Polynomials for mathletes—some comments, some solutions

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Contest problems often contain polynomials. Here are a few ways to look at some of these problems.

1 Facts and definitions

A (real) polynomial of degree n in a single variable x is an expression of the form $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, where the a_i's are real numbers with $a_n \neq 0$. The polynomial $p(x)$ is called *monic* iff $a_n = 1$.

Any polynomial of degree n has (counting repetition) n roots (some perhaps complex).

Lemma 1.1 (Bernoulli's inequality)**.** *For non-zero $x > -1$ and integer $n \geq 2$,*

$$(1+x)^n > 1 + nx.$$

(An easy proof is by induction; another proof is derived from the binomial theorem.)

1.1 Symmetric polynomials

A polynomial in n variables is *symmetric* iff permuting variables gives back the same polynomial. Among the symmetric polynomials in n variables, the *elementary symmetric polynomials* are

$$\begin{aligned} s_1 &= x_1 + x_2 + \cdots + x_n \\ s_2 &= x_1x_2 + x_1x_3 + \cdots + x_{n-1}x_n \\ s_3 &= x_1x_2x_3 + x_1x_2x_4 + \cdots + x_{n-2}x_{n-1}x_n \\ \vdots &\quad \vdots \\ s_n &= x_1x_2\cdots x_n, \end{aligned}$$

where for each $k = 1, 2, \ldots, n$, the elementary symmetric polynomial s_k is formed by summing all possible products of k different variables. Observe that any polynomial in symmetric functions is again a symmetric function. For example, with $n = 2$,

$$s_1^2 - 2s_2 = (x+y)^2 - 2xy = x^2 + y^2$$

is symmetric.

Problem 1.2. *Prove that a symmetric polynomial in n variables can be expressed as a polynomial in elementary symmetric functions.*

Hint: Examine $p(x_1, \ldots, x_{n-1}, 0)$.

2 Look for symmetry

Problem 2.1 (K3, Kopotun's sheet, Fall 2004)**.** *Which of the expressions*

$$(1 + x^2 - x^3)^{100} \quad or \quad (1 - x^2 + x^3)^{100}$$

has the larger coefficient for x^{20} after expanding and collecting terms?

Hint: replace $-x$ with x to gain extra information.

3 Use a modulus

Problem 3.1 (from CMO 1969?)**.** *Show that there are no integer solutions to $a^2 + b^2 - 8c = 6$.*

Solution: Consider $a^2 + b^2$ modulo 8. The squares 0^2, 1^2, 2^2, 3^2,..., 7^2 are (modulo 8): 0,1,4,1,0,1, 4,1, no two of of which add to 6. Since $8c$ is a multiple of 8, there are no solutions. □

For the above problem, is the solution the same if 6 is replaced by 3 or 7?

4 Translate variables

Problem 4.1 (33rd Spanish Mathematical Olympiad)**.** *For positive reals a, b, c, prove that*

$$a^2 + b^2 + c^2 - ab - bc - ca \geq 3(b - c)(a - b),$$

and find when equality holds.

Solution: Putting $x = a - b$ and $y = b - c$,

$$\begin{aligned}
&2(a^2 + b^2 + c^2 - ab - bc - ca) - 6(b - c)(a - b) \\
&= (a - b)^2 + (b - c)^2 + (c - a)^2 - 6(b - c)(a - b) \\
&= x^2 + y^2 + (x + y)^2 - 6xy \\
&= 2(x^2 + y^2 - 2xy) \\
&= 2(x - y)^2 \geq 0,
\end{aligned}$$

and so equality above holds iff $x = y$, that is, when $a - b = b - c$, *i.e.*, when $a + c = 2b$. □

Another problem [I do not know its origin, but it has been done in past years' practice sessions here] is:

Problem 4.2. *Solve the system*

$$\begin{aligned}
x^3 + y &= 3x + 4 \\
2y^3 + z &= 6y + 6 \\
3z^3 + x &= 9z + 8.
\end{aligned}$$

Solution One solution is to first put $x = a + 2$, $y = b + 2$, $z = c + 2$, giving

$$a(a + 3)^2 + b = 0 \tag{1}$$
$$2b(b + 3)^2 + c = 0 \tag{2}$$
$$3c(c + 3)^2 + a = 0. \tag{3}$$

If $a > 0$, then by (1), $b < 0$, and by (3), $c < 0$; but also $b < 0$ implies by (2) that $c > 0$, an impossibility. Similarly, if $a < 0$ one gets a contradiction. So conclude that $a = 0$. Then (1) shows that $b = 0$ and consequently (2) yields $c = 0$. So all of a, b, c are zero, giving $x = y = z = 2$. □

5 Some polynomial factoring

Problem 5.1. *Factor the polynomial $x^{10} + x^5 + 1$ as a product of two lesser degree polynomials.*

Solution:

$$\begin{aligned}
x^{10} + x^5 + 1 &= \frac{(x^5)^3 - 1}{x^5 - 1} \\
&= \frac{x^{15} - 1}{x^5 - 1} \\
&= \frac{(x^3)^5 - 1}{(x - 1)(x^4 + x^3 + x^2 + x + 1)} \\
&= \frac{(x^3 - 1)(x^{12} + x^9 + x^6 + x^3 + 1)}{(x - 1)(x^4 + x^3 + x^2 + x + 1)} \\
&= \frac{(x^2 + x + 1)(x^{12} + x^9 + x^6 + x^3 + 1)}{x^4 + x^3 + x^2 + x + 1} \\
&= (x^2 + x + 1)(x^8 - x^7 + x^5 - x^4 + x^3 - x + 1).
\end{aligned}$$

Problem 5.2 (K6, Kopotun's sheet, Fall 2004)**.** *Factor $a^3 + b^3 + c^3 - abc$.*
–3abc is intended

Hint: one factor is $a + b + c$.

Problem 5.3 (Kopotun's sheet K4, Fall 2004)**.** *Find the remainders upon dividing the polynomial*

$$x + x^3 + x^9 + x^{27} + x^{81} + x^{243}$$

by (a) $x - 1$; (b) $x^2 - 1$, (c) $x^3 - 1$.

Hint: for (b): Work modulo $x^2 - 1$, and so use $x^2 \equiv 1$.

6 Given roots, find a polynomial

Problem 6.1 ([3, p. 46])**.** *Show that*

$$\sqrt[3]{20 + 14\sqrt{2}} + \sqrt[3]{20 - 14\sqrt{2}} = 4.$$

Solutions: One solution is rather simple, however might require inspiration. Notice that

$$\begin{aligned}(2+\sqrt{2})^3 &= 2^3 + 3\cdot 2^2\sqrt{2} + 3\cdot 2\cdot 2 + 2\sqrt{2}\\ &= 20 + 14\sqrt{2},\end{aligned}$$

and similarly, $(2-2\sqrt{2})^3 = 20 - 14\sqrt{2}$, from which the result follows. What if one did not "notice" the above? Set x to the left hand side of the desired identity. Cubing the sum of two cube roots usually leaves one more cube-root to deal with, but in this case, one gets lucky. So that things will fit on one line, put $a = 20+14\sqrt{2}$ and $b = 20-14\sqrt{2}$; note that $a+b = 40$ and $ab = 20^2 - 14^2\cdot 2 = 8$. Then

$$\begin{aligned}x^3 &= a + 3a^{2/3}b^{1/3} + 3a^{1/3}b^{2/3} + b\\ &= 40 + 3(ab)^{1/3}(a^{1/3}+b^{1/3})\\ &= 40 + 3(8)^{1/3}x\\ &= 40 + 6x.\end{aligned}$$

So x satisfies $x^3 - 6x - 40 = 0$. It is trivial to check that $x = 4$ is a root of this equation (verifying the original identity), and upon factoring,

$$x^3 - 6x - 40 = (x-4)(x^2 - 4x + 10),$$

the second factor of which has only complex roots. Since x is evidently real, $x = 4$ is the only solution. □

7 Viete's relations

Most material here is taken from [1], some based on questions from Mathematical Olympiad contests.

Francis Viète (1540–1603) was one of the first to use letters as variables, a practice that allows one to generalize formulae. He wrote *The analytic art*, where he used linear transformations were used to solve some quadratic, cubic, quartic equations.

Theorem 7.1 (Viete's relations)**.** *Let $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$, and let $c_1,\ldots,c_n$ be its roots (real or complex). Then*

$$\begin{aligned}c_1 + c_2 + \cdots + c_n &= -\frac{a_{n-1}}{a_n},\\ c_1c_2 + c_1c_3 + \cdots + c_{n-1}c_n &= \frac{a_{n-2}}{a_n},\\ c_1c_2c_3 + c_1c_2c_4 + \cdots + c_{n-2}c_{n-1}c_n &= -\frac{a_{n-3}}{a_n},\\ &\vdots\\ c_1c_2\cdots c_n &= (-1)^n\frac{a_0}{a_n}.\end{aligned}$$

Problem 7.2. *Prove Viéte's relations.*

Problem 7.3. *If $x+y+z=0$, prove that*

$$\left(\frac{x^2+y^2+z^2}{2}\right)\left(\frac{x^5+y^5+z^5}{5}\right)=\frac{x^7+y^7+z^7}{7}.$$

Outline: Consider a polynomial $p(t)=t^3+pt+q$ with roots $t=x,y,z$. Begin by using the identity

$$x^2+y^2+z^2=(x+y+z)^2-2(xy+xz+yz).$$

By the second Viete relation, $(xy+xz+yz)=p$, so

$$x^2+y^2+z^2=-2p.$$

Adding the three equalities $0=x^3+px+q$, $0=y^3+py+q$, and $0=z^3+pz+q$, (using $x+y+z=0$) gives

$$x^3+y^3+z^3=-3q.$$

Then show that $x^4+y^4+z^4=2p^2$, and use this to derive

$$x^5+y^5+z^5=-p(x^3+y^3+z^3)-q(x^2+y^2+z^2)=5pq,$$

and

$$x^7+y^7+z^7=-p(x^5+y^5+z^5)-q(x^4+y^4+z^4)=-5p^2q-2p^2q=-7p^2q.$$

Problem 7.4 ([1, 1.8.1, p. 29])**.** *If a,b,c are non-zero real numbers satisfying*

$$(ab+bc+ca)^3=abc(a+b+c)^3,$$

then prove that a,b,c are terms in geometric sequence.

Hint: Consider a monic cubic polynomial with roots a,b,c. Also, $(a,b,c)=(a,\alpha a,\alpha^2 a)$ forms a geometric sequence iff $b^2=ac$.

Solution: Let $f(x)=x^3+mx^2+nx+p$ have roots a,b,c. By Viete's relations,

$$\begin{aligned} a+b+c&=m;\\ ab+bc+ca&=n;\\ abc&=-p. \end{aligned}$$

The original equality then becomes $n^3=-pm^3$. Consider the two cases depending upon whether or not $m=0$.

Case 1: Suppose $m\neq 0$. Then $p=(m/n)^3$, and so $f(x)=0$ becomes

$$x^3+mx^2+nx+(n/m)^3=0,$$

and mulitplying through by m^3 gives

$$m^3x^3 + m^4x^2 + nm^3x + n^3 = 0.$$

After a moment of reflection, see that this factors:

$$(mx + n)(m^2x^2 + (m^3 - mn)x + n^2) = 0.$$

So one root is $x_1 = -\frac{n}{m}$, and the other two satisfy (by the last Viete relation for the polynomial in the second parentheses), $x_2x_3 = \frac{n^2}{m^2}$. Hence $x_1^2 = x_2x_3$. Then $(a, b, c) = (x_2, x_1, x_3)$ form a geometric progression (with ratio $\sqrt{c/b}$). □

Problem 7.5 ([1, 1.8.2, p. 30])**.** *Find all solutions in real numbers to the system*

$$\begin{aligned} x + y + z &= 4 \\ x^2 + y^2 + z^2 &= 14 \\ x^3 + y^3 + z^3 &= 34. \end{aligned}$$

Hint: Consider a monic cubic polynomial with roots x, y, z.
Solution: Let $p(t) = t^3 + at^2 + bt + c$ have roots x, y, z. By Viete's relations, $a = -(x+y+z) = -4$, and so $p(t) = t^3 - 4t^2 + bt + c$. From the identity

$$x^2 + y^2 + z^2 = (x + y + z)^2 - 2(xy + yz + zx),$$

it follows that

$$14 = 4^2 - 2(b),$$

and so $b = 1$. Thus, $p(t) = t^3 - 4t^2 + t + c$. The only information not yet used is the third of the original equations; such cubes can arise from putting the roots in $p(t)$:

$$\begin{aligned} x^3 - 4x^2 + x + c &= 0; \\ y^3 - 4y^2 + y + c &= 0; \\ z^3 - 4z^2 + z + c &= 0. \end{aligned}$$

Adding these three equations, using the given equations find $c = 6$. So $p(t) = t^3 - 4t^2 + t + 6$. Observing that $t = -1$ is a root, find the factorization

$$p(t) = (t + 1)(t^2 - 5t + 6) = (t + 1)(t - 2)(t - 3),$$

which has roots -1,2,3. So x, y, z are $-1, 2, 3$ in any order. □

Problem 7.6 ([1, 1.8.3, p. 31], from USAMO and US selection tests)**.** *Let a and b be two roots of $f(x) = x^4 + x^3 - 1$. Prove that ab is a root of $g(x) = x^6 + x^4 + x^3 - x^2 - 1$.*

Solution: Let c and d be the other two roots of $f(x)$. Then Viete's relations give

$$\begin{aligned} a+b+c+d &= -1; \\ ab+ac+ad+bc+bd+cd &= 0; \\ abc+abd+acd+bcd &= 0; \\ abcd &= -1. \end{aligned}$$

Group a with b and c with d by setting $s = a+b$, $t = c+d$, (with foresight) $p = ab$, and $q = cd$. Then the Viete's relations become

$$\begin{aligned} s+t &= -1; \\ p+st+q &= 0 \\ pt+qs &= 0 \\ pq &= -1. \end{aligned}$$

Solving for $t = -1-s$ in the first equation and $q = \frac{-1}{p}$ from the last into the second and third equations gives

$$\begin{aligned} p - s^2 - s - \frac{-1}{p} &= 0; \\ p(1-s) - \frac{s}{p} &= 0. \end{aligned}$$

The last of these two equations yields $s = \frac{-p^2}{p^2+1}$. Putting this into the first of these two equations gives

$$p - \frac{p^4}{(p^2+1)^2} + \frac{p^2}{p^2+1} - \frac{1}{p} = 0.$$

With a little work, this last equation yields

$$p^6 + p^4 + p^3 - p^2 - 1 = 0.$$

Thus $p = ab$ is a root of $g(x)$. □

Problem 7.7 ([1, 1.8.4, p. 32])**.** *Let a, b, c be non-zero reals with $a+b+c \neq 0$. Show that if*

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{a+b+c}, \tag{4}$$

then for all odd positive integers n,

$$\frac{1}{a^n} + \frac{1}{b^n} + \frac{1}{c^n} = \frac{1}{a^n+b^n+c^n}.$$

Hint: Look at a monic cubic polynomial with roots a, b, c.

Solution: For some p, q, r, let $f(x) = x^3 + px^2 + qx + r$ have roots a, b, c. By Viete's relations, $p = -(a+b+c)$, $q = ab+bc+ca$, and $r = -abc$. Putting the left side of (4) over a common denominator and cross multiplying gives

$$(a+b+c)(ab+bc+ca) = abc,$$

and so $pq = r$. Hence

$$\begin{aligned} f(x) &= x^3 + px^2 + qx + pq \\ &= x^2(x+p) + q(x+p) \\ &= (x^2+q)(x+p). \end{aligned}$$

Suppose that one of the roots, say a, is $-p = a+b+c$; then $b+c=0$, and hence $c=-b$. It then remains to show that

$$\frac{1}{a^n} + \frac{1}{b^n} + \frac{1}{(-b)^n} = \frac{1}{a^n + b^n + (-b)^n},$$

which is clearly true since n is odd. The same proof works if one of the roots is b or c. □

Problem 7.8 ([1, 1.8.10], from *Gazetta Matematica*)**.** *Find all solutions in real numbers to the system*

$$x + y + z = 0 \tag{5}$$

$$x^3 + y^3 + z^3 = 18 \tag{6}$$

$$x^7 + y^7 + z^7 = 2058. \tag{7}$$

Hint: Let x, y, z be roots of some polynomial $p(t)$.

Solution: Let $p(t) = t^3 + at^2 + bt + c$ have roots x, y, z. Then by Viete's relations, $x+y+z = -a$, but then equation (5) shows $a = 0$. Hence $p(t) = t^3 + bt + c$. Since x, y, z are roots,

$$\begin{aligned} x^3 + bx + c &= 0, \\ y^3 + by + c &= 0, \\ z^3 + bz + c &= 0. \end{aligned}$$

Adding these equations and appling equations (6) and (5) gives $18 + 3c = 0$; hence $c = -6$. Thus $p(t) = t^3 + bt - 6$.

It remains to find b; this can be done by using (7), if you are persistent. For this, some kind of recursion is useful. For any positive integer n, multiplying each of the last three displayed equations by x^3, y^3 and z^3, respectively,

$$\begin{aligned} x^{n+3} + bx^{n+1} - 6x^n &= 0, \\ y^{n+3} + by^{n+1} - 6y^n &= 0, \\ z^{n+3} + bz^{n+1} - 6z^n &= 0. \end{aligned}$$

Adding these three equations gives

$$x^{n+3} + y^{n+3} + z^{n+3} + b(x^{n+1} + y^{n+1} + z^{n+1}) - 6(x^n + y^n + z^n) = 0.$$

Letting S_k denote $x^k + y^k + z^k$, this last equation becomes, for all $n \geq 1$

$$S_{n+3} + bS_{n+1} - 6S_n = 0.$$

Using the recursion $S_{n+3} = -bS_{n+1} + 6S_n$,

$$\begin{aligned} 2058 = S_7 &= -bS_5 + 6S_4 \\ &= -b(-bS_3 + 6S_2) + 6(-bS_2 + 6S_1) \\ &= b^2 S_3 - 12bS^2 + 36S_1. \end{aligned}$$

Using $S_3 = 18$ and $S_1 = 0$ gives

$$2058 = b^2 \cdot 18 - 12bS_2.$$

To get rid of the S_2, note that

$$S_2 = (x + y + z)^2 - 2(xy + yz + zx) = 0 - 2b = -2b,$$

and so the previous equation becomes

$$2058 = 18b^2 - 12b(-2b) = 42b^2,$$

which reduces to $49 = b^2$. Hence $b = \pm 7$. However, if one uses $b = 7$, the polynomial $t^3 + 7t - 6$ has only one real root (it increases always by a first derivative test), contrary to the requirement that x, y, z are all real. So $b = -7$, in which case one can verify that $t^3 - 7t - 6 = (t + 1)(t + 2)(t - 3)$. Thus the solutions are any permutation of -1,-2, and 3. □

8 More than one variable: scaling and homogeneous functions

There are many meanings of the word "homogeneous"; here is one more. If $f(x_1, x_2, \ldots, x_n)$ is a function of n variables, say that f is *homogeneous of order* d iff for any constant $t > 0$, $f(tx_1, tx_2, \ldots, tx_n) = t^d f(x_1, x_2, \ldots, x_n)$. For example, the expression $f(x, y, z) = \frac{x^3+y^3}{xyz}$ is homogeneous of order 1. The advantage of having an equation with two expressions that are homogeneous of the same order is that, if one needs, one can first prove an equality for, say, a particular 'size' of x, y, z, then later 'scale' the variables to any sizes.

For example, a problem that at first seems quite hard is the following:

Problem 8.1. *Show that for any non-negative real numbers $x_1, x_2, \ldots, x_n$,*

$$\frac{x_1^2 + x_2^2 + \cdots + x_n^2}{n} x_1 x_2 \cdots x_n \leq \left(\frac{x_1 + x_2 + \cdots + x_n}{n} \right)^{n+2}.$$

Notice that both sides are homogeneous of order $n + 2$.

The renowned problem poser/solver Murray S. Klamkin gave this inequality as Problem 1324 in *Mathematics Magazine*, June 1989; in fact, the problem was actually proposed with the added condition $x_1 + x_2 + \ldots + x_n = 1$, and many solutions were received which used the method of Lagrange multipliers (a method from multivariate calculus often used to solve problems with such constraints), however Klamkin gave a solution [5] which was by induction on n. His solution is given here, however with just a few more details supplied. Some simple algebraic steps are still left to the reader.

Solution: Let $x_1, x_2, \ldots, x_n$ be non-negative reals, and let $S(n)$ be the statement

$$\frac{x_1^2 + x_2^2 + \cdots x_n^2}{n} x_1 x_2 \cdots x_n \leq \left(\frac{x_1 + x_2 + \cdots + x_n}{n} \right)^{n+2}.$$

Rewrite $S(n)$ as

$$(x_1^2 + x_2^2 + \cdots x_n^2) x_1 x_2 \cdots x_n \leq n \left(\frac{x_1 + x_2 + \cdots + x_n}{n} \right)^{n+2}.$$

First observe that the inequality is trivial if any of the x_i's are 0, so assume that each $x_i > 0$.
Base steps: For $n = 1$, $S(1)$ says $x_1^3 \leq x_1^3$. For $n = 2$, $S(2)$ says

$$(x_1^2 + x_2^2) x_1 x_2 \leq \frac{1}{8}(x_1 + x_2)^4,$$

which reduces to $0 \leq (x_1 - x_2)^4$, which is certainly true.
Inductive step: Let $k \geq 2$ be fixed and suppose that $S(k)$ holds:

$$S(k): \quad (x_1^2 + x_2^2 + \cdots x_k^2) x_1 x_2 \cdots x_k \leq (k) \left(\frac{x_1 + x_2 + \cdots + x_k}{k} \right)^{k+2}.$$

It remains to prove (using $x = x_{k+1}$)

$$S(k+1): \quad (x_1^2 + \cdots + x_k^2 + x^2) x_1 x_2 \cdots x_k x \leq (k+1) \left(\frac{x_1 + \cdots + x_k + x}{k+1} \right)^{k+3}.$$

Put $A = \dfrac{x_1 + x_2 + \cdots + x_k}{k}$ and $P = x_1 x_2 \cdots x_n$. With this notation, the induction hypothesis assumed is

$$S(k): \quad (x_1^2 + x_2^2 + \cdots x_k^2) P \leq k A^{k+2}$$

and it remains to prove

$$S(k+1): \quad (x_1^2 + x_2^2 + \cdots x_k^2 + x^2) P x \leq (k+1) \left(\frac{kA + x}{k+1} \right)^{k+3}.$$

The left hand side of $S(k+1)$ is

$$\begin{aligned} (x_1^2 + x_2^2 + \cdots x_k^2 + x^2) P x &= (x_1^2 + x_2^2 + \cdots x_k^2) P x + P x^3 \\ &\leq k A^{k+2} x + P x^3 \quad \text{(by } S(k)\text{)}. \end{aligned}$$

So to prove $S(k+1)$, it suffices to prove that

$$k A^{k+2} x + P x^3 \leq (k+1) \left(\frac{kA + x}{k+1} \right)^{k+3}.$$

By the AM-GM inequality, $P \leq A^k$, so it suffices to prove

$$k A^{k+2} x + A^k x^3 \leq (k+1) \left(\frac{kA + x}{k+1} \right)^{k+3}.$$

Now restrict to the situation where the sum $x_1+x_2+\cdots+x_k+x$ is held constant, and prove the result with this added constraint. The general result then follows immediately; observe that for any constant c, the statement $S(n)$ holds for $x_1,\ldots,x_n$ if and only if it holds for $cx_1,\ldots,cx_n$ (the factor c^{n+2} appears on each side). So, consider only those $(x_1,\ldots,x_k,x)\in\mathbb{R}^{k+1}$ for which $x_1+x_2+\cdots+x_k+x=k+1$, that is,

$$kA+x=k+1.$$

So, to prove $S(k+1)$, it suffices to show

$$kA^{k+2}x+A^kx^3\le k+1.$$

The left hand of the above inequality is a function of A (and $x=k+1-kA$, also a function of A), and so this expression is maximized using calculus:

$$\begin{aligned}\frac{d}{dA}[kA^{k+2}x+A^kx^3] &= k(k+2)A^{k+1}x+kA^{k+2}\frac{dx}{dA}+kA^{k-1}x^3+A^k3x^2\frac{dx}{dA}\\ &= k(k+2)A^{k+1}x+kA^{k-1}x^3-k(kA^{k+2}+A^k3x^2).\end{aligned}$$

Putting $A=tx$, this expression becomes (after a bit of algebra)

$$(1-t)(kt^2-2t+1)kt^{k-1}x^{k+2}.$$

Since $k\ge 2$, the above has roots at only $t=0$ and $t=1$, and so the derivative is positive for $0<t<1$ and negative for $t>1$. Thus, $kA^{k+2}x+A^kx^3$ achieves a maximum when $t=1$, that is, when $A=x=1$. Hence,

$$kA^{k+2}x+A^kx^3\le k+1,$$

and so $S(k+1)$ follows, completing the inductive step.

Thus, by mathematical induction, for all $n\ge 1$, the statement $S(n)$ is true. □

References

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