2014 MANITOBA MATHEMATICAL CONTEST

For students in grade 12
9:00 AM – 11:00 AM
Thursday, February 20, 2014

Manitoba Association of Mathematics Teachers

Winnipeg Actuaries' Club

Sponsored by:

The Winnipeg Actuaries' Club

The Manitoba Association of Mathematics Teachers

The Canadian Mathematical Society

The University of Manitoba

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Questions are found on both sides of this sheet. Answer as many as possible, but you are not expected to answer them all. **CALCULATORS ARE NOT PERMITTED**. Numerical answers by themselves, without explanation, will not receive full credit.

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1. (a) In a picket fence each picket is 10 cm wide and the spaces between consecutive pickets are each 5 cm wide. What is the length of a straight fence if it has exactly 50 pickets?

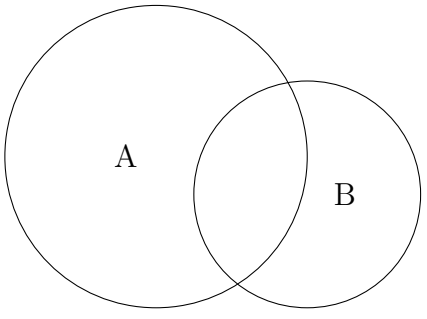
   (b) Two kinds of pickets are available. Some are 10 cm wide and some are 15 cm wide. The spaces between pickets are 5 cm. Wide and narrow pickets alternate. What is the longest fence that can be built using 27 pickets?

2. In $\triangle ABC$, $AB = 3$, $BC = 4$ and $AC = 5$.

   (a) Find the length of the altitude from B to AC.

   (b) This same altitude divides $\triangle ABC$ into two smaller triangles. What is the ratio of their areas?

3. (a) In the diagram, region A is $\frac{4}{5}$ of the large circle and region B is $\frac{5}{7}$ of the smaller circle. What is the ratio of areas A and B?



   (b) There are 100 students in our school of which 50 take chemistry and 40 take physics. If there are 30 students taking both physics and chemistry, how many of these 100 students take neither physics nor chemistry?

4. (a) The average value of 2013 consecutive integers is 2014. What is the smallest of these integers?

   (b) Let $a_1 = 2$, $a_2 = 2a_1$, $a_3 = 2a_2$ and so on. That is, for $n > 1$, $a_n = 2a_{n-1}$. Find and simplify the value of

$$\sqrt[2014]{a_1 \cdot a_2 \cdot a_3 \cdots a_{2014}} \cdot \sqrt{2}.$$

5. (a) Find all possible values of A so the polynomial

$$p(x) = (x^7 - 3x^5 + Ax^4 + 2x^3 + 2x + 1)(x^4 - 3x^3 - Ax^2 + x + 2)$$

   satisfies $p(1) = -12$.

   (b) Express the polynomial $f(x) = 2x^4 - x^3 + 2x^2 + x + 4$ in the form

$$A(x+1)^4 + B(x+1)^3 + C(x+1)^2 + D(x+1) + E.$$

6. A single digit A and a single digit B are placed on the left and right, respectively, of the number 731 to form the five digit number $A731B$. If this five digit number is divisible by 36, find all possible pairs (A, B).

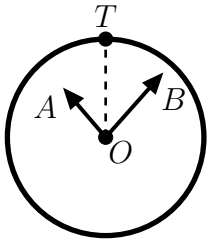
7. For the following system of equations:

$$\begin{array}{rcrcrcrcl} 2a & + & b & + & c & + & d & = & 5 \\ a & + & 2b & + & c & + & d & = & 7 \\ a & + & b & + & 2c & + & d & = & 2 \\ a & + & b & + & c & + & 2d & = & 6 \end{array}$$

   (a) Solve for a, b, c and d.

   (b) Show that there is no solution if every "2" appearing in the system is replaced by "-3"

8. Let O be the centre of a standard clock face and let T be the point at the 12 o'clock position. Let A be the end point of the hour hand and B the end point of the minute hand, as shown. At what time between 10:00 o'clock and 11:00 o'clock, accurate to the nearest second, is $\angle AOT$ equal to $\angle BOT$?



9. Prove that $\sqrt{4 + 2\sqrt{3}} + \sqrt{28 - 10\sqrt{3}}$ is an integer.

10. Consider the equation $7a + 12b = c$ where a, b and c are nonnegative integers. For many values of c it is possible to find one or more pairs (a, b) satisfying the equation. Given $c = 26$, for example, $(a, b) = (2, 1)$ is the only solution.

    (a) If $c = 365$, find all possible solutions (a, b), where a and b are nonnegative integers.

    (b) There are some values of c for which no solutions exist. For example, there is no pair (a, b) such that $7a + 12b = 20$, so $c = 20$ is one such case. Find the largest integer value of c for which there are no nonnegative integer solutions.