2012 University of Manitoba Open Math Challenge (WITH SOLUTIONS)

Are you a "mathlete"? Find out by testing your wits against these problems.

This contest is open UM undergraduate students in any program. Taking part does not commit you to compete on or train for our teams, and costs you nothing but your time. There is no penalty for scoring low. Results remain confidential, except for the overall winner, **a book prize and $100 cash**, and fame. All participants can get private feedback on their own work.

For more details about UM Mathletics, see the last page of this document or visit our web page.

Instructions

Submit *well-written* solutions **before 5:10 PM, Tuesday, Sept. 18** directly to a coach (see final page) or in a sealed envelope taken to the Math Office (MH 342), or by email from your university account before the deadline (typeset or high-quality scan of neat handwriting). For full credit solutions must be complete, clearly and efficiently presented, and properly justified. Begin each solution on a new sheet of paper. Staple solutions in the same order as questions are given; do not submit junk. DO NOT BE DISCOURAGED by these questions—they are not intended to be "easy". HONOUR SYSTEM: Do not solicit or accept assistance from, or provide it to, others. Do not consult any references or use technology to solve these problems.

(Each question is marked out of 10)

1. Solve the following equation

$$\frac{1}{\sqrt{x}+\sqrt{x-4}}+\frac{1}{\sqrt{x}+\sqrt{x+4}}=\frac{1}{2}.$$

Solution: Rationalizing denominators and simplifying, we arrive at

$$\sqrt{x+4}-\sqrt{x-4}=2.$$

Multiplying by $\sqrt{x+4}+\sqrt{x+4}$ gives

$$\sqrt{x+4}+\sqrt{x-4}=4.$$

Adding the last two equations leads to

$$2\sqrt{x+4}=6,$$

so

$$x=5,$$

which satisfies the original equation, so the solution is $x=5$.

2. In a lottery, each ticket is printed with a unique 6-digit number. A ticket is called "almost lucky" if the sum of (some) three of its digits is equal to the sum of the other three. Andrew bought two tickets displaying consecutive integers, both of which were almost lucky. Prove that one of these tickets ends with 0.

Solution: Observe, that the sum of digits of an almost lucky number is always even. If both of the two tickets end with nonzero digits, then sum of all digits of one ticket is one less than the sum of the other one. Hence, both sums cannot be even. We get a contradiction.

3. Evaluate

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x}\ dx$$

Solution: Substitute $x \to \frac{\pi}{2} - x$ to obtain the equivalent integral

$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sin x + \cos x}\ dx;$$

adding gives that twice the integeral is

$$\int_0^{\frac{\pi}{2}} 1\ dx$$

so the answer is $\frac{\pi}{4}$.

4. In a class every boy knows at least a half of all girls and any girl knows at most half of all boys. Prove that every boy knows exactly half of all girls and every girl knows exactly half of all boys. (NOTE: person A knows person B if and only if B knows A.)

Solution: Let's b be the number of boys and g the number of girls. There are bg boy-girl pairs. If the two know each other let us call such a pair *good*. From the first fact we have that the number of good pairs is $\geq \frac{bg}{2}$, and from the second fact, it is $\leq \frac{bg}{2}$. So number of good pairs is $\frac{bg}{2}$. It follows that each boy knows exactly half of all girls, and each girl knows exactly half of all boys.

5. Distinct real numbers a, b, c satisfy the following identities:

$$\frac{1-a^3}{a} = \frac{1-b^3}{b} = \frac{1-c^3}{c}.$$

What are all possible values of $a^3 + b^3 + c^3$?

Solution: Let

$$\frac{1-a^3}{a} = \frac{1-b^3}{b} = \frac{1-c^3}{c} = k.$$

Then, a, b, c are the solutions of

$$x^3 + kx - 1 = 0.$$

Now, $a^3+b^3+c^3 = 3abc+(a+b+c)^3-3(a+b+c)(ab+ab+bc) = 3$, since $a+b+c = 0$. (By Viet's relations and relating symmetric functions.)

6. Prove that the following inequality holds for every $a \in \mathbb{R}$.

$$a + a^3 - a^4 - a^6 < 1.$$

Solution: We rewrite the inequality to get

$$a(1 + a^2)(1 - a^3) < 1.$$

We distinguish three cases:
1) $a \leq 0$. Then, $LHS \leq 0 < 1$.

2) $1 \geq a > 0$. We can estimate

$$(a + a^3)(1 - a^3) \leq (1 + a^3)(1 - a^3) = 1 - a^6 < 1.$$

3) $a > 1$. Then, $LHS \leq 0 < 1$.

7. Prove that, for any $n \geq 6$, an equilateral triangle can be divided into n smaller equilateral triangles.

Solution: Let's prove this fact by induction. Base for $n = 6, 7, 8$ is almost trivial. Now suppose we proved statement for n. Then lets take any division of equilateral triangle into n smaller triangles, chose one triangle from division and divide it to 4 smaller triangles. We obtain division into n+3 triangles.

8. Function f defined on the plane is such that for any equilateral triangle ABC we have $f(A)+f(B)+f(C)=0$. Does it follow that, for every point X in the plane, $f(X)=0$?

Solution: For a regular hexagon $ABCDEF$ with centre X the condition gives

$$f(A)+f(B)+f(X)=0, \tag{1}$$
$$f(B)+f(C)+f(X)=0, \tag{2}$$
$$\vdots \tag{3}$$
$$f(F)+f(A)+f(X)=0. \tag{4}$$

Further,

$$f(A)+f(C)+f(E)=0 \tag{5}$$

and

$$f(B)+f(D)+f(F)=0. \tag{6}$$

Adding (1) to (4) and subtracting twice (5) and (6) gives $6f(X)=0$, so $f(X)=0$. So the answer is "yes".

9. Consider a semicircle with base (diameter) on the x-axis, and the two corners at $A = (-x, 0)$ and $B = (x, 0)$. In the first quadrant, there are four points, C, D, E, F on the semicircle so that the distances from each to both A and B are integers; also AB has integer length. What is the smallest such value of x?

Solution: 65/2, because (briefly, to be filled out)

$$\begin{aligned} 65^2 &= 63^2 + 16^2 && (7) \\ &= 60^2 + 25^2 && (8) \\ &= 56^2 + 33^2 && (9) \\ &= 52^2 + 39^2, && (10) \end{aligned}$$

and 65 is the smallest number whose square is the sum of four squares in four ways. [This last fact can be proved as follows: the two smallest reduced pythagorean triples are 3,4,5 and 5,12,13. Thus, 5×13 gives two of the expressions. Also, $65 = 8^2 + 1 = 7^2 + 4^2$ gives the other two expressions.]

10. Solve the following equation:

$$9^x + 2^x = 8^x + 3^x$$

Solution: Two obvious solutions are $x = 0$ and $x = 1$.

Divide both sides by 3^x and rearrange it to express in terms of an exponential function of x as follows:

$$f(x) = 3^x + \left(\frac{2}{3}\right)^x - \left(\frac{8}{3}\right)^x - 1.$$

Thus $f(0) = f(1) = 0$. If there is another solution, then Roll's Theorem provides (at least) two distinct solutions to the equation $f'(x) = 0$. Thus,

$$f'(x) = 3^x \ln 3 + \left(\frac{2}{3}\right)^x \ln\frac{2}{3} - \left(\frac{8}{3}\right)^x \ln\frac{8}{3} = 0.$$

Multiplying by $\left(\frac{3}{8}\right)^x$ gives

$$\left(\frac{9}{8}\right)^x \ln 3 + \left(\frac{1}{4}\right)^x \ln\frac{2}{3} - \ln\frac{8}{3} = 0.$$

So function $g(x) = \left(\frac{9}{8}\right)^x \ln 3 + \left(\frac{1}{4}\right)^x \ln\frac{2}{3} - \ln\frac{8}{3}$ has at least two zeroes. Again, Rolle's Theorem implies that g' has at least one zero. But

$$g'(x) = \left(\frac{9}{8}\right)^x \ln 3 \ln\frac{9}{8} + \left(\frac{1}{4}\right)^x \ln\frac{2}{3} \ln\frac{1}{4} > 0$$

—a contradiction. So the given equation has only solutions $x = 0$ and $x = 1$.