University of Manitoba, Mathletics 2009

Session 8: Geometry

1 Facts and definitions

Things to remember:

- Proper drawing helps.
- Vector geometry and dot product.
- Similar triangles.
- Complex numbers for planar problems.
- Calculus — applications of integration.
- Symmetry.
- Coordinate systems: cartesian, polar, spherical, cylindrical.

Heron's formula: the area of the triangle with sides a, b, c is

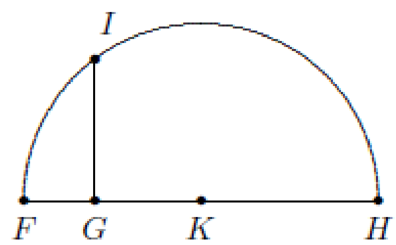
$$A = \frac{1}{4}\sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}.$$

Pappus's theorem: the volume of a solid of revolution is equal to the product of the area of the revolving region times the distance through which the center of mass is rotated.

Area of the spherical cap $\{(x,y,z) \,|\, x^2+y^2+z^2=1,\ z \geq z_0\}$ is $2\pi(1-z_0)$.

2 Examples

Example 8.1: (NCS/MAA 2003) In "La Géométrie", Descartes gives the following geometric construction of a square root: "If the square root of GH is desired, I add, along the same straight line, FG equal to unity; then bisecting FH at K, I describe the circle FIH about K as center, and draw from G a perpendicular and extend it to I, and GI is the required root." Assuming, as in the figure below, that $GH > 1$, prove that the length of GI is the required root.



Solution: Angle FIH is right as FH is a diameter. Hence $\angle GFI = \frac{\pi}{2} - \angle IHG = \angle GIH$. So the right triangles GFI and GIH are similar, and $\dfrac{GF}{GI} = \dfrac{GI}{GH}$, i.e., $GI^2 = GF \cdot GH = GH$. Note: we have not used the assumption $GH > 1$.

Example 8.2: (Putnam 2006) Find the volume of the region of points (x, y, z) such that

$$(x^2 + y^2 + z^2 + 8)^2 \leq 36(x^2 + y^2).$$

Solution: In cylindrical coordinates $r = \sqrt{x^2 + y^2}$ our region is

$$(r - 3)^2 + z^2 \leq 1.$$

This is a solid of revolution (torus). By Pappus's theorem, the volume is the product of the area of the planar region (disc), which is π in our case, times the distance through which the center of mass is rotated, which is 6π. Answer: $6\pi^2$.

Example 8.3: (Putnam 2004) Let n be a positive integer, $n \geq 2$, and put $\theta = 2\pi/n$. Define points $P_k = (k, 0)$ in the xy-plane, for $k = 1, \ldots, n$. Let R_k be the map that rotates the plane counterclockwise by the angle θ about the point P_k. Let R denote the map obtained by applying, in order, R_1, then R_2, …, then R_n. For an arbitrary point (x, y), find, and simplify, the coordinates of $R(x, y)$.

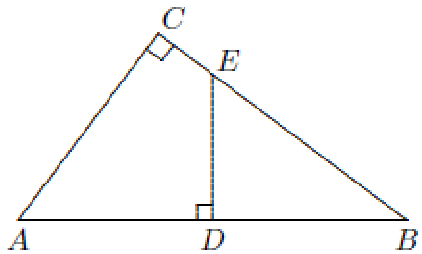
Solution: Identify the xy-plane with the complex plane $\mathbb{C}$, so that P_k is k. Denoting $\zeta = e^{2\pi i/n}$, we get $R_k(z) = k + \zeta(z - k)$ — a linear function in z with leading coefficient ζ. Hence, the composition $R(z) = (R_n \circ \cdots \circ R_2 \circ R_1)(z)$ is a linear function in z with leading coefficient $\zeta^n = 1$. So $R(z) = z + t$, $t \in \mathbb{C}$, is a translation. Since $R_1(1) = 1$, we have $1 + t = (R_n \circ \cdots \circ R_2)(1)$. But also $(R_n \circ \cdots \circ R_2)(1) = (R_{n-1} \circ \cdots \circ R_1)(0) + 1$ by the symmetry of the definition of R_i. Hence

$$R_n(1 + t) = (R_n \circ \cdots \circ R_1)(0) + R_n(1) = t + R_n(1),$$

so $R_n(t) = t$ and $t = n$. Answer: $R(x, y) = (x + n, y)$.

3 Problems for discussion

Discussion problem 8.1: (NCS/MAA 2004) In the figure $AB = 20$, $AC = 12$, $AD = DB$, angles ACB and ADE are right angles. Find the area of the quadrilateral $ADEC$.



Discussion problem 8.2: (Putnam 2007) Find the least possible area of a convex set in the plane that intersects both branches of the hyperbola $xy = 1$ and both branches of the hyperbola $xy = -1$.

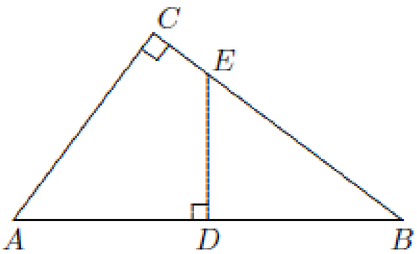
Discussion problem 8.3: (NCS/MAA 1997) In the rectangle $ABCD$, sides AD and CD have lengths 10 and 15, respectively. The point P lies inside the rectangle, and the lengths of AP and BP are, respectively, 12 and 9. Prove that triangle APD is isosceles.

Discussion problem 8.4: (Putnam 2004) For $i = 1, 2$ let T_i be a triangle with side lengths a_i, b_i, c_i, and area A_i. Suppose that $a_1 \le a_2$, $b_1 \le b_2$, $c_1 \le c_2$, and that T_2 is an acute triangle. Does it follow that $A_1 \le A_2$?

Discussion problem 8.5: (Putnam 1998) Let H be the unit hemisphere $\{(x, y, z) : x^2 + y^2 + z^2 = 1, z \ge 0\}$, C the unit circle $\{(x, y, 0) : x^2 + y^2 = 1\}$, and P the regular pentagon inscribed in C. Determine the surface area of that portion of H lying over the planar region inside P, and write your answer in the form $A \sin\alpha + B \cos\beta$, where A, B, α, β are real numbers.

4 Solutions for discussion problems

Discussion problem 8.1: (NCS/MAA 2004) In the figure $AB = 20$, $AC = 12$, $AD = DB$, angles ACB and ADE are right angles. Find the area of the quadrilateral $ADEC$.



Solution: The area is $\frac{117}{2}$. From the Pythagorean Theorem, $BC = 16$. From similar triangles, $\frac{ED}{BD} = \frac{AC}{BC} = \frac{12}{16}$, so $ED = BD \cdot \frac{3}{4} = 10 \cdot \frac{3}{4} = \frac{15}{2}$. The area of the triangle ABC is $\frac{1}{2} \cdot 12 \cdot 16 = 96$, and the area of the triangle DBE is $\frac{1}{2} \cdot ED \cdot 10 = \frac{75}{2}$. Then the area of the quadrilateral $ADEC$ is $96 - \frac{75}{2} = \frac{117}{2}$.

Discussion problem 8.2: (Putnam 2007) Find the least possible area of a convex set in the plane that intersects both branches of the hyperbola $xy = 1$ and both branches of the hyperbola $xy = -1$.
Solution: The minimum is 4, achieved by the square with vertices $(\pm 1, \pm 1)$.

To prove that 4 is a lower bound, let S be a convex set of the desired form. Choose $A, B, C, D \in S$ lying on the branches of the two hyperbolas, with A in the upper right quadrant, B in the upper left, C in the lower left, D in the lower right. Then the area of the quadrilateral $ABCD$ is a lower bound for the area of S.

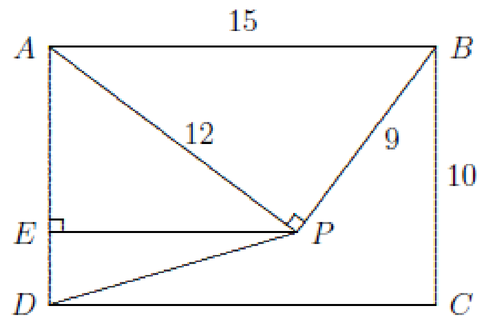
Write $A = (a, 1/a)$, $B = (b, -1/b)$, $C = (-c, -1/c)$, $D = (-d, 1/d)$ with $a, b, c, d > 0$. Then the area of the quadrilateral $ABCD$ is

$$\frac{1}{2}(a/b + b/c + c/d + d/a + b/a + c/b + d/c + a/d),$$

which by the arithmetic-geometric mean inequality is at least 4.
Alternative solution: Choose A, B, C, D as in the first solution. Note that both the hyperbolas and the area of the convex hull of $ABCD$ are invariant under the transformation $(x, y) \mapsto (xm, y/m)$ for any $m > 0$. For m small, the counterclockwise angle from the line AC to the line BD approaches 0; for m large, this angle approaches π. By continuity, for some m this angle becomes $\pi/2$, that is, AC and BD become perpendicular. The area of $ABCD$ is then $AC \cdot BD$.

It thus suffices to note that $AC \geq 2\sqrt{2}$ (and similarly for BD). This holds because if we draw the tangent lines to the hyperbola $xy = 1$ at the points $(1, 1)$ and $(-1, -1)$, then A and C lie outside the region between these lines. If we project the segment AC orthogonally onto the line $x = y = 1$, the resulting projection has length at least $2\sqrt{2}$, so AC must as well.

Discussion problem 8.3: (NCS/MAA 1997) In the rectangle $ABCD$, sides AD and CD have lengths 10 and 15, respectively. The point P lies inside the rectangle, and the lengths of AP and BP are, respectively, 12 and 9. Prove that triangle APD is isosceles.



Solution: Note that ABP is a right triangle since $15^2 = AB^2 = AP^2 + BP^2 = 12^2 + 9^2$. If E is the foot of the perpendicular from P to AD, the triangle PEA is similar to the triangle ABP, with sides $\frac{4}{5}$ as long. Hence, the lengths of AE, EP and ED are $\frac{36}{5}$, $\frac{48}{5}$ and $\frac{14}{5}$, respectively. From the Pythagorean Theorem, the length of DP is $\frac{1}{5}\sqrt{14^2 + 48^2} = 10$, which shows that the triangle ADP is isosceles.

Discussion problem 8.4: (Putnam 2004) For $i = 1, 2$ let T_i be a triangle with side lengths a_i, b_i, c_i, and area A_i. Suppose that $a_1 \leq a_2$, $b_1 \leq b_2$, $c_1 \leq c_2$, and that T_2 is an acute triangle. Does it follow that $A_1 \leq A_2$?
Solution: Yes, it does follow. For $i = 1, 2$, let P_i, Q_i, R_i be the vertices of T_i opposite the sides of length a_i, b_i, c_i, respectively. Since the angle measures in any triangle add up to π, some angle of T_1 must have measure less than or equal to its counterpart in T_2. Without loss of generality assume that $\angle P_1 \leq \angle P_2$. Since the latter is acute (because T_2 is acute), we have $\sin\angle P_1 \leq \sin\angle P_2$. By the Law of Sines,

$$A_1 = \frac{1}{2} b_1 c_1 \sin\angle P_1 \leq \frac{1}{2} b_2 c_2 \sin\angle P_2 = A_2.$$

Discussion problem 8.5: (Putnam 1998) Let H be the unit hemisphere $\{(x, y, z) : x^2 + y^2 + z^2 = 1, z \geq 0\}$, C the unit circle $\{(x, y, 0) : x^2 + y^2 = 1\}$, and P the regular pentagon inscribed in C. Determine the surface area of that portion of H lying over the planar region inside P, and write your answer in the form $A\sin\alpha + B\cos\beta$, where A, B, α, β are real numbers.
Solution: We use the fact that the surface area of the "sphere cap" $\{(x, y, z) \,|\, x^2 + y^2 + z^2 = 1,\ z \geq z_0\}$ is $2\pi(1 - z_0)$. Now the desired surface area is just 2π minus the surface areas of five identical halves of sphere caps; these caps, up to isometry, correspond to z_0 being the distance from the center of the pentagon to any of its sides, i.e., $z_0 = \cos\frac{\pi}{5}$. Thus the desired area is $2\pi - \frac{5}{2}\left(2\pi(1 - \cos\frac{\pi}{5})\right) = 5\pi\cos\frac{\pi}{5} - 3\pi$ (i.e., $B = \pi/2$).

5 Take home problems

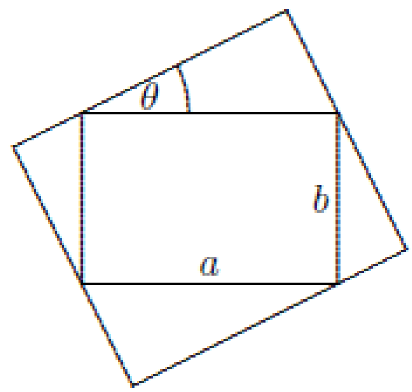
Take home problem 8.1: (NCS/MAA 2003) A rectangle with sides a and b is circumscribed by another rectangle of area m^2. Determine all possible values of m in terms of a and b.

Take home problem 8.2: (Napoleon's theorem) Given a triangle, erect equilateral triangles on all its edges. Show that the centers of the three equilateral triangles form themselves the vertices of an equilateral triangle.

Take home problem 8.3: (Putnam 2008) What is the largest possible radius of a circle contained in a 4-dimensional hypercube of side length 1?

6 Take home solutions

Take home problem 8.1 solution:

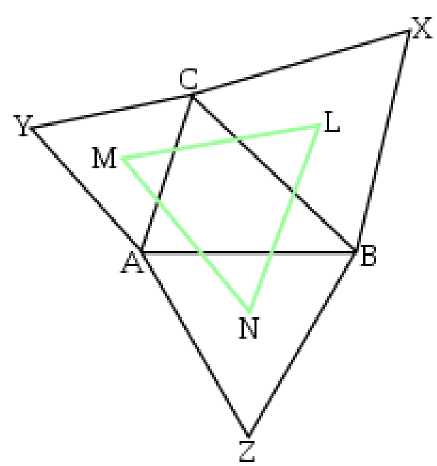


Let θ, $0 \le \theta \le \frac{\pi}{2}$ be the angle formed by side a of the given rectangle and one of the sides of the circumscribing rectangle. The outer rectangle is partitioned into five parts, consisting of the inner rectangle and two pairs of congruent right triangles. Hence,

$$m^2 = ab + (a\sin\theta)(a\cos\theta) + (b\sin\theta)(b\cos\theta) = ab + \frac{1}{2}(a^2+b^2)\sin(2\theta).$$

The range of this function on the interval $[0, \frac{\pi}{2}]$ is $[ab, \frac{1}{2}(a+b)^2]$, so m takes all values in the interval $[\sqrt{ab}, \frac{a+b}{\sqrt{2}}]$.

Take home problem 8.2 solution:



Denote all the points as in the diagram. Each point in the plane can be associated with a complex number, which we will denote by the corresponding lower-case letter. It is easy to see that $x - b = e^{-i\frac{\pi}{3}}(c-b)$ and $l - b = \frac{1}{3}((x-b)+(c-b))$. A simple computation implies

$$l = \Big(\frac{1}{2} - \frac{1}{2\sqrt{3}}i\Big)c + \Big(\frac{1}{2} + \frac{1}{2\sqrt{3}}i\Big)b.$$

Similarly, by symmetry,

$$m = \left(\frac{1}{2} - \frac{1}{2\sqrt{3}}i\right)a + \left(\frac{1}{2} + \frac{1}{2\sqrt{3}}i\right)c,$$
$$n = \left(\frac{1}{2} - \frac{1}{2\sqrt{3}}i\right)b + \left(\frac{1}{2} + \frac{1}{2\sqrt{3}}i\right)a.$$

Verifying that $l - n = e^{-i\frac{\pi}{3}}(m-n)$ completes the proof.

Take home problem 8.3 solution: The largest possible radius is $\frac{\sqrt{2}}{2}$. It will be convenient to solve the problem for a hypercube of side length 2 instead, in which case we are trying to show that the largest radius is $\sqrt{2}$.

Choose coordinates so that the interior of the hypercube is the set $H = [-1,1]^4$ in $\mathbb{R}^4$. Let C be a circle centered at the point P. Then C is contained both in H and its reflection across P; these intersect in a rectangular paralellepiped each of whose pairs of opposite faces are at most 2 unit apart. Consequently, if we translate C so that its center moves to the point $O = (0,0,0,0)$ at the center of H, then it remains entirely inside H.

This means that the answer we seek equals the largest possible radius of a circle C contained in H *and centered at* O. Let $v_1 = (v_{11}, \ldots, v_{14})$ and $v_2 = (v_{21}, \ldots, v_{24})$ be two points on C lying on radii perpendicular to each other. Then the points of the circle can be expressed as $v_1 \cos\theta + v_2 \sin\theta$ for $0 \le \theta < 2\pi$. Then C lies in H if and only if for each i, we have

$$|v_{1i}\cos\theta + v_{2i}\sin\theta| \le 1 \qquad (0 \le \theta < 2\pi).$$

In geometric terms, the vector (v_{1i}, v_{2i}) in $\mathbb{R}^2$ has dot product at most 1 with every unit vector. Since this holds for the unit vector in the same direction as (v_{1i}, v_{2i}), we must have

$$v_{1i}^2 + v_{2i}^2 \le 1 \qquad (i = 1, \ldots, 4).$$

Conversely, if this holds, then the Cauchy-Schwarz inequality and the above analysis imply that C lies in H.

If r is the radius of C, then

$$\begin{aligned} 2r^2 &= \sum_{i=1}^4 v_{1i}^2 + \sum_{i=1}^4 v_{2i}^2 \\ &= \sum_{i=1}^4 (v_{1i}^2 + v_{2i}^2) \\ &\le 4, \end{aligned}$$

so $r \le \sqrt{2}$. Since this is achieved by the circle through $(1,1,0,0)$ and $(0,0,1,1)$, it is the desired maximum.