2009 University of Manitoba Open Math Challenge

Are you a "mathlete"? Find out by testing your wits against these problems.

This contest is for undergraduate students in any program. Participating does not commit you to compete on or train for our teams. It costs you nothing but your time. There is no penalty for scoring low; it is for your information—and ours (we use it to select a team for the Putnam competition). Results remain confidential, except for the overall winner who receives **a book prize and $100 cash**, but all participants receive feedback.

The back page contains information about mathletics at U of M this year.

Instructions

Submit (well-)written solutions **before 4:30 PM, Friday, Sept. 18** directly to a coach (see final page) or in a sealed envelope taken to the Math Office (MH 342), or by email before the deadline (either properly typeset or high-quality scan of neat handwriting).

DO NOT BE DISCOURAGED by these questions—they are not intended to be "easy"; *solving just one is an achievement!* Give full solutions in *good form*, each answer adequately justified clearly, efficiently and logically for full credit. Begin each solution on a new sheet of paper. Staple solutions in the same order as questions are given; do not submit junk.

HONOUR SYSTEM: Do not solicit or accept assistance from, or provide it to, others. Do not consult any references or use technology to solve these problems.

(Each question is marked out of 10)

1. For every positive integer n define a_{n+1} to be the sum of 2009-th powers of the decimal digits of a_n, where $a_1 = 2009$ (hence $a_2 = 2^{2009} + 9^{2009}$). Are all the terms of the sequence $\{a_n\}_{n\geq 1}$ distinct numbers?

2. A *3-colouring of the plane* is an assignment of one of three colours (say, red (R), white (W) and blue (B)) to every point of the plane. Let a line L in the plane be given. Find, and clearly describe, all possible 3-colourings of the plane such that every point of L is coloured (assigned) the colour red, and no line has points of all three colours.

3. Find all integers n with the property that every multiple of n is a sum of n consecutive integers.

4. A cake of uniform thickness has some convex shape. Prove that there exist two vertical cuts, in perpendicular directions, that cut the cake into four pieces of exactly the same volume.

5. Prove that if all vertices of a triangle lie on the hyperbola $xy = 1$, then the orthocenter of the triangle (the point of intersection of the altitudes) lies on the same hyperbola.

6. Prove that the product of any seven consecutive positive integers greater than 4 is divisible by a prime greater than 7.

7. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(f(f(x)))) = x$ for all $x \in \mathbb{R}$. Prove that $f(x) = x$ for all $x \in \mathbb{R}$.

8. Prove that for any polynomial $p(x)$ of degree ≤ 3 there exists a point $x \in [-1, 1]$ satisfying $|x^4 - p(x)| \geq \frac{1}{8}$.

Mathletics 2009

TRAINING

- **Training seminar every Tuesday at 5:00-6:30 PM** throughout the Fall term, starting with an ORGANIZATIONAL MEETING Sept. 15 in MH 418. Open to all undergraduates. **Pizza will be served for participants.**
- **Two all-day training sessions, Nov.7 and Nov. 28**.
- **Lunch is provided** on practise days and, on contest days, at fine local restaurants (our treat). We pay all costs associated with both contests:
- **CAN'T ATTEND?** Email us and we'll work something out.

CONTESTS

NCS/MAA team competition, Sat. Nov. 14, 9:00–12:00: In teams of 3, mathletes collaborate to solve 10 questions of varying difficulty. Any students wishing to compete will be placed on a team—we try to have at least 4 teams each year. This regional competition usually involves about 70 teams from Canada and the North Central U.S.; U of M has done very well in the past, almost winning a few times but never ranking first—perhaps you will be on our first team to do so.

Putnam Competition, Sat. Dec. 5, 9:00–5:00: Students attempt 6 problems in each of two 3 hour sessions with a 2 hour break between. The Putnam is considered the most challenging contest of its type in the world. Prizes include cash awards and scholarships, and placing in the top 500 (out of about 4000 North American students writing each year) attracts the attention of Grad schools around the world. We select 3 students whose scores are used to compile a "team" ranking; all others write solely for personal (grief or) glory.

Coaches and contact info

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