University of Manitoba Open Math Challenge

15 September 2014

Are you a mathlete? Find out by testing your wits against these problems. This contest is open to undergraduate students in any UM program. Participation carries no obligation and costs you nothing but your time. You receive your results confidentially except that the overall winner receives a book prize and $100 cash, and fame. Interested participants will receive feedback on their own work. For more details about UM Mathletics visit our web page: `http://server.math.umanitoba.ca/~craigen/manitobamathletics/mathteam.html`

Instructions:

Submit well-written solutions before 5:00 PM Monday 22 September 2014 to D. Gunderson (MH 521) or, in a sealed envelope to the Math Office (MH 342), or by email to the above address from your university account before the deadline (typeset or high-quality scan of neat handwriting). Solutions will be judged according to correctness, completeness, clarity, elegance, and proper justification. Begin each solution on a new sheet of paper. Staple solutions in the same order as questions are given. You are not expected to solve all questions; submit solutions only for those on which you have made significant progress; do not submit work you consider to be worthless. DO NOT BE DISCOURAGED by these questionsthey are not designed to be easy. Each question is worth 10 marks. HONOUR SYSTEM: Do not solicit or accept assistance from, or provide it to, others. Do not consult any references or use technology to solve these problems.

Problem 1. *Suppose that a convex polygon has 2014 sides. What is the sum of the interior angles measured in radians?*

Problem 2. *Consider a board with dimensions* 2014×2014 *that has been marked off into* 1×1 *squares in the standard grid fashion. Is there a way to cover the board using* 1×4 *strips? If so, describe one way to place the strips. If not, show why not. (Strips can not overlap, nor can they hang over the edge of the big board; all strips are either vertical or horizontal.)*

Problem 3. *Let* $n \geq 3$ *and suppose that each of* $x_1, x_2, \ldots, x_n$ *are either 1 or* -1*. Show that if*

$$x_1x_2 + x_2x_3 + x_3x_4 + \cdots x_{n-1}x_n + x_nx_1 = 0,$$

then n *is divisible by 4.*

Problem 4. *Suppose that there are two kinds of days for the month of September: wet, or cold. If it is cold on one day, the probability that is cold the following day is 3/4. If it is wet on one day, the probability that it is cold the following day is 1/3. If 19 September is cold, what is the probability that it will be cold on 22 September?*

Problem 5. *Find two positive integers* x *and* y *so that together they use each of the digits 0,1,...9 precisely once, and there is an integer* n *so that* $x = n^2$ *and* $y = n^3$*. Find* n*, and show that* n *is unique.*

Problem 6. *Construct an infinite set* X *of positive integers so that no sum of distinct integers in* X *(including a single element sum) is a perfect power, that is, is of the form* y^k*, where* y *and* $k > 1$ *are positive integers.*

Problem 7. *For* $n \geq 2$*, let* $x_1, x_2, \ldots, x_n$ *be positive numbers so that for every* $1 \leq k \leq n-1$*, the inequality* $x_k \leq x_{k+1} \leq 2x_k$ *holds. Show that in a sum of the form* $S = \pm x_1 \pm x_2 \pm \cdots \pm x_n$*, the signs can be chosen so that* $0 \leq S \leq x_1$*.*

Problem 8. *Prove that if* $0 \leq a \leq 2$*, then*

$$(1 + \sqrt[5]{a})^5 + (1 + \sqrt[5]{2 - a})^5 \leq 64.$$

Problem 9. *Determine whether or not the following limit exists, and if does, compute it:*

$$\lim_{n \to \infty} \left(\frac{n}{1 + 9n^2} + \frac{n}{4 + 9n^2} + \frac{n}{9 + 9n^2} + \cdots + \frac{n}{n^2 + 9n^2} \right).$$

Problem 10. *A perfect power is a number of the form* m^ℓ*, where both* m *and* $\ell > 1$ *are positive integers. For* $k \geq 3$*, an* k*-term arithmetic progression is a sequence of integers of the form* $a, a + d, a + 2d, \ldots, a + (k-1)d$*, where* $a \in \mathbb{Z}$ *and* $d \in \mathbb{Z}^+$*. Show that for each* $k \geq 3$*, there exists a* k*-term arithmetic progression consisting only of perfect powers. Give an example with* $k = 5$ *terms.*