Tools and problem solving strategies for mathletes

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In each of ten meetings, a certain topic, theme, technique, or style of question is focussed on. For each topic,

1. Some general facts, techniques, or theorems are reviewed.

2. Optimally, three examples are given (with solution) demonstrating facts from part 1.

3. At least five more medium difficulty problems to be worked on in class, three of which will have strong indications as to technique applied, and two more with no direct hint or suggested approach, also to be worked on in class.

4. In addition, at least two problems are given for homework, one challenging one, and one to be written up for the following week, similar to one discussed in class. The appropriate instructor will give remarks on the submitted written solution.

5. All solutions will be available for all instructors, with perhaps a copy for the students to be distributed at the end of the session.

6. For each week, the handout is prepared by a member (or two) of the committee. Time permitting, at the beginning of each meeting, one or two outstanding solutions (from the previous week) can be presented by a student.

1 Schedule

Tentatively, the schedule is [where initials indicate the person(s) responsible for material preparation]. The goal is to have at least two instructors present for each weekly meeting.

- 1. Mathematical induction. [DSG]
- 2. Combinatorics I (including PHP and inclusion/exclusion).[DSG]
- 3. Polynomials.[DSG and AP]
- 4. Number theory.[KK]
- 5. Inequalities.[AP]
- 6. Analysis.[KK]
- 7. Sequences and recursion.[RC]
- 8. Geometry. [DSG,]
- 9. Homogeneity [DSG], invariance and parity [AP].
- 10. Combinatorics II (including generating functions.[RC]

2 General strategies

This section contains some general strategies that might be helpful in finding a solution for a contest problem.

2.1 Make a drawing

Problem ([4], problem 49, p.228): Given a sequence a_0, a_1, \ldots, a_n with $a_0 = a_n = 0$ and $a_{k-1} - 2a_k + a_{k+1} \ge 0$ for $k = 1, \ldots, n-1$, prove that for every $i, a_i \le 0$.

The hint in [4] was to look at a diagram with points (k, a_k) , and a dotted line connecting these points; it is convex because $a_{k+1} - a_k \ge a_k - a_{k-1}$.

2.2 Consider extreme cases

2.3 Exaggerate the problem

2.4 Look for symmetry

Often replacing x with -x yields extra information. For example (from K3, Kirill's sheet 1, Fall 2004):

Problem: which of the expressions

$$(1+x^2-x^3)^{100}$$
 or $(1-x^2+x^3)^{100}$

has the larger coefficient for x^{20} after expanding and collecting terms?

2.5 Argue by contradiction

Problem (Allrussian Mathematical Olympiad, 1990): Given a set of positive numbers, the sum of the pairwise products of its elements is equal to 1, show that it is possible to eliminate one number so that the sum of the remaining numbers is less than $\sqrt{2}$.

2.6 Check your answer

Some equations yield solutions that do not apply to the original problem, so it is wise to actually try your answers in the stated problem. Furthermore, in the heat of battle, it is very easy to quickly move on to the subsequent problem without double checking your results.

2.7 Solve an analogous, simpler problem

To solve $(3x+7)^{x^2-9} = 1$ in integers, instead look at when $a^b = 1$. This only occurs when a = 1, a = -1 and b is even, or when b = 0 and $a \neq 0$. The case a = 1 occurs when x = -2. The case a = -1 never occurs. If b = 0, then $x \in \{-3, 3\}$, and in either case $a \neq 0$. So, $x \in \{-2, -3, 3\}$.

2.8 Use different point of view

Two trains are headed toward each other at different speeds, and a fly starts at the nose of one train, flies to the other, and continues back and forth until the train crashes. How far did the fly fly? Instead of computing the distances each way and summing a series, look at simply how long the fly was in the air.

2.9 Find a pattern

Problem (based on [4], problem 27, p.226): Let $a_1 = a_2 = 1$, $a_3 = -1$, and for $n \ge 4$, $a_n = a_{n-1}a_{n-3}$. Find a_{2009} .

Solution: Compute the first 10 terms of the sequence:

$$\underbrace{1, 1, -1, -1, -1, 1, -1, 1}_{\text{period}}, 1, -1, \dots$$

Observe that the sequence is periodic with period 7. Since 2009 is divisible by 7, then $a_{2009} = -1$.

2.10 Break the problem into cases

Did you know that all primes larger than 3 are adjacent to a multiple of 6? So it might behoove one to look at the two kinds of primes separately.

2.11 Use exhaustion

Often a problem can break down into four or eight cases, each of which is easily checked. Exhaust all cases. Here is a simple example.

Problem (from CMO 1969?): Show that there are no integer solutions to $a^2 + b^2 - 8c = 6$. Solution: Work modulo 8, and consider $a^2 + b^2$ modulo 8. The squares 0^2 , 1^2 , 2^2 , 3^2 ,..., 7^2 are 0,1,4,1,0,1, 4,1, no two of of which add to 6.

2.12 Solve small cases

Working out the smallest of examples is often the right approach to discovering a larger pattern.

2.13 Consider a more general problem

(See section on Induction for an example.)

2.14 Consider a variation or strengthening of the problem

Problem: ([5, number 647, p. 230]).

Any two squares of side-length 0.9 inside a circle of radius 1 must overlap.

Solution: It suffices to prove that any such square contains the centre of the circle. Since any square sitting on a diameter is largest when centred, any largest square on the diameter has (by applying Pythagoras with hypotenuse 1, legs x/2 and x) side-length $x = \sqrt{4/5} < .9$. If a square does not contain the centre, there is a diameter for which the square is on one side, and so must have a side-length smaller than .9.

2.15 Work backwards

2.16 State direct implications of given conditions

Problem: Find all pairs x, y of natural numbers with x < y that satisfy

$$x^y = y^x$$
.

Solution: First observe that the prime factors of each x, y must be the same, say $x = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ and $y = p_1^{b_1} p_2^{b_2} \cdots p_k^{b_k}$. Then for each $i, a_i y = b_i x$, and so $a_i < b_i$. Thus y is divisible by x, say, y = cx for some positive integer c. Then $x^y = y^x$ becomes $x^{cx} = (cx)^x$. Upon taking x-th roots, $x^c = cx$ and $x^{c-1} = c$. Now use y > x to see that c > 1 and so x > 1. Now if c > 2, (and $x \ge 2$), then $x^{c-1} \ge 2^{c-1} > c$. Also, if x > 2 and c = 2, $x^{c-1} = x > 2 = c$, another contradiction. So, k = 2, x = 2 gives the only solution, namely, when y = 2x = 4.

2.17 Look for year

For example, $2009 = 7^2 \cdot 41$ could be in question or answer. This is a common theme in contest problems.

2.18 Translate variables

In the 33rd Spanish Mathematical Olympiad, the question was: for positive reals a, b, c, prove that

$$a^{2} + b^{2} + c^{2} - ab - bc - ca \ge 3(b - c)(a - b),$$

and find when equality holds. Putting x = a - b and y = b - c,

$$\begin{aligned} &2(a^2+b^2+c^2-ab-bc-ca)-6(b-c)(a-b)\\ &= (a-b)^2+(b-c)^2+(c-a)^2-6(b-c)(a-b)\\ &= x^2+y^2+(x+y)^2-6xy\\ &= 2(x^2+y^2-2xy)\\ &= 2(x-y)^2 \ge 0, \end{aligned}$$

and so equality above holds iff x = y, that is, when a - b = b - c, *i.e.*, when a + c = 2b.

Another problem [I do not know its origin, but it has been done in past years' practice sessions here] is to solve the system

$$x^{3} + y = 3x + 4$$

$$2y^{3} + z = 6y + 6$$

$$3z^{3} + x = 9z + 8.$$

One solution is to first put x = a + 2, y = b + 2, z = c + 2, giving

$$a(a+3)^2 + b = 0 \tag{1}$$

$$2b(b+3)^2 + c = 0 (2)$$

$$3c(c+3)^2 + a = 0. (3)$$

If a > 0, then by (1), b < 0, and by (3), c < 0; but also b < 0 implies by (2) that c > 0, an impossibility. Similarly, if a < 0 one gets a contradiction. So we can only conclude that a = 0. Then (1) shows that b = 0 and consequently (2) yields c = 0. So all of a, b, c are zero, giving x = y = z = 2.

2.19 Greedy algorithms

Often, the easiest of solutions is obtained by making the first choice that is available, and then repeating.

Problem: Consider a finite collection of lines drawn in the plane, no three of which are concurrent. Where two lines intersect, draw a point. Two points are adjacent if they are consecutive points on some line. Prove that it is possible to colour the points, each point receiving one of blue, green, and red, so that no two adjacent points are coloured the same.

Solution: By slightly rotating the picture, one can assume that there are no two points one above the other. Colour the points greedily from left to right. Upon colouring any point, there are at most two neighbours (to the left) that have already been coloured, leaving at least one colour available for the present point.

2.20 Differentiate/integrate

3 Tools

This may be a beginning to the ten documents mentioned above.

3.1 Mathematical induction

(See also the section on homogeneous functions and scaling.)

Problem (from Tournament of the Towns, 1987): Prove that for any natural number n,

$$\sqrt{2\sqrt{3\sqrt{4\ldots\sqrt{(n-1)\sqrt{n}}}}} < 3.$$

Comment: Solving a more general problem is easier. Prove that for every $n \in \mathbb{Z}^+$ and every non-negative real number a,

$$\sqrt{a+1\sqrt{a+2+\dots+\sqrt{a+n}}} < a+3.$$

Let S(n) be the statement that for any non-negative real a,

$$\sqrt{a+1+\sqrt{a+2+\cdots+\sqrt{a+n}}} < 3.$$

BASE STEP: S(1) says $\sqrt{a+1} < a+3$, which is verifiable since $a+1 < (a+3)^2 \Leftrightarrow 0 < a^2+5a+8$, which is true for $a \ge 0$.

INDUCTIVE STEP: Fix some $k \ge 0$, and suppose that S(k) is true, that is, for any non-negative a,

$$\sqrt{a+1} + \sqrt{a+2} + \dots + \sqrt{a+k} < a+3$$

is true. It remains to prove S(k+1), namely that for every non-negative b,

$$\sqrt{b+1} + \sqrt{b+2} + \dots + \sqrt{b+k} + \sqrt{b+k+1} < b+3.$$

Indeed, using a = b + 1,

$$\sqrt{b+1} + \sqrt{b+2} + \dots + \sqrt{b+k+\sqrt{b+k+1}} = \sqrt{b+1} + \sqrt{a+1} + \dots + \sqrt{a+k-1} + \sqrt{a+k} + \sqrt{a+k-1} + \sqrt{a+k-1} + \sqrt{a+k-1} + \sqrt{a+k-1} + \sqrt{a+k-1} + \sqrt{a+k} + \sqrt{a+k-1} + \sqrt{a+$$

where the last inequality follows since $2b + 5 < (b + 3)^2 = b^2 + 6b + 9$ and $b^2 \ge 0$. This proves S(k+1), concluding the inductive step.

Hence, by MI, S(n) is true for all $n \ge 1$.

Problem (difficult, from IMO 1988): If a, b and $q = (a^2 + b^2)/(ab + 1)$ are non-negative integers, then $q = [gcd(a, b)]^2$. Prove this by induction on ab.

Problem: Prove that any positive integer can be expressed uniquely as a sum of distinct non-consecutive Fibonacci numbers.

Problem: Planes in three dimensional space are said to be in *general position* if no three planes share a common line and no two planes are parallel. Prove that the maximum number of regions three dimensional space is divided into by n planes in general position is

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3},$$

and the number of infinite (unbounded) regions is

$$2\binom{n}{0} + 2\binom{n}{2}.$$

4 Combinatorics I

4.1 Pigeonhole principle

Problem: 10 points are on a disk of diameter 5. Prove that there are two points within $\sqrt{2}$ of each other. Solution: Partition the disk into 9 pieces, one circular piece in the middle and eight equal sectors. Each of the pieces will have diameter less than $\sqrt{2}$.

4.2 Inclusion/exclusion

5 Polynomials

5.1 Factoring polynomials

Problem: Factor the polynomial $x^{10} + x^5 + 1$ as a product of two lesser degree polynomials. Solution:

$$\begin{aligned} x^{10} + x^5 + 1 &= \frac{(x^5)^3 - 1}{x^5 - 1} \\ &= \frac{x^{15} - 1}{x^5 - 1} \\ &= \frac{(x^3)^5 - 1}{(x - 1)(x^4 + x^3 + x^2 + x + 1)} \\ &= \frac{(x^3 - 1)(x^{12} + x^9 + x^6 + x^3 + 1)}{(x - 1)(x^4 + x^3 + x^2 + x + 1)} \\ &= \frac{(x^2 + x + 1)(x^{12} + x^9 + x^6 + x^3 + 1)}{x^4 + x^3 + x^2 + x + 1} \\ &= (x^2 + x + 1)(x^8 - x^7 + x^5 - x^4 + x^3 - x + 1). \end{aligned}$$

Problem (taken from K6, Kirill's sheet 1, Fall 2004): Factor $a^3 + b^3 + c^3 - abc$. Hint: one factor is a + b + c.

Here is another, taken from Kirill's problem sheet, K4, Fall 2004: Problem: Find the remainders upon dividing the polynomial

$$x + x^3 + x^9 + x^{27} + x^{81} + x^{243}$$

by (a) x - 1; (b) $x^2 - 1$, (c) $x^3 - 1$. Hint: for (b), work modulo $x^2 - 1$, and so use $x^2 \equiv 1$.

5.2 Viete's relations

This material is taken from [1], some based on questions from Mathematical Olympiad contests. Let $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$, and let c_1, \ldots, c_n be its roots (real or complex). Then

$$c_1 + c_2 + \dots + c_n = -\frac{a_{n-1}}{a_n},$$

$$c_1 c_2 + c_1 c_3 + \dots + c_{n-1} c_n = \frac{a_{n-2}}{a_n},$$

$$c_1 c_2 c_3 + c_1 c_2 c_4 + \dots + c_{n-2} c_{n-1} c_n = -\frac{a_{n-3}}{a_n},$$

$$\vdots$$

$$c_1 c_2 \cdots c_n = (-1)^n \frac{a_0}{a_n}$$

Problem: Prove Viéte's relations.

Problem: If a, b, c are non-zero real numbers satisfying

$$(ab + bc + ca)^3 = abc(a + b + c)^3$$
,

then prove that a, b, c are terms in geometric sequence. [Hint: consider a monic cubic polynomial with roots a, b, c.]

Problem: Find all solutions in real numbers to the system

$$x + y + z = 4$$

$$x^{2} + y^{2} + z^{2} = 14$$

$$x^{3} + y^{3} + z^{3} = 34.$$

Hint: consider a monic cubic polynomial with roots x, y, z.

Problem (from USAMO and US selection tests): Let a and b be two roots of $p(x) = x^4 + x^2 + 1$. Prove that ab is a rood of $q(x) = x^6 + x^4 + x^3 - x^2 + 1$.

Problem: Let a, b, c be non-zero reals with $a + b + c \neq 0$. Show that if

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{a+b+c},$$

then for all odd positive integers n,

$$\frac{1}{a^n} + \frac{1}{b^n} + \frac{1}{c^n} = \frac{1}{a^n + b^n + c^n}$$

Hint: look at a monic cubic polynomial with roots a, b, c.

Problem (from *Gazetta Matematica*): Find all solutions in real numbers to the system

$$x + y + z = 0$$

$$x^{3} + y^{3} + z^{3} = 18$$

$$x^{7} + y^{7} + z^{7} = 2058$$

6 Fibonacci sequence

Define $F_0 = 0$, $F_1 = 1$, and for $n \ge 2$, $F_n = F_{n-2} + F_{n-1}$. (see also section on induction)

7 Inequalities

7.0.1 AM-GM inequality

For $a_1, a_2, ..., a_n \ge 0$,

$$\frac{a_1 + a_2 + \dots + a_n}{n} \ge (a_1 a_2 \cdots a_n)^{1/n},$$

with equality iff all a_i 's are identical.

Problem ([4], 44, p. 228): Does there exist an infinite sequence $\{b_n\}$ of positive reals such both $\sum b_n$ and $\sum \frac{1}{n^2 b_n}$ are convergent?

Solution: No. By the AM-GM inequality, $\sum (b_n + \frac{1}{n^2 b_n}) \ge \sum \frac{2}{n}$, which diverges.

Problem: For $a, b, c \ge 0$, prove $a^3 + b^3 + c^3 \ge a^2b + b^2c + c^2a$. (Hint: show that $2a^3 + b^3 \ge 3a^2b$ and then add cyclically.)

Problem: For $a_1, \ldots, a_n \ge 0$, prove $a_1^5 + \cdots + a_n^5 \ge a_1^3 a_2 a_3 + a_2^3 a_3 a_4 + \cdots + a_n^3 a_1 a_2$.

7.0.2 Cauchy's inequality

For reals $a_1, \ldots, a_n, b_1, \ldots, b_n$, if $\mathbf{u} = (\mathbf{a_1}, \ldots, \mathbf{a_n})$ and $\mathbf{v} = (\mathbf{b_1}, \ldots, \mathbf{b_n})$, then $\|\mathbf{u} \bullet \mathbf{v}\| \le \|\mathbf{u}\| \cdot \|\mathbf{v}\|$, or equivalently (by squaring both sides)

$$(a_1b_1 + \dots + a_nb_n)^2 \le (a_1^2 + \dots + a_n^2)(b_1^2 + \dots + b_n^2),$$

where equality holds iff (a_1, \ldots, a_n) and (b_1, \ldots, b_n) are proportional.

Problem: For a+b+c=1, with each of a, b, c at least -1/4, prove that $\sqrt{4a+1}+\sqrt{4b+1}+\sqrt{4c+1} \le 21$.

Solution: Apply the Cauchy-Schwarz inequality with $\mathbf{u} = (1, 1, 1)$ and $\mathbf{v} = (\sqrt{4a+1}, \sqrt{4b+1}, \sqrt{4c+1})$ to obtain

$$(LHS)^{2} = (\mathbf{u} \bullet \mathbf{v})^{2}$$

$$\leq \|\mathbf{u}\|^{2} \|\mathbf{v}\|^{2}$$

$$= 3(4a + 1 + 4b + 1 + 4c + 1)$$

$$= 3(4(a + b + c) + 3)$$

$$= 3(4 + 3) = 21.$$

(Also, equality iff a = b = c = 1/3.)

Problem: For
$$x_1, \dots, x_n > 0$$
, prove $\frac{x_1^2}{x_1 + x_2} + \frac{x_2^2}{x_2 + x_3} + \dots + \frac{x_n^2}{x_n + x_1} \ge \frac{x_1 + \dots + x_n}{2}$
Hint: $a_k = \frac{x_k}{\sqrt{x_k + x_{k+1}}}, \ b_k = \sqrt{x_k + x_{k+1}}.$

7.0.3 Bernoulli's inequality

For non-zero x > -1 and integer $n \ge 2$, $(1 + x)^n > 1 + nx$. (An easy proof is by induction; another is by staring at the binomial theorem.)

7.0.4 Convex functions, Jensen's inequality

For $x_1, \ldots, x_n \in I$, where $I \subset \mathbb{R}$ is an interval, if f is a continuous function on I which is convex (concave up), then

$$f(\frac{x_1 + \dots + x_n}{n}) \le \frac{f(x_1) + \dots + f(x_n)}{n}.$$

If f is strictly convex, then equality holds iff all x_i 's are equal.

If the function is concave (concave down), then the sign of the inequality is reversed.

If the function is twice differentiable, it is convex iff $f'' \ge 0$ on the interval. In practice, it is usually easier to show that f' is monotone increasing. The sum of two convex (concave) functions is a convex (concave) function. Multiplication by a positive number also preserves convexity (concavity). Problem (Putnam, see [2], p. 37): Let x_1, \ldots, x_n be reals in [0, 1]. Find the maximum of the sum $\sum_{i < j} |x_i - x_j|$. (Hint: examine f(x) = |x - a| for fixed a.)

Problem: If $a, b \ge 0$, a + b = 2, prove $(1 + \sqrt[5]{a})^5 + (1 + \sqrt[5]{b})^5 \le 64$. Problem: If a, b, c > 0, prove $\left(\frac{a+b+c}{3}\right)^{a+b+c} \le a^a b^b c^c$. (Hint: $f(x) = x \ln x$.) Problem: If a, b, c > 0, prove $\frac{a}{a+3b+3c} + \frac{b}{3a+b+3c} + \frac{c}{3a+3b+c} \ge \frac{3}{7}$. (Hint: Let S = a+b+c. Take f(x) = x/(3S-2x).) Problem: If $a_1, \ldots, a_n \ge 1$, prove $\sum_{k=1}^n \frac{1}{1+a_k} \ge \frac{n}{1+\sqrt[n]{a_1} \cdots a_n}$. (Hint: $f(x) = 1/(1+e^x)$.) Problem: Let α, β, γ be angles of a triangle. Prove:

$$\begin{aligned} \sin \alpha + \sin \beta + \sin \gamma &\leq \frac{3\sqrt{3}}{2}, \\ \sin \alpha \cdot \sin \beta \cdot \sin \gamma &\leq \frac{3\sqrt{3}}{8}, \\ \cos \alpha \cdot \cos \beta \cdot \cos \gamma &\leq \frac{1}{8}, \\ \sec \frac{\alpha}{2} + \sec \frac{\beta}{2} + \sec \frac{\gamma}{2} &\geq 2\sqrt{3}. \end{aligned}$$

Problem: For $a_1, \ldots, a_n, b_1, \ldots, b_n \ge 0$, prove

$$((a_1+b_1)\cdot\ldots\cdot(a_n+b_n))^{\frac{1}{n}} \ge (a_1\cdot\ldots\cdot a_n)^{\frac{1}{n}} + (b_1\cdot\ldots\cdot b_n)^{\frac{1}{n}}.$$

Hint: Can be reduced (careful with zeros!) to

$$(1+x_1)\cdot\ldots\cdot(1+x_n) \ge (1+\sqrt[n]{x_1\cdot\ldots\cdot x_n})^n$$

for positive variables. Then Jensen's with $f(x) = \ln(1 + e^x)$.

Weighted Jensen's inequality. If $x_1, \ldots, x_n \in I$, where I is an interval, $\lambda_1, \ldots, \lambda_n > 0$, $\lambda_1 + \cdots + \lambda_n = 1$, f is convex on I, then

$$\lambda_1 f(x_1) + \dots + \lambda_n f(x_n) \ge f(\lambda_1 x_1 + \dots + \lambda_n x_n).$$

If f is strictly convex, then equality holds iff all x_i 's are equal.

Corollary: weighted AM-GM inequality. If $x_1, \ldots, x_n \ge 0, \lambda_1, \ldots, \lambda_n > 0, \lambda_1 + \cdots + \lambda_n = 1$, then

$$\lambda_1 x_1 + \dots + \lambda_n x_n \ge x_1^{\lambda_1} \cdot \dots \cdot x_n^{\lambda_n},$$

equality holds iff $x_1 = x_2 = \cdots = x_n$.

Problem: For $a_1, \ldots, a_n > 0$ with $a_1 \cdot \ldots \cdot a_n = 1$, prove

$$a_1 + \sqrt{a_2} + \dots + \sqrt[n]{a_n} \ge \frac{n+1}{2}.$$

(Hint: $\sqrt[k]{a_k} = k \sqrt[k]{\frac{a_k}{k^k}}$.)

7.0.5 Power mean inequality

Let $x_1, \ldots, x_n \ge 0, \lambda_1, \ldots, \lambda_n > 0, \lambda_1 + \cdots + \lambda_n = 1$. For $t \in \mathbb{R}, t \ne 0$, define the weighted mean M_t of order t as $M_t := (\lambda_1 x_1^t + \cdots + \lambda_n x_n^t)^{1/t}$. Also $M_0 := x_1^{\lambda_1} \cdot \ldots \cdot x_n^{\lambda_n} = \lim_{t \to 0} M_t$, $M_{-\infty} := \min\{x_1, \ldots, x_n\} = \lim_{t \to -\infty} M_t$, $M_{\infty} := \max\{x_1, \ldots, x_n\} = \lim_{t \to \infty} M_t$. Then

$$M_s \le M_t$$
, if $-\infty \le s < t \le \infty$

Problem: Prove power mean inequality using weighted Jensen's inequality. (Hint: first do it for $s \neq 0, t > 0$.)

Problem: For a, b, c > 0, prove $\frac{a^{10} + b^{10} + c^{10}}{a^5 + b^5 + c^5} \ge \left(\frac{a+b+c}{3}\right)^5$. (Hint: rewrite using M_{10}, M_5 and $M_{1.}$)

Problem: For $a_1, \ldots, a_n > 0$, prove $a_1^{n+1} + \cdots + a_n^{n+1} \ge a_1 \cdot \ldots \cdot a_n \cdot (a_1 + \ldots + a_n)$. Hint: rewrite using M_{n+1}, M_1 and M_0 .

Alternative way: use AM-GM to show that

$$\frac{2a_1^{n+1}}{n+1} + \frac{a_2^{n+1}}{n+1} + \dots + \frac{a_n^{n+1}}{n+1} \ge a_1^2 \cdot a_2 \cdot \dots \cdot a_n,$$

and add cyclically such inequalities.

7.0.6 Triangle inequality

For
$$\mathbf{x} = (x_1, \dots, x_n), \mathbf{y} = (y_1, \dots, y_n) \in \mathbb{R}^n,$$

 $\left(\sum_{k=1}^n (x_k + y_k)^2\right)^{1/2} \le \left(\sum_{k=1}^n x_k^2\right)^{1/2} + \left(\sum_{k=1}^n y_k^2\right)^{1/2},$

where equality holds iff \mathbf{x} and \mathbf{y} are scalar multiples of one another.

Minkowski's triangle inequality. If p > 1 and $\mathbf{x} = (x_1, \ldots, x_n), \mathbf{y} = (y_1, \ldots, y_n) \in [0, \infty)^n$, then

$$\left(\sum_{k=1}^{n} (x_k + y_k)^p\right)^{1/p} \le \left(\sum_{k=1}^{n} x_k^p\right)^{1/p} + \left(\sum_{k=1}^{n} y_k^p\right)^{1/p},$$

equality holds iff \mathbf{x} and \mathbf{y} are scalar multiples of one another.

Problem: Prove Minkowski's triangle inequality using weighted Jensen's inequality.

7.0.7 Hölder's inequality

If p, q > 1, $\frac{1}{p} + \frac{1}{q} = 1$ and $a_1, \ldots, a_n, b_1, \ldots, b_n$ are any real or complex numbers, then

$$\left(\sum_{k=1}^{n} |a_k|^p\right)^{1/p} \left(\sum_{k=1}^{n} |b_k|^q\right)^{1/q} \ge \sum_{k=1}^{n} |a_k b_k|,$$

equality holds iff $|a_k|^p$ are proportional to $|b_k|^q$, k = 1, ..., n.

Problem: Prove Hölder's inequality using weighted Jensen's inequality.

Problem: For
$$x_1, \dots, x_n, y_1, \dots, y_n > 0$$
, prove $\frac{a_1^{n+1}}{b_1^n} + \dots + \frac{a_n^{n+1}}{b_n^n} \ge \frac{(a_1 + \dots + a_n)^{n+1}}{(b_1 + \dots + b_n)^n}$
Hint: $p = n + 1$, $q = \frac{n+1}{n}$, $a_k = \frac{x_k}{y_k^{\frac{n}{n+1}}}$, $b_k = y_k^{\frac{n}{n+1}}$.

7.0.8 Maximum of convex in every variable function on an *n*-box

If $F : [a_1, b_1] \times \cdots \times [a_b, b_n] \to \mathbb{R}$ is convex in every $x_k \in [a_k, b_k]$, then

$$F(x_1,\ldots,x_n) \le \max_{t_k \in \{a_k,b_k\}, k=1,\ldots,n} F(t_1,\ldots,t_n)$$

In other words, F achieves its maximum at one of the 2^n vertices of the *n*-box $[a_1, b_1] \times \cdots \times [a_n, b_n]$.

Problem: For $a, b, c \in [0, 1]$, prove

$$\frac{a}{b+c+1} + \frac{b}{c+a+1} + \frac{c}{a+b+1} + (1-a)(1-b)(1-c) \le 1.$$

Problem: For $a, b, c \in [1, 2]$, prove $(a + 5b + 9c)\left(\frac{1}{a} + \frac{5}{b} + \frac{9}{c}\right) \le 225$.

7.0.9 Karamata's inequality

Let f be a convex function on an interval $I, a_1, \ldots, a_n, b_1, \ldots, b_n \in I$, and

$$a_1 \ge a_2 \ge \dots \ge a_n,$$

$$b_1 \ge b_2 \ge \dots \ge b_n,$$

$$a_1 \ge b_1,$$

$$a_1 + a_2 \ge b_1 + b_2,$$

$$\vdots$$

$$a_1 + \dots + a_{n-1} \ge b_1 + \dots + b_{n-1},$$

$$a_1 + \dots + a_n = b_1 + \dots + b_n$$

 $((a_1, ..., a_n) \succ (b_1, ..., b_n), (a_1, ..., a_n)$ majorizes $(a_1, ..., a_n))$. Then

$$f(a_1) + \dots + f(a_n) \ge f(b_1) + \dots + f(b_n).$$

Problem: For $0 \le A, B, C, D, E, F \le \frac{\pi}{2}, A + B + C + D + E + F = 2\pi$, prove

$$\sin A + \sin B + \sin C + \sin D + \sin E + \sin F \ge 4.$$

7.0.10 Estimating with non-convex function

Idea: make the function convex - use convex (concave) hull.

Problem: Let $x_1, \ldots, x_n \in [-1, 1]$ be such that $x_1^3 + \cdots + x_n^3 = 0$. Prove that $x_1 + \cdots + x_n \leq \frac{n}{4}$. Hint: For $f(x) = \sqrt[3]{x}$ on [-1, 1], construct its concave hull: g(x) will be equal to the tangent line to f(x) passing through (-1, -1) between -1 and the x-coordinate of the tangent point, and g(x) = f(x) everywhere else. Then g is concave and $g(x) \geq f(x)$ for $x \in [-1, 1]$. Use Jensen's inequality after this. $g(0) = \frac{1}{4}$.

8 Geometry

8.1 Area and volume of a sphere

A sphere with radius r has area $4\pi r^2$ and volume $\frac{4}{3}\pi r^3$.

8.2 Heron's formula

A triangle with side lengths a, b, and c, and semiperimeter $s = \frac{1}{2}(a+b+c)$ has area $\sqrt{s(s-a)(s-b)(s-c)}$.

8.3 Theorems of Ceva and Menelaus

Menelaus' theorem: Let ABC be a triangle, and let D, E, F be points on the lines containing sides BC, CA, AB respectively. D, E, F are collinear iff

$$\frac{|BD|}{|DC|} \cdot \frac{|CE|}{|EA|} \cdot \frac{|AF|}{|FB|} = 1.$$

A *cevian* of a triangle is a line segment that joins a vertex of the triangle to a point on the opposite side.

Ceva's theorem: Given a triangle ABC, three cevians AY, BZ, and CX are concurrent iff

$$\frac{|AX|}{|XB|} \cdot \frac{|BY|}{|YC|} \cdot \frac{|CZ|}{|ZA|} = 1.$$

Consequences of Ceva's theorem: medians of a triangle are concurrent (all meet in a point); internal angle bisectors of a triangle are concurrent; altitudes of a triangle are concurrent.

8.4 Law of cosines

For a triangle with side lengths a, b, c and angle θ across from side with length c,

$$c^2 = a^2 + b^2 - 2ab\cos\theta.$$

Note that Pythagoras's theorem is a special case when $\theta = \pi/2$.

8.5 Interior angles of a convex polygon

A convex polygon with n sides (and n vertices) has interior angles which sum to $(n-2)\pi$.

Problem: Let ABCDE be a regular pentagon and M a point in its interior such that $m \angle MBA = m \angle MEA = 42^{\circ}$. Prove that $m \angle CMD = 60^{\circ}$.

8.6 Pick's theorem

For present purposes, a *lattice point* lattice point is a point $(x, y) \in \mathbb{R}^2$ in the real cartesian plane whose coordinates x, y are integers. (Lattice points are also used in Exercise ??.) In other words, a lattice point is an element of \mathbb{Z}^2 .

To calculate the area of an arbitrary polygon might be very cumbersome, however if the polygon has vertices that are lattice points, then finding its area is nearly trivial by the spectacular 1899 result of Georg Alexander Pick (1859–1942) [10]. For a simple (non-intersecting)

polygon P on lattice points, let I(P) be the number of lattice points on the interior of P, and let B(P) be the number of lattice points occurring on the boundary of P.

Theorem 8.1 (Pick's theorem). Let P be a polygon whose vertices are lattice points. Then the area of P is

$$A(P) = I(P) + \frac{1}{2}B(P) - 1.$$
(4)

For example, in Figure 1, there are 4 interior points, and 9 boundary points, and the area is $4 + \frac{1}{2}9 - 1 = \frac{15}{2}$.

Figure 1: Pick's theorem: I(P) = 4, B(P) = 9; Area =7.5.

8.7 Power of an inside point

Let A,B,C,D be points in order on a circle. If the intersection of the two chords AC and BD is a point inside the circle, then $|AI| \cdot |IC| = |BI| \cdot |ID|$.

8.8 Power of an outside point

8.9 Opposite angles of a cyclic quadrilateral are supplementary

(A cyclic quadrilateral is one whose vertices lie on a circle, and supplementary angles add to π .)

8.10 An angle inscribed in a circle is half the central angle

If O is the center of a circle, and A, B, C are points on the circle, then $m \angle ABC = \frac{1}{2}m \angle AOC$.

8.11 Ptolemy's theorem

In a convex quadrilateral ABCD,

$$|AC| \cdot |BD| = |AB| \cdot |CD| + |BC| \cdot |AD|.$$

As an example [4, No. 53, §12.3, p. 321], here is an application: An equilateral triangle ABC is inscribed in a circle, and an arbitrary point M is chosen on the arc BC. Prove that |MA| = |MB| + |MC|.

Solution: By Ptolemy's theorem, $|BC| \cdot |AM| = |AC| \cdot |BM| + |AB| \cdot |CM|$; now divide through by the length |AB| = |BC| = |CA|.

8.12 Problems in space and combinatorial geometry

Three lines are said to be *concurrent* iff they share a common point; similarly define planes to be concurrent. Points are in *general position* iff no three are on a line. Lines are in *general position* iff no two are parallel and no three are concurrent. The following problems are in no particular order with regards to difficulty.

1. For $n \in \mathbb{Z}^+$, let f(n) be the maximum number of pieces a circular cake can be cut into with precisely *n* vertical cuts. What is f(n)?

2. A circular cake is decorated by placing *n* raisins on the perimeter, then the cake is cut into pieces by cuts joining every pair of raisins. For each *n*, what is the maximum number g(n) so that (with careful arrangement of the raisins) the cake be cut into g(n) pieces?

3. Prove that any cake, regardless of shape (but uniform in thickness) can be cut into four pieces of equal area using two (vertical) cuts, where the cuts are perpendicular to each other. (Try this with a 3,4,5 triangle first?)

4. Some n points are on the plane, where no two distances between points are the same. Connect each point to its nearest point with a straight line segment. Prove that the resulting figure does not contain:

- (a) any closed polygon;
- (b) intersecting segments;
- (c) any point with six or more neighbours.

5. Three spheres intersect in a point P, but no line containing P is tangent to all three spheres. Prove that the spheres intersect in an additional common point.

6. A cop chased a robber into a circular swimming pool, and is now treading water at the center, and the cop is at the edge. The cop will not enter the pool. If the robber can reach a point on the edge of pool before the cop does, the robber gets away. The robber can swim one quarter as fast as the cop can run (around the perimeter). Is there a strategy so the robber can make it to the edge of the pool and get away? Prove your answer.

Solution outline: The robber need only swim around in a circle (with same center as pool) of radius slightly less than 1/4 of the pool's radius. While swimming in this circle, one can attain a point antipodal to that of the cop, as the robber's angular velocity is always larger than the cop's. Once the robber has the cop on the opposite side of the pool, the robber then swims directly to the edge and escapes.

7. Very similar to the above problem, a Roman and a lion are in a circular arena, and they can both run at the same speed. Is there a strategy for the lion to catch the Roman? [This problem is hard.]

9 Scaling and homogeneous functions

There are many usages of the word "homogeneous"; here is one more. If $f(x_1, x_2, \ldots, x_n)$ is a function of *n* variables, we say that *f* is *homogeneous of order d* if for any constant t > 0, $f(tx_1, tx_2, \ldots, tx_n) = t^d f(x_1, x_2, \ldots, x_n)$. For example, the expression $f(x, y, z) = \frac{x^3 + y^3}{xyz}$ is homogeneous of order 1. The advantage of having an equation with two expressions that are homogeneous of the same order is that, if one needs, one can first prove an equality for, say, a particular 'size' of x, y, z, then later 'scale' the variables to any sizes. For example, a problem that at first seems quite hard is the following: show that for any non-negative real numbers x_1, x_2, \ldots, x_n ,

$$\frac{x_1^2 + x_2^2 + \dots + x_n^2}{n} x_1 x_2 \cdots x_n \le \left(\frac{x_1 + x_2 + \dots + x_n}{n}\right)^{n+2}.$$

Notice that both sides are homogeneous of order n + 2; that can be quite a useful observation.

The renowned problem poser/solver Murray S. Klamkin gave this inequality as Problem 1324 in *Mathematics Magazine*, June 1989: In fact, the problem was actually proposed with the added condition $x_1 + x_2 + \ldots + x_n = 1$, and many solutions were received which used the method of Lagrange multipliers (a method from multivariate calculus often used to solve problems with such constraints), however Klamkin gave a solution in [8] which was by induction on n. His solution is given here, however with just a few more details supplied. Some simple algebraic steps are still left to the reader.

Let x_1, x_2, \ldots, x_n be non-negative reals, and let S(n) be the statement

$$\frac{x_1^2 + x_2^2 + \dots + x_n^2}{n} x_1 x_2 \dots + x_n \le \left(\frac{x_1 + x_2 + \dots + x_n}{n}\right)^{n+2}$$

It will be convenient to rewrite S(n) as

$$(x_1^2 + x_2^2 + \dots + x_n^2)x_1x_2 \dots + x_n \le (n) \left(\frac{x_1 + x_2 + \dots + x_n}{n}\right)^{n+2}.$$

First observe that the inequality is trivial if any of the x_i 's are 0, so we will assume that each $x_i > 0$.

BASE STEPS: For n = 1, S(1) says $x_1^3 \le x_1^3$. For n = 2, S(2) says

$$(x_1^2 + x_2^2)x_1x_2 \le \frac{1}{8}(x_1 + x_2)^4,$$

which reduces to $0 \leq (x_1 - x_2)^4$, which is certainly true.

INDUCTIVE STEP: Let $k \ge 2$ be fixed and suppose that S(k) holds:

$$S(k): (x_1^2 + x_2^2 + \dots + x_k^2) x_1 x_2 \dots x_k \le (k) \left(\frac{x_1 + x_2 + \dots + x_k}{k}\right)^{k+2}.$$

We would like to prove (using $x = x_{k+1}$)

$$S(k+1): \quad (x_1^2 + \dots + x_k^2 + x^2)x_1x_2 \cdots x_kx \le (k+1)\left(\frac{x_1 + \dots + x_k + x}{k+1}\right)^{k+3}$$

Put $A = \frac{x_1 + x_2 + \dots + x_k}{k}$ and $P = x_1 x_2 \cdots x_n$. With this notation, we now have assumed

$$S(k): \quad (x_1^2 + x_2^2 + \dots + x_k^2)P \le kA^{k+2}$$

and would like to prove

$$S(k+1): \quad (x_1^2 + x_2^2 + \dots + x_k^2 + x^2) Px \le (k+1) \left(\frac{kA+x}{k+1}\right)^{k+3}$$

The left hand side of S(k+1) is

$$\begin{array}{rcl} (x_1^2+x_2^2+\cdots x_k^2+x^2)Px & = & (x_1^2+x_2^2+\cdots x_k^2)Px+Px^3 \\ & \leq & kA^{k+2}x+Px^3 \ \ (\mbox{by }S(k)). \end{array}$$

So to prove S(k+1), it suffices to prove that

$$kA^{k+2}x + Px^3 \le (k+1)\left(\frac{kA+x}{k+1}\right)^{k+3}$$

By the AM-GM inequality (Theorem ??), $P \leq A^k$, so it suffices to prove

$$kA^{k+2}x + A^kx^3 \le (k+1)\left(\frac{kA+x}{k+1}\right)^{k+3}$$

Now restrict to the situation where the sum $x_1 + x_2 + \cdots + x_k + x$ is held constant, and prove the result with this added constraint. The general result then follows immediately; observe that for any constant c, the statement S(n) holds for x_1, \ldots, x_n if and only if it holds for cx_1, \ldots, cx_n (the factor c^{n+2} appears on each side). So, consider only those $(x_1, \ldots, x_k, x) \in \mathbb{R}^{k+1}$ for which $x_1 + x_2 + \cdots + x_k + x = k + 1$, that is,

$$kA + x = k + 1.$$

So, to prove S(k+1), it suffices to show

$$kA^{k+2}x + A^kx^3 \le k+1.$$

The left hand of the above inequality is a function of A (and x = k + 1 - kA, also a function of A), and so we maximize the expression using calculus:

$$\begin{aligned} &\frac{d}{dA}[kA^{k+2}x + A^kx^3] \\ &= k(k+2)A^{k+1}x + kA^{k+2}\frac{dx}{dA} + kA^{k-1}x^3 + A^k3x^2\frac{dx}{dA} \\ &= k(k+2)A^{k+1}x + kA^{k-1}x^3 - k(kA^{k+2} + A^k3x^2). \end{aligned}$$

Putting A = tx, this expression becomes (after a bit of algebra)

$$(1-t)(kt^2 - 2t + 1)kt^{k-1}x^{k+2}.$$

Since $k \ge 2$, the above has roots at only t = 0 and t = 1, and so the derivative is positive for 0 < t < 1 and negative for t > 1. Thus, $kA^{k+2}x + A^kx^3$ achieves a maximum when t = 1, that is, when A = x = 1. Hence,

$$kA^{k+2}x + A^kx^3 \le k+1,$$

and so S(k+1) follows, completing the inductive step.

Thus, by mathematical induction, for all $n \ge 1$, the statement S(n) is true.

10 Parity

Problem: (my adaptation of [5, Prob. 247], which is for reals)

Suppose that 2n + 1 integers satisfy the following property: any 2n of these numbers can be split into two sets of n each so that the sum of each set is the same. Prove that all numbers are equal.

Solution: The sum of any 2n of these numbers is even, so all numbers have the same parity (take 2n of the numbers, interchange one with the spare one, and the sum must remain even). Subtracting one of the numbers from all numbers, get a list of 2n + 1 numbers, all of which are even (and one is 0). This list of new shifted numbers still satisfies the property. Dividing each number in the new list by 2 gives another list of 2n + 1 numbers with the same property. This new list again contains 0 and all have the same parity (even) by the above argument. So create a new list, again with 0, all even, divide by 2, ... The only way that this division by 2 can continue indefinitely is if all are 0, which means the original list had all numbers equal.

[This also shows that the same is true when all numbers are rational by first multiplying by the greatest common denominator. To see that this holds for reals, consider the reals as an infinite dimensional vector space over the rationals, pick a basis, and use the rational coordinates as above. A second solution was also given for the real case: Assume not all equal and the property holds. For each element x_i , get an equation in the remaining 2n numbers, put all on one side, get a homogeneous system of 2n + 1 equations in 2n + 1 unknowns (in each row of the coefficient matrix, -1 occurs n times, 1 occurs n times, and 0 occurs once). The solution space to this system contains $V = \text{span}\{(1, 1, \ldots, 1)\}$. By assumption, the system contains a solution not in V. Because the matrix has integer entries, by Gaussian elimination, there exists a solution with rational entries that is not in V, but that was shown to be impossible above.]

11 Extras

Is there a common theme to these problems?

1. Prove that the set $A = \{3, 7, 11, 15, \ldots\}$ contains infinitely many primes.

2. Let a_1, \ldots, a_7 be real numbers in the open interval (1, 13). Prove that there exist three of these a_i 's that are side-lengths of a (non-trivial) triangle.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a periodic function such that $\{f(n) : n \in \mathbb{Z}^+\}$ is infinite. Prove that the period of f is irrational.

- 4. Does there exist a function $f : \mathbb{R} \to \mathbb{R}$ satisfying the following?
 - (a) there exists M > 0 such that for every $x, -M \le f(x) \le M$;
 - (b) f(1) = 1;
 - (c) for $x \neq 0$,

$$f\left(x+\frac{1}{x^2}\right) = f(x) + \left(f(\frac{1}{x})\right)^2.$$

5. In a circular arrangement of n symbols consisting of 0's or 1's, prove that if the number of 1's exceeds n - n/k, there must be a string of k consecutive 1's.

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