2015 UMOMC

9^{TH} annual **University of Manitoba Open Math Challenge**

15 September 2015

Are you a "mathlete"? Find out by testing your wits against these problems. This contest is open to undergrad students in any UM program. Participation costs you nothing but your time. You receive your results confidentially except that the overall winner receives a book prize and $100 cash, and fame. Interested participants will receive feedback on their own work. For more details about UM Mathletics visit our web page: `http://server.math.umanitoba.ca/~craigen/manitobamathletics/mathteam.html`

Instructions:

Submit well-presented solutions before 5:00 PM Tuesday 22 September 2015 to R. Craigen (MH 523), or D. Gunderson (MH 521) or, in a sealed envelope to the Math Office (MH 342), or by email either coach from your UM email account before the deadline (typeset or high-quality scan of neat handwriting). Solutions will be judged according to correctness, completeness, clarity, elegance, and proper justification. Each question is worth 10 marks. Begin each solution on a new page. Staple solutions in the same order as questions are given.

HONOUR SYSTEM: Do not solicit or accept assistance from, or provide it to, others. Do not consult references or use technology to solve these problems.

You are not expected to solve all questions; submit solutions only for those on which you have made significant progress; do not submit work you consider to be worthless. DO NOT BE DISCOURAGED by these questions—they are not designed to be "easy".

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Problem 1. *The integers 1–25 are placed around the outside of a wheel in any order. Show that there must be three adjacent numbers whose sum is at least 39.*

Solution to Problem 1: Suppose the contrary. Then the sum of all 25 triples of adjacent numbers is $s < 25 \cdot 39$. However, every number is used three times in this sum, so we must also have $s = 3(1+2+\cdots+25) = 3\frac{25\cdot 26}{2} = 25 \cdot 39$—a contradiction. (Essentially PHP from first principles without explicitly invoking the principle.)

Problem 2. *X is a regular 2015-gon and Y is a regular 2016-gon. Both polygons have edges of length 2014. The area of the region between the inscribed and circumscribed circles of X is A and the area of the annulus similarly defined by Y is B. Show that $A = B$ and find this common value.*

Solution to Problem 2: Let PQR be a segment of length 2014, tangent to the inner circle of an annulus at Q, with P and R lying on the outer circle. If the radii of the two circles are r and s respectively, then by Pythagoras' Theorem, $r^2 + 1007^2 = s^2$, so $s^2 - r^2 = 1007^2$. The area of the annulus is

$$\pi s^2 - \pi r^2 = \pi(s^2 - r^2) = 1007^2\pi,$$

which we see to be independent of r and s, depending only on the length 2014 of the segment. Taking PR to be the side of either polygon shows that $A = B = 1007^2\pi$.

Problem 3. *Find all instances of twenty-five consecutive perfect squares whose sum is a perfect square.*

Solution to Problem 3: Write and simplify the described sum as follows:

$$\begin{aligned} m^2 &= (n-12)^2 + \cdots + (n-1)^2 + n^2 + (n+1)^2 + \cdots + (n+12)^2 \\ &= 25n^2 + 2(1^2 + 2^2 + \cdots + 12^2) \\ &= 25n^2 + 2\frac{n(n+1)(2n+1)}{6} \\ &= 25(n^2 + 52). \end{aligned}$$

Taking $m = 5k$ we have $k^2 = n^2 + 52$, or $(k+n)(k-n) = 52 = 2^2 \cdot 13$. Since $k+n$ and $k-n$ have the same parity, they must both be even, so $(k-n, k+n) \in \{\pm(2,26), \pm(26,2)\}$, and so $n = \pm 12$. In both cases, the perfect squares are $0^2, 1^2, 2^2, \ldots, 24^2$ (and since $k = 14$, their sum is $(5 \cdot 14)^2 = 70^2 = 4900$).

Problem 4. *2015 nonoverlapping disks, all of radius 1cm, are strewn on a region of a plane. An elastic band is stretched tight around them, enclosing all 2015 disks in a single convex closed curve C, consisting of straight line segments joined by curved segments. If the total length of the straight line portions is 10 m, what is the exact total length of the curve?*

Solution to Problem 4: The problem boils down to determining the total length of the non-linear portions of the curve, which are all seen to be circular arcs $C_1, C_2, \ldots, C_n$ (for some $n \leq 2015$) with radius 1 cm. Let L be a line in the plane with C lying entirely on one side of L. With parallel motion, move L toward C until they meet. One of two things will occur: either L will be tangent with an entire straight segment making up C, or it will be tangent at a uniquely determined (by the orientation of L and on which of its two sides C is found) point on one of $C_1, \ldots, C_n$. In a finite number of orientations, L is parallel to a linear face of C. It follows that $C_1, \ldots, C_n$ can be reassembled, using parallel motion in the plane, into a circle of radius 1 cm, omitting only a finite number of points. The sum of the lengths of $C_1, \ldots, C_n$ is therefore 2π cm $= \frac{\pi}{50}$ m, the circumference of this circle, and so the length of C is $10 + \frac{\pi}{50}$ m.

(NOTE: we could add one of the angle or arc-length arguments from student solutions)

Problem 5. *A corner is sliced from a rectangular block of wood with a planar cut, forming a tetrahedron having three mutually perpendicular faces of areas A, B, C respectively. The area of the face formed by the planar cut is D. Prove that $A^2 + B^2 + C^2 = D^2$.*

Solution to Problem 5: Let $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ be vectors formed by the three perpendicular edges, with common initial vertex at the vertex where the they meet. The "cut" face is a triangle, two of whose edges are defined by the vectors $\mathbf{x} = \mathbf{v} - \mathbf{u}$ and $\mathbf{y} = \mathbf{w} - \mathbf{u}$. The area of that face is obtained from $\mathbf{x}, \mathbf{y}$ using the cross product formula and the fact that $\mathbf{u} \times \mathbf{u} = \mathbf{0}$:

$$\begin{aligned} D &= \frac{1}{2}||\mathbf{x} \times \mathbf{y}|| \\ &= \frac{1}{2}||(\mathbf{v} - \mathbf{u}) \times (\mathbf{w} - \mathbf{u})|| \\ &= \frac{1}{2}||\mathbf{v} \times \mathbf{w} - \mathbf{v} \times \mathbf{u} - \mathbf{u} \times \mathbf{w}||. \end{aligned}$$

The three vectors in the above expression are mutually orthogonal, so by Pythagoras' Theorem we have

$$D^2 = \frac{1}{4}(||\mathbf{v}\times\mathbf{w}||^2 + ||\mathbf{v}\times\mathbf{u}||^2 + ||\mathbf{u}\times\mathbf{w}||^2) = \left(\frac{1}{2}||\mathbf{v} \times \mathbf{w}||\right)^2 + \left(\frac{1}{2}||\mathbf{v} \times \mathbf{u}||\right)^2 + \left(\frac{1}{2}||\mathbf{u} \times \mathbf{w}||\right)^2$$

which (again by the cross product formula for area of a triangle) is equal to $A^2 + B^2 + C^2$. *Comment: this is doable by a more brute force approach by anyone with a moderate ability with algebra. I don't see the value in adding the brute force approaches. However, it might be considered simpler to start by taking the three vectors to be $(a, 0, 0), (0, b, 0)$ and $(0, 0, c)$ and doing explicit calculations.*

Problem 6. *Two congruent ellipses have semi-minor axes a and semi-major axes b and are initially externally tangent to each other and positioned so that their foci A, B, C and D are colinear and lie in that sequential order. The ellipse with foci A and B remains fixed while the ellipse with foci C and D rolls along it, without slipping. Describe, with justification, the locus of each of C and D.*

Solution to Problem 6: With the second ellipse in general position let P be the point of tangency of the two ellipses. By symmetry segment BP makes the same angle with the common tangent T as segment CP. By the reflective property of ellipses this is equal to the angle made by DP and T. It follows that B, P and D are colinear. Further, by symmetry

$|AP| = |DP|$ so $|BD| = |BP| + |PD| = |BP| + |AP| = 2b$, since the sum of the distance from the foci of an ellipse to any point on the ellipse is constant, easily seen to be $2b$. It follows that the locus of C is a circle of radius $2b$ centered at B and the locus of C is a circle of radius $2b$ centered at A.

Problem 7. *Let $f : \mathbb{R} \to \mathbb{R}$ denote a function whose first derivative f' is continuous, and satisfies the equation*

$$(f(x))^2 = \int_0^x [f(s)^2 + f'(s)^2]ds + 2015$$

for all real numbers x. Find all such functions f.

Solution to Problem 7: Differentiating with the chain rule and the FTC we obtain that $2f(x)f'(x) = f(x)^2 + f'(x)^2$ so that $(f(x) - f'(x))^2 = 0$. It follows that $f(x) = f'(x)$, for which it is well known the only solutions are multiples of e^x. Setting $f(x) = Ce^x$ and evaluating the given expression for $x = 0$ yields $f(0)^2 = 2015$.

Answer: $f(x) = \pm\sqrt{2015}e^x$.

Problem 8. *Show that there are infinitely many positive integers x so that $2x^2 + 1$ is a perfect square.*

Solution to Problem 8: There is at least one solution, namely $x = 2$. Let a be a solution, i.e., $2a^2 + 1 = b^2$. Let $x = 2ab$ and $k = 2b^2 - 1$. Then

$$2x^2 + 1 = 2(2ab)^2 + 1 = 8a^2(2a^2 + 1) + 1 = 16a^4 + 8a^2 + 1 = (4a^2 + 1)^2.$$

Since $x = 2ab > a$, this is strictly larger solution than the one from which it was obtained. Iterating this procedure yields an infinite number of distinct solutions.

Comment: it may be of value to show Suraj's approach, which gives a simple, direct derivation of this solution rather than pulling it from a hat.

Problem 9. *Show that if an $n \times n$ matrix has entries $1, 2, \ldots, n^2$ in any order, then there are two neighbouring entries (i.e., appearing consecutively in some row or column) which differ by at least n.*

Solution to Problem 9: Let m be the smallest number so that some row/column is filled with numbers $\leq m$. WLOG, assume that the first row beginning with m in position (1,1) is such a row. Since none of the columns $2, 3, \ldots n$ uses only numbers from 1 to m, every column but the first has two neighbouring entries, one of which is at most $m - 1$ and the other at least $m + 1$. The max of all numbers in these pairs is at least $m + n - 1$, and the other number in that pair is at most $m - 1$, so the difference of those two numbers is at least n. $\square$

Comment: it was a good point that the choice to put m in the $(1, 1)$ position is not so much because there is no loss in generality (it depends what one means by "generality") but as a matter of convenience so we needn't discuss the less elegant cases. It is quite true, regardless, that once one sees this case it is clear how to handle the messier ones; this was merely a device to avoid talking about them. Should the presentation change?

Problem 10. *Show that there are no solutions to*

$$m^3 = n^4 - 4,$$

where m and n are both integers.

Solution to Problem 10:

First observe that if such solutions exist, either both m and n are even or both are odd.

Suppose first that m and n are even, say $m = 2k$ and $n = 2\ell$. Then equation (10) becomes

$$8k^3 = 16\ell^4 - 4,$$

which, after division by 4, yields $2k^3 = 4\ell^4 - 1$, which is impossible since the left side is even, and the right side is odd.

So suppose that both m and n are odd, say $m = 2k + 1$ and $n = 2\ell + 1$. Then equation (10) becomes

$$(2k + 1)^3 = (n^2 - 2)(n^2 + 2) = (4\ell^2 + 4\ell - 1)(4\ell^2 + 4\ell + 3).$$

The right side is congruent to 1 mod 4. So is the left. However, the only prime that divides both a number N and $N - 4$ is 2, and since all terms are odd, this is impossible. So, any prime dividing the left side divides only one of $n^2 - 2$ or $n^2 + 2$, so all primes dividing the right side occur three times in only one of these terms. So both $n^2 - 2$ and $n^2 + 2$ are perfect cubes. However, perfect cubes differ by at least 7, not 4, so no such n exists. $\square$

(General Comment: In addition to finding typos and places to tighten up presentations we should consider if there are worthy alternative proofs in the students' papers or other things we should add to give them credit).