University of Manitoba, Mathletics 2009

Session 5: Inequalities

1 Facts and definitions

AM-GM inequality: For $a_1, a_2, \dots, a_n \geq 0$,

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq (a_1 a_2 \cdots a_n)^{1/n},$$

with equality iff all a_i's are equal.
Cauchy's inequality: For reals $a_1, \dots, a_n, b_1, \dots, b_n$, if $\mathbf{u} = (a_1, \dots, a_n)$ and $\mathbf{v} = (b_1, \dots, b_n)$, then $|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\| \cdot \|\mathbf{v}\|$, or equivalently

$$(a_1 b_1 + \cdots + a_n b_n)^2 \leq (a_1^2 + \cdots + a_n^2)(b_1^2 + \cdots + b_n^2),$$

where equality holds iff $(a_1, \dots, a_n)$ and $(b_1, \dots, b_n)$ are proportional.
Convex functions, Jensen's inequality: For $x_1, \dots, x_n \in I$, where $I \subset \mathbb{R}$ is an interval, if f is a continuous function on I which is convex (concave up), then

$$f\Big(\frac{x_1 + \ldots + x_n}{n}\Big) \leq \frac{f(x_1) + \ldots + f(x_n)}{n}.$$

If f is strictly convex, then equality holds iff all x_i's are equal.

If the function is concave (concave down), then the sign of the inequality is reversed.

If the function is twice differentiable, it is convex iff $f'' \geq 0$ on the interval. In practice, it is usually easier to show that f' is monotone increasing. The sum of two convex (concave) functions is a convex (concave) function. Multiplication by a positive number also preserves convexity (concavity).
Weighted Jensen's inequality: If $x_1, \dots, x_n \in I$, where I is an interval, $\lambda_1, \dots, \lambda_n > 0$, $\lambda_1 + \cdots + \lambda_n = 1$, f is convex on I, then

$$\lambda_1 f(x_1) + \cdots + \lambda_n f(x_n) \geq f(\lambda_1 x_1 + \cdots + \lambda_n x_n).$$

If f is strictly convex, then equality holds iff all x_i's are equal.
Corollary: weighted AM-GM inequality: If $x_1, \dots, x_n \geq 0$, $\lambda_1, \dots, \lambda_n > 0$, $\lambda_1 + \cdots + \lambda_n = 1$, then

$$\lambda_1 x_1 + \cdots + \lambda_n x_n \geq x_1^{\lambda_1} \cdot \ldots \cdot x_n^{\lambda_n},$$

equality holds iff $x_1 = x_2 = \cdots = x_n$.
Power mean inequality: Let $x_1, \dots, x_n \geq 0$, $\lambda_1, \dots, \lambda_n > 0$, $\lambda_1 + \cdots + \lambda_n = 1$. For $t \in \mathbb{R}$, $t \neq 0$, define the weighted mean M_t of order t as $M_t := (\lambda_1 x_1^t + \cdots + \lambda_n x_n^t)^{1/t}$. Also $M_0 := x_1^{\lambda_1} \cdot \ldots \cdot x_n^{\lambda_n} = \lim_{t \to 0} M_t$, $M_{-\infty} := \min\{x_1, \dots, x_n\} = \lim_{t \to -\infty} M_t$, $M_\infty := \max\{x_1, \dots, x_n\} = \lim_{t \to \infty} M_t$. Then

$$M_s \leq M_t, \quad \text{if } -\infty \leq s < t \leq \infty.$$

2 Examples

Example 5.1: For $a, b, c \geq 0$, prove $a^3 + b^3 + c^3 \geq a^2b + b^2c + c^2a$.

Solution: AM-GM inequality implies $2a^3 + b^3 = a^3 + a^3 + b^3 \geq 3\sqrt[3]{a^6b^3} = 3a^2b$. Similarly, we obtain $2b^3 + c^3 \geq 3b^2c$ and $2c^3 + a^3 \geq 3c^2a$. Adding all three inequalities, we get the required one.

Example 5.2: For each positive integer n, find the smallest possible value of the expression $(x_1 + \cdots + x_n)(x_1^{-1} + \cdots + x_n^{-1})$, over all possible $x_1, \ldots, x_n > 0$.

Solution: With $x_1 = x_2 = \ldots = x_n = 1$, our expression is equal to n^2. So, it is enough to prove that $(x_1 + \cdots + x_n)(x_1^{-1} + \cdots + x_n^{-1}) \geq n^2$ for any $x_1, \ldots, x_n > 0$. This follows immediately from the Cauchy's inequality with $a_k = \sqrt{x_k}$ and $b_k = 1/\sqrt{x_k}$, $k = 1, \ldots, n$.

Example 5.3: If $a, b \geq 0$, $a + b = 2$, prove $(1 + \sqrt[5]{a})^5 + (1 + \sqrt[5]{b})^5 \leq 64$.

Solution: The function $f(x) = (1 + \sqrt[5]{x})^5$ is concave on $[0, \infty)$ as $f'(x) = 5(1 + \sqrt[5]{x})^4 \cdot \frac{1}{5} \cdot x^{-4/5} = (x^{-1/5} + 1)^4$ is decreasing there. By Jensen's inequality, $(1 + \sqrt[5]{a})^5 + (1 + \sqrt[5]{b})^5 = f(a) + f(b) \leq 2f(1) = 64$.

Example 5.4: For $a, b, c > 0$, prove $\dfrac{a^{10} + b^{10} + c^{10}}{a^5 + b^5 + c^5} \geq \left(\dfrac{a + b + c}{3}\right)^5$.

Solution: (Hint: rewrite using M_{10}, M_5 and M_1.) Let M_t be the mean of order t of a, b, c, so $a^t + b^t + c^t = 3M_t^t$. Our inequality becomes $\dfrac{3M_{10}^{10}}{3M_5^5} \geq M_1^5$, which is equivalent to $M_{10}^{10} \geq M_1^5 M_5^5$. To obtain this inequality we multiply the fifth powers of the power mean inequalities $M_{10} \geq M_1$ and $M_{10} \geq M_5$.

3 Problems for discussion

Discussion problem 5.1: For $a_1, \dots, a_n \geq 0$, prove

$$a_1^5 + \cdots + a_n^5 \geq a_1^3 a_2 a_3 + a_2^3 a_3 a_4 + \cdots + a_n^3 a_1 a_2.$$

Discussion problem 5.2: For $x_1, \dots, x_n > 0$, prove

$$\frac{x_1^2}{x_1 + x_2} + \frac{x_2^2}{x_2 + x_3} + \cdots + \frac{x_n^2}{x_n + x_1} \geq \frac{x_1 + \cdots + x_n}{2}.$$

Discussion problem 5.3: If $a, b, c > 0$, prove $\dfrac{a}{a + 3b + 3c} + \dfrac{b}{3a + b + 3c} + \dfrac{c}{3a + 3b + c} \geq \dfrac{3}{7}$.

Discussion problem 5.4: If $a_1, \dots, a_n \geq 1$, prove $\displaystyle\sum_{k=1}^{n} \frac{1}{1 + a_k} \geq \frac{n}{1 + \sqrt[n]{a_1 \cdot \ldots \cdot a_n}}$.

Discussion problem 5.5: For $a_1, \dots, a_n > 0$ with $a_1 \cdot \ldots \cdot a_n = 1$, prove

$$a_1 + \sqrt{a_2} + \cdots + \sqrt[n]{a_n} \geq \frac{n+1}{2}.$$

4 Solutions for discussion problems

Discussion problem 5.1: For $a_1, \dots, a_n \ge 0$, prove

$$a_1^5 + \cdots + a_n^5 \ge a_1^3 a_2 a_3 + a_2^3 a_3 a_4 + \cdots + a_n^3 a_1 a_2.$$

Solution: Let $a_{n+1} = a_1$ and $a_{n+2} = a_2$. AM-GM inequality gives

$$\frac{3a_j^5 + a_{j+1}^5 + a_{j+2}^5}{5} \ge \sqrt[5]{a_j^{15} a_{j+1}^5 a_{j+2}^5} = a_j^3 a_{j+1} a_{j+2}, \quad j = 1, \dots, n.$$

Adding the above inequalities, we obtain the required one.

Discussion problem 5.2: For $x_1, \dots, x_n > 0$, prove

$$\frac{x_1^2}{x_1 + x_2} + \frac{x_2^2}{x_2 + x_3} + \cdots + \frac{x_n^2}{x_n + x_1} \ge \frac{x_1 + \cdots + x_n}{2}.$$

Solution: Let $x_{n+1} = x_1$. Applying Cauchy's inequality with $a_k = \dfrac{x_k}{\sqrt{x_k + x_{k+1}}}$, $b_k = \sqrt{x_k + x_{k+1}}$, $k = 1, \dots, n$, we get

$$\Big(\frac{x_1^2}{x_1 + x_2} + \cdots + \frac{x_n^2}{x_n + x_1}\Big)\Big((x_1 + x_2) + (x_2 + x_3) + \cdots + (x_n + x_1)\Big) \ge (x_1 + \cdots + x_n)^2,$$

which becomes the inequality that we need to prove after division by $2(x_1 + \cdots + x_n)$.

Discussion problem 5.3: If $a, b, c > 0$, prove $\dfrac{a}{a + 3b + 3c} + \dfrac{b}{3a + b + 3c} + \dfrac{c}{3a + 3b + c} \ge \dfrac{3}{7}$.
Solution: Fix $S = a + b + c$. Take $f(x) = x/(3S - 2x)$. The inequality becomes $f(a) + f(b) + f(c) \ge 3f(S/3)$, i.e., Jensen's inequality. It remains to show that f is convex on $[0, S)$ (as $a, b, c > 0$, each of them is smaller than S). Indeed, it is clear that $f'(x) = \frac{3S}{(3S-2x)^2}$ is increasing on $[0, S)$ (even on $[0, \frac{3}{2}S)$).

Discussion problem 5.4: If $a_1, \dots, a_n \ge 1$, prove $\displaystyle\sum_{k=1}^{n} \frac{1}{1 + a_k} \ge \frac{n}{1 + \sqrt[n]{a_1 \cdot \ldots \cdot a_n}}$.

Solution: Consider $f(x) = 1/(1+e^x)$. Straightforward computation shows that $f'' = \dfrac{e^x(2e^x - 1)}{(1 + e^x)^3} > 0$ for $x \ge 0$, so that f is convex on $[0, \infty)$. Since $\ln a_k \ge 0$, $k = 1, \dots, n$, by Jensen's inequality

$$\sum_{k=1}^{n} \frac{1}{1 + a_k} = \sum_{k=1}^{n} f(\ln a_k) \ge n f((\ln a_1 + \cdots + \ln a_n)/n) = \frac{n}{1 + \sqrt[n]{a_1 \cdot \ldots \cdot a_n}}.$$

Discussion problem 5.5: For $a_1, \dots, a_n > 0$ with $a_1 \cdot \ldots \cdot a_n = 1$, prove

$$a_1 + \sqrt{a_2} + \cdots + \sqrt[n]{a_n} \ge \frac{n + 1}{2}.$$

Solution: We write $\sqrt[k]{a_k} = k \sqrt[k]{\dfrac{a_k}{k^k}}$, $k = 1, \dots, n$, and use the weighted AM-GM inequality:

$$\sum_{k=1}^{n} k \sqrt[k]{\frac{a_k}{k^k}} \ge \frac{n(n+1)}{2} \Big(\frac{a_1 \cdot \ldots \cdot a_n}{1^1 \cdot 2^2 \cdot 3^3 \cdot \ldots \cdot n^n}\Big)^{\frac{2}{n(n+1)}} = \frac{n+1}{2} \Big(\frac{n^1 \cdot n^2 \cdot n^3 \cdot \ldots \cdot n^n}{1^1 \cdot 2^2 \cdot 3^3 \cdot \ldots \cdot n^n}\Big)^{\frac{2}{n(n+1)}} \ge \frac{n+1}{2}.$$

5 Take home problems

Take home problem 5.1: If $a, b, c > 0$, prove $\left(\frac{a+b+c}{3}\right)^{a+b+c} \leq a^a b^b c^c$.
Take home problem 5.2: For $a_1, \ldots, a_n, b_1, \ldots, b_n \geq 0$, prove

$$((a_1 + b_1) \cdot \ldots \cdot (a_n + b_n))^{\frac{1}{n}} \geq (a_1 \cdot \ldots \cdot a_n)^{\frac{1}{n}} + (b_1 \cdot \ldots \cdot b_n)^{\frac{1}{n}}.$$

Take home problem 5.3: For $a_1, \ldots, a_n > 0$, prove $a_1^{n+1} + \ldots + a_n^{n+1} \geq a_1 \cdot \ldots \cdot a_n \cdot (a_1 + \ldots + a_n)$.

6 Take home solutions

Take home problem 5.1 solution: As all the numbers are positive, we can take the logarithm on both sides to arrive at the equivalent inequality

$$a \ln a + b \ln b + c \ln c \geq (a+b+c) \ln((a+b+c)/3) = 3((a+b+c)/3) \ln((a+b+c)/3),$$

which is Jensen's inequality with $f(x) = x \ln x$ for a, b, c, if f is convex on $(0, \infty)$. Indeed, $f'(x) = \ln x - 1$ is increasing on $(0, \infty)$.

Take home problem 5.2 solution: If for some k we have $a_k = 0$, then the inequality becomes obvious. Hence, we can assume that all a_k are strictly positive, $k = 1, \dots, n$. Dividing our inequality by $(a_1 \cdot \ldots \cdot a_n)^{\frac{1}{n}}$, we obtain an equivalent inequality

$$\left(\left(1 + \frac{b_1}{a_1}\right) \cdot \ldots \cdot \left(1 + \frac{b_n}{a_n}\right)\right)^{\frac{1}{n}} \geq 1 + \left(\frac{b_1}{a_1} \cdot \ldots \cdot \frac{b_n}{a_n}\right)^{\frac{1}{n}}.$$

Take the logarithm on both sides and denote $y_k = b_k/a_k \geq 0$, $k = 1, \dots, n$. Now we need to prove

$$\frac{1}{n}(\ln(1+y_1) + \ldots + \ln(1+y_n)) \geq \ln(1 + \sqrt[n]{y_1 \cdot \ldots \cdot y_n}).$$

The function $f(x) = \ln(1+e^x)$ is convex on $[0, \infty)$ because $f'(x) = 1 - \frac{1}{1+e^x}$ is increasing on $[0, \infty)$. Hence, by Jensen's inequality

$$\frac{\ln(1+y_1) + \ldots + \ln(1+y_n)}{n} = \frac{f(\ln y_1) + \ldots + f(\ln y_n)}{n} \geq f(\ln((y_1 \cdot \ldots \cdot y_n)^{\frac{1}{n}})) = \ln(1 + \sqrt[n]{y_1 \cdot \ldots \cdot y_n}).$$

Take home problem 5.3 solution: Let M_t be the power mean of order t for $a_1, \dots, a_n$. Our inequality can be rewritten as $nM_{n+1}^{n+1} \geq M_0^n \cdot nM_1$, which follows immediately from the power mean inequalities $M_{n+1} \geq M_0$ (and so $M_{n+1}^n \geq M_0^n$) and $M_{n+1} \geq M_1$.
Alternative way: AM-GM inequality implies

$$\frac{a_k^{n+1}}{n+1} + \sum_{j=1}^{n} \frac{a_j^{n+1}}{n+1} \geq a_k \cdot \prod_{j=1}^{n} a_j,$$

for $k = 1, \dots, n$. Adding these inequalities, we obtain the required one.