## $10^{\text {th }}$ annual University of Manitoba Open Math Challenge

Are you a "mathlete"? Find out by testing your wits against these problems. This contest is open to undergrad students in any UM program. Participation costs you nothing but your time. You receive your results confidentially except that the overall winner receives a book prize and $\$ 100$ cash, and fame. Interested participants will receive feedback on their own work. For more details about UM Mathletics visit our web page: http://server.math.umanitoba.ca/~craigen/manitobamathletics/ mathteam/

## Instructions:

Submit well-presented solutions before 5:00 PM 20 September 2016 to R. Craigen (MH 523), or K. Gunderson (MH 464) or, in a sealed envelope to the Math Office (MH 420), or by email either coach from your UM email account before the deadline (typeset or high-quality scan of neat handwriting). Solutions will be judged according to correctness, completeness, clarity, elegance, and proper justification. Each question is worth 10 marks. Begin each solution on a new page. Staple solutions in the same order as questions are given.

HONOUR SYSTEM: Do not solicit or accept assistance from, or provide it to, others. Do not consult references or use technology to solve these problems.

You are not expected to solve all questions; submit solutions only for those on which you have made significant progress; do not submit work you consider to be worthless. DO NOT BE DISCOURAGED by these questions - they are not intended to be "easy".

1. Find the last two digits of the numerical value of the power-tower $2016^{2016} . .^{20}$ where "2016" occurs 2016 times in the expression.
2. For which values of $n \geq 1$ is it true that

$$
\left(a_{1}-1\right)\left(a_{2}-2\right)\left(a_{3}-3\right) \cdots\left(a_{n}-n\right)
$$

is even, for every arrangement $a_{1}, a_{2}, \ldots, a_{n}$ of the integers $1,2, \ldots, n$ ?
3. Suppose $n_{1}, n_{2}, \ldots, n_{2016}$ are distinct positive integers. Show that, if none has a prime factor greater than 30 , then the product of some two of them is a perfect square. Also show that if prime factors up to 31 are permitted then this property is no longer the case.
4. Show that the polynomial

$$
Q(x)=A \frac{x(x-1)}{2}+B x+C
$$

takes integer values at every integer $x$ iff $A, B, C \in \mathbb{Z}$.
5. In $\triangle A B C$ points $D, E$ and $F$ are on sides $B C, C A$ and $A B$ respectively, and with $\angle A F E=\angle B F D ; \angle B D F=\angle C D E$, and $\angle C E D=\angle A E F$ as shown. Given that $|A B|=5,|B C|=8$ and $|C A|=7$, determine $|B D|$.

6. Let $a, b, c$ be the lengths of the sides of a triangle. Prove that

$$
\sqrt{a+b-c}+\sqrt{a+c-b}+\sqrt{b+c-a} \leq \sqrt{a}+\sqrt{b}+\sqrt{c}
$$

7. Show that for any two real numbers $a, b>0$, the function

$$
\frac{1}{x}+\frac{1}{x-a}+\frac{1}{x+b}
$$

has two real roots: one in the interval $(-2 b / 3,-b / 2)$ and the other in $(a / 2,2 a / 3)$.
8. Prove that for every positive integer $n$,

$$
1392^{n}+993^{n}-321^{n}-48^{n}
$$

is divisible by 2016 .
9. Find all integers $p$ for which the equations

$$
x+p y=2016 \quad \text { and } \quad x+y=p^{z}
$$

have solutions with $x, y, z$ all positive integers. Show that for those choices of $p$, there is a unique solution.
10. A rectangular network of streets (as illustrated) makes up a city $m>1$ blocks by $n>1$ blocks. Show that the number of possible walking tours along the streets, beginning in the southwest corner and ending at the northeast corner, such that no intersection is visited twice, is at most $2^{m n}-2^{(m-1)(n-1)}$.


