Mathletics Tools II

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1 Using homogeneity to simplify complex problems

An expression or relation involving quantities $a, b, c, \ldots$ is **homogeneous** if its validity is not affected when $a, b, c, \ldots$ are all multiplied by a common scalar factor, $\lambda > 0$.

Thus the relation

$$F(a, b, c, \ldots) = 0$$

is homogeneous if it is equivalent, for all $\lambda > 0$, to

$$F(\lambda a, \lambda b, \lambda c, \ldots) = 0.$$

An inequality (of any type ... and other kinds of relations) might also be homogeneous:

$$F(a, b, c, \ldots) > 0 \Leftrightarrow \forall \lambda > 0, F(\lambda a, \lambda b, \lambda c, \ldots) > 0,$$

$$F(a, b, c, \ldots) \leq 0 \Leftrightarrow \forall \lambda > 0, F(\lambda a, \lambda b, \lambda c, \ldots) \leq 0, \ldots$$

For example, the relation $ab + bc + ac = a^2 + b^2 + c^2$ is homogeneous because for all $\lambda > 0$,

$$\begin{aligned}(\lambda a)(\lambda b) + (\lambda b)(\lambda c) + (\lambda a)(\lambda c) &= (\lambda a)^2 + (\lambda b)^2 + (\lambda c)^2 \\ \Leftrightarrow \lambda^2(ab + bc + ac) &= \lambda(a^2 + b^2 + c^2) \\ \Leftrightarrow ab + bc + ac &= a^2 + b^2 + c^2.\end{aligned}$$

The first exercise in our set shows how homogeneity may be used to transform a problem involving a difficult-looking relation into a completely different sort of problem.

2 Convex sets and the convex hull

A set in the plane (or Euclidean space of any dimension) is **convex** if the segment joining any two elements in the set is entirely contained in the set.

Thus, a solid ball in $\mathbb{R}^3$ is convex, but the sphere which forms its surface is not.

Examples of convex sets in $\mathbb{R}^2$: (sketch in your own)

Examples of non-convex sets in $\mathbb{R}^2$: (sketch in your own)

Theorem: The segment with endpoints A and B is given by

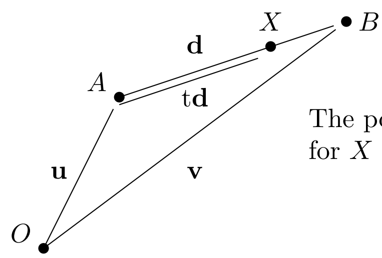
$$\{sA + tB \mid s + t = 1, s, t \geq 0\} = \{(1 - t)A + tB \mid 0 \leq t \leq 1\}$$

(Observe, for example, that $X = (1 - t)A + tB$ is the point A when $t = 0$, the point B when $t = 1$ and the point halfway between, $\frac{1}{2}A + \frac{1}{2}B$, when $t = \frac{1}{2}$.)

Proof: (assumes familiarity with first year linear algebra, and uses vector notation—my apologies, I'm currently using a graphics editor that has a problem with arrowheads):
Let $\mathbf{u} = \overrightarrow{OA}$ and $\mathbf{v} = \overrightarrow{OB}$ (the position vectors for points A, B. Then write $\mathbf{d} == \overrightarrow{AB} = \mathbf{v} - \mathbf{u}$. Thus by vector addition the position vector for the point between A and B at distance $t|\mathbf{d}|$ from A $(0 \leq s \leq 1)$ is

$$\mathbf{u} + s\mathbf{d} = s(\mathbf{v} - \mathbf{u})$$

Diagramatically:



The position vector for X is $\mathbf{u} + t\mathbf{d}$.

This formula works for A, B in the number line $\mathbb{R}$ ... or in $\mathbb{R}^2$ or in $\mathbb{R}^3$ or for that matter in $\mathbb{R}^n$, $n \in \mathbb{Z}^+$.

The **convex hull** of points $A_1, A_2, \ldots, A_n$ the set of **convex combinations** of them:

$$\{s_1 A_1 + s_2 A_2 + \cdots + s_n A_n \mid s_1 + s_2 + \cdots + s_n = 1, s_i \geq 0, i = 1, \ldots, n\}$$

Theorem: The convex hull of any set of points in $\mathbb{R}^n$ is convex.

Proof: Consider arbitrary $X, Y \in H$, the convex hull of $A_1, A_2, \ldots, A_n$. That is, for coefficients s_i, t_i with $s_1 + s_2 + \cdots + s_n t_1 + t_2 + \cdots + t_n = 1$, we have

$$X = s_1 A_1 + s_2 A_2 + \cdots + s_n A_n, \quad Y = t_1 A_1 + t_2 A_2 + \cdots + t_n A_n$$

Now consider an arbitrary point on the segment $\overline{XY}$, say

$$sX + tY = s(s_1 A_1 + s_2 A_2 + \cdots + s_n A_n) + t(t_1 A_1 + t_2 A_2 + \cdots + t_n A_n) = (ss_1 + tt_1)A_1 + (ss_2 + tt_2)A_2 + \cdots + (ss_n + tt + n)$$

where $s + t = 1$. The sum of the coefficients in this expression is

$$ss_1 + tt_1 + ss_2 + tt_2 + \cdots + ss_n + tt + n = s(s_1 + \cdots + s_n) + t(t_1 + \cdots + t_n) = s \cdot 1 + t \cdot 1 = 1$$

(since $s_1 + \cdots + s_n = t_1 + \cdots + t_n = s + t = 1$). It follows that $sX + tY \in H$, proving the result. $\square$

3 Convex functions

A function $y = f(x)$ is **convex** on interval I if the region above its graph for $x \in I$ is a convex (infinite) set in $\mathbb{R}^2$. (It doesn't matter if I is an open, close, half-open, finite or infinite interval!)

This definition amounts to "concave up" in calculus—and the tools of calculus are one way to test for convexity.

f is **concave** on interval I if $-f$ is convex—or equivalently, the region below I is convex, or (in calculus terms) the graph of $y = f(x)$ is "concave down".

For example, $f(x) = x^2$ is convex. on any interval. $\sin x$ is concave on $[0, \pi]$, and convex on $[-\pi, 0]$ but neither on $(-\frac{\pi}{2}, \frac{\pi}{2})$.

We can similarly define **concave** functions, which correspond to "concave down" in calculus.

4 Jensen's inequality: A multi-purpose inequalities tool

Now observe that the average (i.e., arithmetic mean) of a set of points, $\frac{1}{n}(A_1 + \ldots + A_n)$, is a convex combination of them—and so lies in their convex hull. This is all we require to prove the following powerful tool!

Theorem: Suppose f is a convex function on interval I, $x_1, x_2, \ldots, x_n \in I$. Write $\overline{x} := \frac{\sum_i x_i}{n}$ and $\overline{f} := \frac{\sum_i f(x_i)}{n}$. Then

$$f(\overline{x}) \le \overline{f}$$

Proof: For each i write $X_i = (x_i, f(x_i))$, giving n points on the curve $y = f(x)$. Their average,

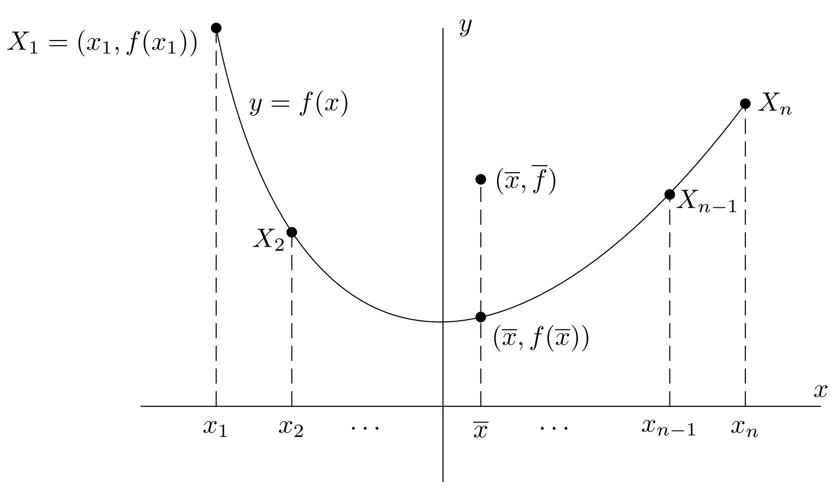
$$\frac{1}{n}(X_1 + \cdots + X_n) = (\overline{x}, \overline{f})$$

is a convex combination of these points, whose convex hull lies above the curve because f is convex on I. It follows that

$$f(\overline{x}) \le \overline{f}$$

□

The proof can be seen in a glance using the following diagram, which has the advantage of enabling you to use both convex and concave cases without getting confused as to which is which (even if you forget whether "convex" means "concave up" or "concave down" — as long as you know which way the graph bends, you're "good to go":



The case of equality in Jensen's is obtained under fairly obvious conditions, which we will not state here in generality—if the graph of f includes line segments then the statement is a bit tricky, but if not, then we say f is "strictly convex"–ie., $f'' > 0$ on I, and equality occurs when $x_1 = x_2 = \cdots = x_n$).

You can restate Jensen's inequality for concave functions—just reverse the inequality. Or, as suggested above, just rely on the diagram to guide you to whichever version of the result may apply.

5 Generalizations of Jensen

If you grasp the use of convexity in Jensen's inequality you will understand that other versions of the result may be stated. For example, there is no reason we must restrict ourselves to the ordinary arithmetic mean, $\frac{1}{n}(X_1 + \cdots + X_n)$ of points $X_i = (x_i, f(x_i))$.

Suppose we write

$$\overline{x} := a_1x_1 + a_2x_2 + \cdots + a_nx_n, \text{ and } \overline{f} := a_1f(x_1) + a_2f(x_2) + \cdots + a_nf(x_n)$$

where $a_1 + \cdots + a_n = 1$—that is, $\overline{f}$ is the same convex combination of the values $f(x_i)$ as $\overline{x}$ is of the values x_i—then the stated result still applies, using the fact that this **weighted mean** of the points $X_1, \ldots, X_n$ is in their convex hull.

Further you'll realise that a similar diagram can be used to obtain a similar result for a function of two variables $f(x, y)$ whose graph (in $\mathbb{R}^3$) is a convex (or concave) surface. Here you'll work with points $X_i = (x_i, y_i, f(x_i, y_i))$ in space.

Similarly one might extend this to higher dimensions and functions of more variables. The task of proving convexity may, however, be a bit of a challenge.

6 Master these tools and watch for places to use them!

Try going back over problems in all the prior worksheets and ask yourself—particularly for inequality problems (but by no means limited to these)—whether there is some homogeneous relation which may be rescaled to simplify things, or if something can be turned into a Jensen inequality question.

Also get in the habit of noticing situations where these tools might come in handy!

7 Problems

1. Prove that, for all $a, b, c \in \mathbb{R}^+$,

$$E = \frac{2a}{b+c} + \frac{2b}{a+c} + \frac{2c}{a+b} \geq 3$$

Solution: *E is homogeneous in variables a, b, c. Therefore we can assume WLOG that the variables are scaled so that $a+b+c = 1$, and so $E = \frac{2a}{1-a} + \frac{2b}{1-b} + \frac{2c}{1-c} = f(a)+f(b)+f(c)$, where $f(x) = \frac{2x}{1-x}$.*

We prove that $f(a) + f(b) + f(c) \geq 3$.

Differentiating twice we obtain $f''(x) = \frac{4}{(1-x)^{-3}}$, so f is a convex function (i.e., $y = f(x)$ is concave down) for $x < 1$. Now since $a, b, c > 0$ and $a + b + c = 1$ we infer that $0 < a, b, c < 1$ and so lie in an interval in which f is convex. So by Jensen's inequality, $f(\overline{x}) \leq \overline{f}$. That is,

$$f\left(\frac{a+b+c}{3}\right) \leq \frac{f(a) + f(b) + f(c)}{3}.$$

Thus, $f(\frac{1}{3}) = \frac{2 \cdot \frac{1}{3}}{1-\frac{1}{3}} = 1 \leq f(a) + f(b) + f(c)$, or $f(a) + f(b) + f(c) \geq 3$.

2. A, B and C are the angles of a triangle. Show that

$$\sin A + \sin B + \sin C \leq \frac{3\sqrt{3}}{2}.$$

Hint $f(x) = \sin x$ is concave on $[0, \pi]$

3. Suppose $a + b + c = 1$ and $a, b, c > 0$. Show that

$$\left(a + \frac{1}{a}\right)^2 + \left(b + \frac{1}{b}\right)^2 + \left(c + \frac{1}{c}\right)^2 \geq \frac{100}{3}$$

Hint: $f(x) = \left(x + \frac{1}{x}\right)^2$

4. Use Jensen's inequality to prove the AM/GM inequality

Hint: $f(x) = \ln x$ (or e^x)

5. Use Jensen's inequality to prove the QM/AM inequality

Hint: $f(x) = x^2$ (or $\sqrt{x}$)

6. Use Jensen's inequality to prove the AM/HM inequality

Hint: $f(x) = \frac{1}{x}$

7. (2001 IMO) For $a, b, c \in \mathbb{R}^+$, prove that

$$\frac{a}{\sqrt{a^2 + 8bc}} + \frac{b}{\sqrt{b^2 + 8ac}} + \frac{c}{\sqrt{c^2 + 8bc}} \geq 1.$$

Hint: Use homogeneity and scale appropriately (hint, hint: this time don't use $a + b + c$—make something else equal to 1!). Pick an appropriate function and recast in terms which can be solved by Jensen's inequality.

8. Let n be any integer, and let $a_1, a_2, \ldots, a_n > 0$. Define the function

$$M(t) = \left(\frac{a_1^t + a_2^t + \cdots + a_n^t}{n}\right)^{\frac{1}{t}}.$$

Theorem (Power Mean Inequality): Let x, y be any real numbers, subject to the condition that, if negative, they must be integers. If $x < y$ then $M(x) \leq M(y)$. Use Jensen's inequality to prove the Power-Mean Inequality.

Hint: Consider $f(x) = x^{\frac{x}{y}}$. You'll have to do something else too...I suggest introducing another list of numbers in addition to $a_1, a_2, \ldots, a_n$ ...

9. Prove that the set of convex combinations of points A, B, C in $\mathbb{R}^2$ (or in any number of dimensions for that matter!) is the triangle with vertices A, B, C.

Hint: Consider $X = sA + tB + uC$ where $s, t, u \geq 0$ and $s + t + u = 1$, and let $s = \lambda s'$ and $t = \lambda t'$, with λ chosen so that $s' + t' = 1$. What can you say about the point $Y = s'A + t'B$? What can you say about the sum $\lambda + u$? What can you say about the point $Z = \lambda Y + uC$?

10. Use the power mean inequality to prove as many parts of the QAGH (QM $\geq$ AM $\geq$ GM $\geq$ HM) inequality as you can. Can you think of a way to complete the last couple of parts?

11. It should be a pretty obvious fact, but it's worth the effort: prove (rigorously) that the convex hull of any finite set of points is contained in every convex set containing those points.

    (In fact this is true also for infinite sets of points, but we did not discuss the meaning of convex combinations in this case, and you're more likely to se the finite case in mathematics contests)