# FOURTEENTH ANNUAL NORTH CENTRAL SECTION MAA TEAM CONTEST

November 13, 2010, 9:00 a.m. to 12:00 noon

NO BOOKS, NOTES, CALCULATORS, COMPUTERS OR NON-TEAM-MEMBERS may be consulted.

**PLEASE BEGIN EACH PROBLEM ON A NEW SHEET OF PAPER.** Team identification and problem number should be given at the top of each sheet of paper submitted.

Each problem counts 10 points. Partial credit for significant but incomplete work. For full credit, answers must be fully justified. But in some cases this may simply mean showing all work and reasoning. Have fun!

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# 1. $\lfloor \log_3 n \rfloor$ is a multiple of 3.

An integer n is drawn at random from the first 2010 positive integers. What is the probability that  $\lfloor \log_3 n \rfloor$  is a multiple of 3? (For real numbers x,  $\lfloor x \rfloor$  denotes the greatest integer less than or equal to x.)

# 2. Find vertices given midpoints.

Find the vertices of the pentagon ABCDE, if the midpoints of the sides AB, BC, CD, DE, and EA are, respectively, P = (1, -7), Q = (8, 4), R = (3, 10). S = (1, 4) and T = (-2, -5).

# 3. Real part of 1/(1-z).

Show that the real part of 1/(1-z) is constant for all complex numbers z with |z| = 1 and  $z \neq 1$ .

# 4. Length of a segment.

A circle is inscribed in an equilateral triangle ABC as shown at the right. The segment DE is tangent to the circle, perpendicular to AB, and its endpoints D and E lie on BC and AB, respectively. If the radius of the circle is 1, find the length |BE| and express it as simply as you can with rational denominator.



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## 5. Sum is 1/2010.

Find two distinct pairs (m, n) of positive integers satisfying

$$\sum_{k=m}^{n} \frac{1}{k(k+1)} = \frac{1}{2010}.$$

## 6. Both have value 7.

Let  $f(x) = ae^{2x} + be^{x} + cx + d$ , where a, b, c and d are real numbers. Given that

$$f(2) + f(5) < 7 < f(3) + f(4),$$

prove that there are real numbers r and s such that r + s = 7 and f(r) + f(s) = 7.

## 7. Limit of a sequence.

Evaluate

$$\lim_{n \to \infty} \int_{1}^{2} (\cos x) (\operatorname{Arctan} \, nx) dx,$$

where Arctan is the principal value arctangent function.

### 8. An unbounded sequence.

Let  $a_0$  be an odd integer,  $a_0 > 5$ , and define the sequence  $\{a_n\}$  recursively by

$$a_{n+1} = a_n^2 - 5$$
 if  $a_n$  is odd,  
 $a_{n+1} = \frac{a_n}{2}$  if  $a_n$  is even.

Prove that the sequence is unbounded.

#### 9. Divisible by 2010.

Show that for every positive integer n,  $1492^n - 1678^n - 1827^n + 2013^n$  is divisible by 2010.

## 10. Three real roots.

Consider the cubic polynomial  $P(x) = x^3 + ax^2 + bx + c$ , where a, b, c are non-zero real numbers satisfying b < c/a < 0. Prove that the equation P(x) = 0 has three distinct real roots.