2007 University of Manitoba Open Math Challenge

Instructions

Are you a "mathlete"? Here's how to find out.

Undergraduate students in any discipline: test your wits against the following problems. Submit solutions to one of the coaches (see over) directly or in a sealed envelope in our mailboxes in the Math Office **before 4:30 PM, Fri., Oct. 5**. Participants will receive feedback but results will remain confidential, except for the overall winner, who will receive **a book prize and $100 cash**.

Submitting solutions does not commit you to compete on or train for our teams. It costs you nothing but your time and there is no penalty for scoring low; it is simply for your information—and ours: it helps us rank students for the Putnam team in December.

DO NOT BE DISCOURAGED by these questions—they are not intended to be "easy"; *solving even one is an achievement!* Give full solutions in *good form*, each answer adequately justified clearly, efficiently and logically for full credit.

Begin each solution on a new sheet of paper. Staple solutions in the same order as questions are given; do not submit junk. HONOUR SYSTEM: Do not solicit or accept assistance from, or provide it to, others. Do not consult any references or use technology to solve these problems.

(Each question marked out of 10)

1. A circular coin of radius 2 cm rolls all the way around the outside of a circular can of radius 10 cm. Through how many complete rotations does the coin turn in the process? How (if at all) would your answer change if the coin rolled around the *inside* of the can?

2. Seven different positive integers a, b, c, d, e, f, g are given whose sum is 101. Prove that every three of these sum to less than the other four.

3. Solve following the system of two equations in <u>positive integers</u> a, b and c:
$$\begin{aligned} a^3 - b^3 - c^3 &= 3abc \\ a^2 &= 2(b + c). \end{aligned}$$

4. Let a, b, c be distinct integers. Prove that there is no polynomial $p(x)$ with integer coefficients such that $p(a) = b$, $p(b) = c$ and $p(c) = a$. (HINT: You may use the well-known fact that if x, y are integers and $f(x)$ a polynomial with integer coefficients then $x - y$ is a divisor of $f(x) - f(y)$.)

5. How many different pairs of integers (m, n) are there, with $0 \leq m < n \leq 1000$, such that $m^2 + n^2$ is a multiple of 49?

6. Prove that, for any integer $n \geq 2$, $\sqrt{1 + \sqrt{2 + \cdots + \sqrt{n}}}$ is irrational. (HINT: consider two cases: n a square; n not a square.)

7. Let $f(x) = \frac{x+x^3+x^7}{3}$; show that f is invertible and find the value of
$$E = \int_0^1 f(1 - x)dx + \int_0^1 f^{-1}(x)dx.$$

8. Show that, given finitely many points on the plane, not all on the same line, there exists a line containing *exactly* two of the points. (HINT: not as easy as you think!)

9. Show that, given finitely many points in space, not all on the same line, there exists a line containing exactly two of the points.

10. Let S be the set of pairs (x, y), where $x, y \in \{0, 1, 2, 3, 4\}$. The *translation of (x, y) by (a, b)* is $(x+a, y+b)$, with numbers reduced modulo 5. The *translation of a set by (a, b)* is the set obtained by translating all of its elements. For example, the translation of $\{(1, 4), (2, 2)\}$ by $(3, 2)$ is $\{(4, 1), (0, 4)\}$.

Let π denote translation by $(1, 1)$. In the diagram below, let X be the set of points indicated by bold **A**. Its translates $\pi X, \pi^2 X, \pi^3 X, \pi^4 X$ are similarly indicated by B, C, D, E.

$$\begin{array}{c|ccccc}
4 & D & E & E & D & D \\
3 & D & D & C & C & C \\
2 & C & B & B & B & C \\
1 & \mathbf{A} & \mathbf{A} & \mathbf{A} & B & B \\
0 & E & E & \mathbf{A} & \mathbf{A} & E \\
\hline
 & 0 & 1 & 2 & 3 & 4
\end{array}$$

A subset X of five elements of S is a *π-tile* if (i) $S = X \cup \pi X \cup \cdots \cup \pi^4 X$ and (ii) X is *connected*—which means that there is a "path" in X from element of X to any other element consisting of a series of translations by $(0, \pm 1)$ or $(\pm 1, 0)$. For example, $\{(0, 0), (0, 1), (0, 3), (1, 0), (1, 3)\}$ is not a π-tile, for although (i) is satisfied: (ii) is violated because there is no path in X from $(0, 0)$ to $(1, 3)$.

Translates of a single set are *equivalent*. How many inequivalent π-tiles are there?

Mathletics training 2007

Our training seminars are every Tuesday at 2:30 PM, starting Oct. 2 in the Machray Lounge, MH 440E. Open to all undergraduate students.

CAN'T ATTEND? Email us and we'll make arrangements to suit you.

In addition we schedule two all-day training sessions, **Nov. 3 and Nov. 24**. Lunch is provided here on practise days and, on contest days, at fine local restaurants (our treat). We also pay all costs associated with both contests:

NCS/MAA team competition, Sat. Nov. 10, 9:00–12:00: In teams of 3, mathletes collaborate to solve 10 questions of varying difficulty. Any students wishing to compete will be placed on a team—we try to have at least 4 teams each year. This regional competition usually involves about 70 teams from Canada and the North Central U.S.; U of M has done very well in the past, almost winning a few times but never ranking first—perhaps you will be on our first team to do so.

Putnam Competition, Sat. Dec. 1, 9:00–5:00: Students attempt 6 problems in each of two 3 hour sessions with a 2 hour break between. The Putnam is considered the most challenging contest of its type in the world. Prizes include cash awards and scholarships, and placing in the top 500 (out of about 4000 North American students writing each year) attracts the attention of Grad schools around the world. We select 3 students whose scores are used to compile a "team" ranking; all others write solely for personal (grief or) glory.

Coaches and contact info

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- Web page: http://server.maths.umanitoba.ca/~craigen/manitobamathletics/