possible ways in which values can be assigned to A, B and C so that the resulting calculation is correct.

(a) Prove that $X = 9$ and $Y = 8$, regardless of the particular values of A, B and C.

(b) How many ordered triples (A, B, C) are possible?

Solution:

(a) $(100A+10B+C)-(100C+10B+A) = 99(A-C)$. Therefore $1XY$ is a multiple of 99 between 100 and 199 (inclusive). It follows that $1XY = 198$—that is, $X = 9$, $Y = 8$, as required.

(b) From our analysis above, $A - C = 2$, so $A = C + 2$. Further, since the letters must represent distinct symbols neither A nor C can be 8 or 9. This implies that

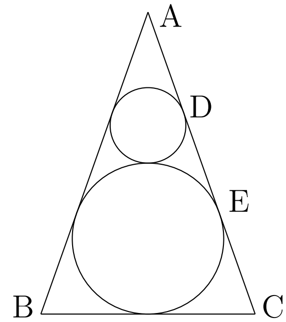
$$(A, C) \in \{(3,1), (4,2), (5,3), (6,4), (7,5)\}.$$

Further, B can be any nonzero digit *other* than $A, C, 8$ or 9. For each of the five possible pairs (A, C) above, this gives five possibilities for B. It follows that there are $5 \cdot 5 = 25$ distinct possible triples (A, B, C).

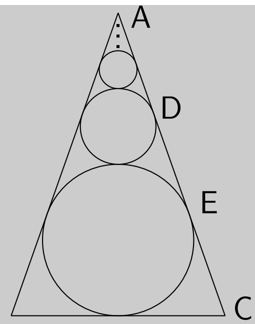
Comments:

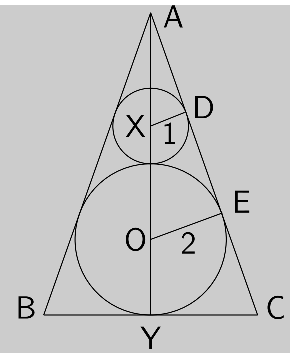
Reasoning involving the relative sizes of A and C, borrowing and so on, is possible, but quite awkward and marks were lost for poor or ambiguous explanations. This approach was also prone to various kinds of errors. Many students failed to use the assumption that the letters represented distinct digits, overcounting as a result.

9. Circles of radius 1 and 2 are externally tangent. Isosceles triangle ABC is inscribed around them as shown. Side AC is tangent to the small circle at D and the large circle at E. Prove that $AD = DE = EC$.



Solution:





Scaling the diagram about point A provides a sequence of circles whose radii are in geometric sequence, converging on A as illustrated on the left. Let the large circle meet BC at Y. Then $AY = 4 + 2 + 1 + \frac{1}{2} + \cdots = 4(1 + \frac{1}{2} + \frac{1}{4} + \cdots) = 4 \cdot 2 = 8$.

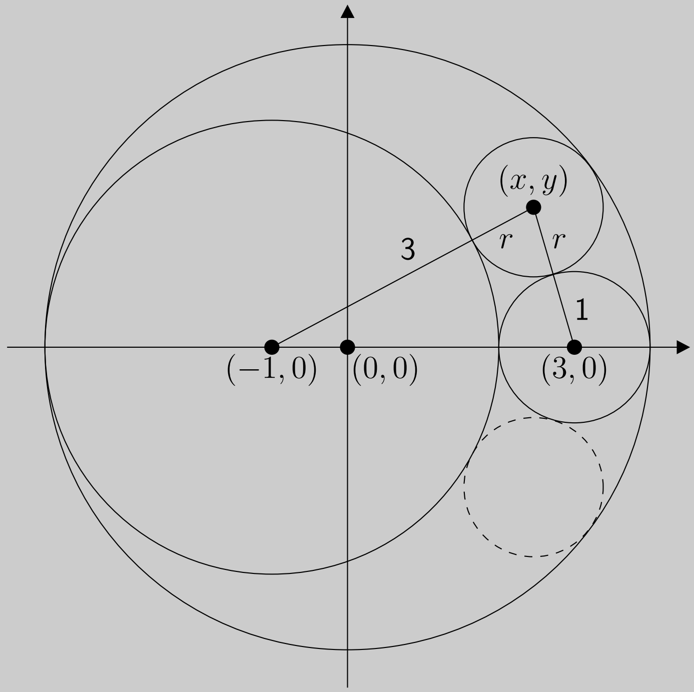
Take $0, X$ to be the centers of the circles of radius 1, 2 respectively. Right triangles $\triangle AXD$ and $\triangle A0E$ have hypotenuses $AX = 3$, $AO = 6$. So $AD = \sqrt{3^2 - 1^2} = 2\sqrt{2}$, $AE = \sqrt{6^2 - 2^2} = 4\sqrt{2}$. Similar triangles gives $YC = 2\sqrt{2}$, whence $AC = \sqrt{8^2 + 8} = 6\sqrt{2}$ and the result follows.

Comments:

There were several methods used. An easy first step notes that $\triangle AXD \sim \triangle AOE$, so $OE = 2XD$ implies $AE = 2AD$, whence $AD = DE$. Showing that this common length is equal to AE remains and requires much more work with this approach, as it is generally necessary to determine OY or AY, and to use the fact that $\triangle AXD \sim \triangle ACY$

10. The three following circles are tangent to each other: the first has centre $(0,0)$ and radius 4, the second has centre $(3,0)$ and radius 1, and the third has centre $(-1,0)$ and radius 3. Find the radius of a fourth circle tangent to each of these 3 circles.

Solution:



The obvious equations are, for center (x, y) and radius r,

$$\begin{aligned} x^2 + y^2 &= (4 - r)^2 \\ (3 - x)^2 + y^2 &= (r + 1)^2 \\ (x + 1)^2 + y^2 &= (r + 3)^2. \end{aligned}$$

Subtracting the first equation from the second and third yields, respectively,

$$\begin{aligned} 9 - 6x &= 10r - 15 \\ 1 + 2x &= 14r - 7. \end{aligned}$$

Adding three times the second of these to the first gives $12 = 52r - 36$, so $r = \frac{12}{13}$.

Comments:

The problem deliberately was given without a diagram, to challenge students to correctly interpret it and sketch an appropriate diagram, a prerequisite to obtaining appropriate equations. There are two "fourth" circles matching the description due to symmetry, but for the same reason it doesn't matter which is used—their radii are equal. Another possibility arises if one considers a certain degeneracy in which a circle is deemed to be tangent to itself: then any of the three circles would match the description and the answer is not unique. No students appeared to pursue this line of reasoning, however, and it was unnecessary, therefore, to deal with this "case".

Coordinates are given as a convenience; the problem can be solved without this. In general circles of radii $a, b, a + b$ give a fourth circle with radius $r = \dfrac{ab(a + b)}{(a^2 + ab + b^2)}$.