# University of Manitoba Mathletics 2005 Open Math Challenge Contest 

## Instructions

This quiz is one of the tools we use to select students for the Putnam team. You (all aspiring mathletes) should participate whether or not you wish to be on the team. Participants will receive feedback but your individual results will be communicated to noone but you. This diagnostic test costs you nothing and there is no penalty for scoring low; it is simply for your (and our) information.

Give full solutions in good form (don't submit scratch work); each answer should be adequately justified in a clear and logical fashion. Begin each solution on a new sheet of paper. Do not solicit or accept assistance from others.

Staple in order and submit to one of the coaches (Craigen or Kopotun) or place in a sealed envelope in our mailboxes in the Math Department before 4:30 pm, Friday Sept. 7.

1. Find the least integer $n$ which is the sum of 2005 consecutive positive integers and is also the sum of 2006 consecutive positive integers.
2. Let $M$ and $N$ be numbers having 3 digits, all nonzero, when expressed in base 8 . Let $M^{\prime}$ and $N^{\prime}$ have the same base 8 digits as $M$ and $N$ respectively, but not necessarily in the same order. Prove that $M+N-$ $M^{\prime}-N^{\prime}+1029$ is not prime (1029, when written in base 8 , is 2005).
3. Let $f(x)=x+x^{3}+x^{5}$. Given that $f: \mathbb{R} \rightarrow \mathbb{R}$ is an invertible function and that $f^{-1}(x)$ has the form

$$
f^{-1}(x)=\sum_{k=0}^{\infty} a_{k} x^{k}=a_{0}+a_{1} x+a_{2} x^{2}+\cdots
$$

find the value of $a_{5}$.
4. Suppose $a(2004 a+1)=b(2005 b+1)$, where $a, b$ are nonnegative integers. Prove that $a-b$ is a perfect square.
5. Prove that, for all $x, y, z \geq 0, \frac{x}{2}+\frac{y}{3}+\frac{z}{6} \geq x^{\frac{1}{2}} y^{\frac{1}{3}} z^{\frac{1}{6}}$.
6. Is it possible for $\left\{a_{n}\right\}_{n=1}^{\infty}$ and $\left\{b_{n}\right\}_{n=0}^{\infty}$ to be nonincreasing positive sequences such that (i) $\sum_{n=0}^{\infty} a_{n}$ converges; (ii) $\sum_{n=0}^{\infty} b_{n}$ diverges; and (iii) $a_{n}=b_{n}$ for infinitely many values of $n$ ?

## Solutions

1. Write $n=a+(a+1)+\cdots+(a+2004)=b+(b+1)+\cdots+(b+2005)=$ $2005 a+\frac{2004 \cdot 2005}{2}=2006 b+\frac{2005 \cdot 2006}{2}$. Thus, $2005(a-1)=2006 b$. Since $(2005,2006)=1$ it follows that $2005 \mid b$, so that $b \geq 2005$. The smallest value of $n$, then, is given by $b=2005, a=2007$-that is, $n=2005 \cdot 2007+$ $1002 \cdot 2005=2005 \cdot 3009=6033045$.
2. For all $n$, and $k \geq 0, n^{k} \equiv 1 \bmod n-1$. Therefore the sum of the digits of any positive integer in base $n$ is congruent to that number, modulo $n-1$, and so also to any number obtained by permuting its digits. It follows that $M+N-M^{\prime}-N^{\prime}+2005 \equiv\left(M-M^{\prime}\right)+\left(N-N^{\prime}\right) \equiv 0 \bmod 7$. Further, the largest 3 -digit number in base 8 is $8^{3}-1=511$, so $M+N-M^{\prime}-N^{\prime}+1029>$ $1029-2 \times 511=7$. Therefore this number cannot be prime.
3. Straightforward calculation: Begin with $f^{-1}(f(x))=a_{0}+a_{1}\left(x+x^{3}+\right.$ $\left.x^{5}\right)+a_{2}\left(x+x^{3}+x^{5}\right)^{2}+\cdots=x$. Ignore all terms of degree $>5$. Gather terms having common odd powers of $x$ and equate. We obtain

$$
a_{1} x+\left(a_{1}+a_{3}\right) x^{3}+\left(a_{1}+3 a_{3}+a_{5}\right) x^{5}=x
$$

from which it follows immediately that $a_{1}=1, a_{3}=-1$ and $a_{5}=2$. [Can also be done by differentiating implicitly.]
4. Rewrite the relation as $(a-b)(2004 a+2004 b+1)=b^{2}$. Consider the GCD of the two factors on the left: $(a-b, 2004 a+2004 b+1)=(a-b, 2004 a+$ $2004 b+1-2004(a-b))=(a-b, 4008 b+1)$. Now $4008 b+1$ is clearly relatively prime to $b^{2}$, and so relatively prime to $a-b$, which is a divisor of $b^{2}$; so the above GCD is $1 . a-b$ is a square because it is one of two relatively prime numbers whose product is a square.
5. $f(t)=\ln t$ is a concave function, so Jensen's inequality gives, for $x, y, z>$ $0, \ln \left(\frac{x}{2}+\frac{y}{3}+\frac{z}{6}\right) \geq \frac{\ln x}{2}+\frac{\ln y}{3}+\frac{\ln z}{6}$. Since $g(t)=e^{t}$ is an increasing function, we have $\frac{x}{2}+\frac{y}{3}+\frac{z}{6} \geq e^{\frac{\ln x}{2}+\frac{\ln y}{3}+\frac{\ln z}{6}}=x^{\frac{1}{2}} y^{\frac{1}{3}} z^{\frac{1}{6}}$, as required. If any of $x, y, z$ are 0 then the RHS is 0 but the LHS is $\geq 0$, so the inequality remains valid.
6. Yes, it is possible.

Define $x_{n}:=2^{\frac{-n(n+1)}{2}}, f_{n}:=2^{\frac{n(n-1)}{2}}, n \geq 0$. Let $\left\{a_{k}\right\}$ be the decreasing sequence in which $x_{n}$ appears $f_{n}$ times, and let $\left\{b_{k}\right\}$ be the decreasing sequence in which $x_{n}$ appears $f_{n+1}$ times. Then $\sum a_{n}=\sum_{n=0}^{\infty} f_{n} x_{n}=$ $\sum 2^{-n}$, a convergent (geometric) series, while $\sum b_{n}=\sum_{n=0}^{\infty} f_{n+1} x_{n}=$ $\sum 1=\infty$. Further, the last instance of $x_{n}$ in sequence $\left\{a_{k}\right\}$ occurs in the same position (i.e., $k=\sum_{i=1}^{n} f_{i}$ ) as the the first instance of $x_{n}$ in the sequence $\left\{b_{k}\right\}$, so the sequences coincide infinitely often, as required.
[Many other pairs of sequences work, using the same basic approach: repeated elements.]

