2013 MANITOBA MATHEMATICAL CONTEST

For students in grade 12
9:00 AM – 11:00 AM
Thursday, February 21, 2013

Sponsored by:
The Winnipeg Actuaries' Club
The Manitoba Association of Mathematics Teachers
The Canadian Mathematical Society
The University of Manitoba

Questions are found on both sides of this sheet. Answer as many as possible, but you are not expected to answer them all. **CALCULATORS ARE NOT PERMITTED**. Numerical answers by themselves, without explanation, will not receive full credit.

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1. (a) The angles of a triangle are in the ratio $2 : 3 : 4$. Find the largest of the three angles.

   (b) The sides of a triangle are integers. It has positive area and its perimeter is 12. How many such triangles exist?

2. (a) Solve $|x^2 - 6| = 2$.

   (b) The roots of the quadratic equation
   $$3x^2 + bx + c = 0$$
   are $2 \pm \sqrt{3}$. Find the values of b and c.

3. In each part, solve for x.

   (a) $4x = x^3 + 3x^2$.

   (b) $\dfrac{3}{x^2 - 1} = \dfrac{1}{x - 1}$.

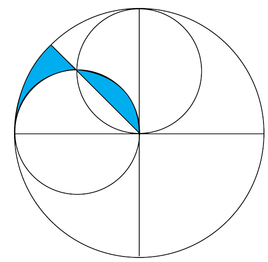
4. (a) A historian divides the past 2013 years into 63 time intervals (each consisting of a positive integer number of years). Explain why two of these intervals must have the same length.

   (b) Consider the following game of dice: Player A rolls a standard die with faces numbered 1,2,3,4,5,6. Player B rolls a modified die with faces numbered 3,4,5,6,7,8. With what exact probability will player B roll the higher value?

5. (a) Given that $ab = 10$, $bc = 12$ and $ad = 5$, evaluate $(a + c)(b + d)$.

   (b) A rectangle is cut into four smaller rectangles by two perpendicular lines. Four of the five numbers 6, 9, 10, 12 and 15 are the areas of the four smaller rectangles. Which one is not, and why not?

6. The sketch shown is from the files of Leonardo da Vinci. Two perpendicular diameters divide a circle into four parts. On each of these diameters a circle of half the diameter is drawn, tangent to the original circle and meeting at its centre. A radius to the large circle is drawn through the intersection points of these smaller circles. Show that the two shaded regions are of the same area.



7. Given that $x^2 + y^2 = 4$ and $x + y = 1$, find all possible values of $x^3 + y^3$.

8. Let A, B, C, X, Y represent distinct, non-zero digits. Consider the following subtraction (and specific example, taking $(A, B, C) = (4, 5, 2)$):

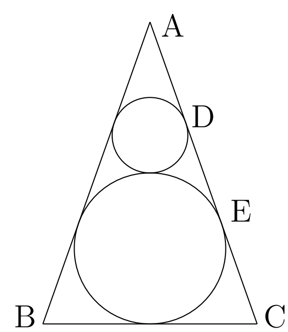
$$\begin{array}{rccc} & A & B & C \\ - & C & B & A \\ \hline & 1 & X & Y \end{array} \qquad \text{Example:} \qquad \begin{array}{rccc} & 4 & 5 & 2 \\ - & 2 & 5 & 4 \\ \hline & 1 & 9 & 8 \end{array}$$

   There are many possible ways in which values can be assigned to A, B and C so that the resulting calculation is correct.

   (a) Prove that $X = 9$ and $Y = 8$, regardless of the particular values of A, B and C.

   (b) How many ordered triples (A, B, C) are possible?

9. Circles of radius 1 and 2 are externally tangent. Isosceles triangle ABC is inscribed around them as shown. Side AC is tangent to the small circle at D and the large circle at E. Prove that $AD = DE = EC$.



10. The three following circles are tangent to each other: the first has centre $(0, 0)$ and radius 4, the second has centre $(3, 0)$ and radius 1, and the third has centre $(-1, 0)$ and radius 3. Find the radius of a fourth circle tangent to each of these 3 circles.