University of Manitoba Mathletics

Miscellaneous problems, 24 November 2009

Many Putnam exams have at least one or two fairly straightforward questions. Here are a few that might bring together some of the concepts looked at this year. Almost all of the problems below were from a Putnam given in a year ending with a 9, from 1939–1999.

Example 1: (Question A1, March 1939 Putnam) Find the length of the curve $y^2 = x^3$ from the origin to the point when the tangent makes an angle of 45 degrees with the positive x-axis.

Example 2: (1949 Putnam, 1st question in afternoon) Each rational p/q (p, q relatively prime positive integers) in the open interval $(0, 1)$ is covered by a closed interval of length $\frac{1}{2q^2}$, whose centre is at p/q. Prove that $\sqrt{2}/2$ is not covered by any of the above closed intervals.

Example 3: (1959 Putnam, 1st afternoon) Let each of m distinct points on the positive x-axis be joined to n distinct points on the positive y-axis by straight line segments. Obtain a formula for the number of intersection points (excluding endpoints), assuming that no three of the segments are concurrent.

Example: (A1 1969 Putnam) Let $f(x, y)$ be a polynomial with real coefficients in the real variables x and y defined over the entire x-y plane. What are the possibilities for the range of $f(x, y)$?

Example: (B1 1969 Putnam) The positive integer $n + 1$ is divisible by 24. Show that the sum of all the positive divisors n (including 1 and n) is also divisible by 24.

Example: (1979 Putnam A1) Find the set of positive integers with sum 1979 and maximum possible product.

Example: (1989 Putnam, A1) Which members of the sequence 101, 10101, 1010101, . . . are prime?

Example: (1999 Putnam) Find polynomials $f(x)$, $g(x)$ and $h(x)$, if they exist, such that for all x,

$$|f(x)| - |g(x)| + h(x) = \begin{cases} -1 & \text{if } x < -1, \\ 3x + 2 & \text{if } -1 \leq x \leq 0, \text{ or} \\ -2x + 2 & \text{if } x > 0. \end{cases}$$

Discussion problem 1: For $t \geq 0$, define the *Fermat numbers* $F_t = 2^{2^t} + 1$. Then $F_0 = 3$, $F_1 = 5$, $F_2 = 17$, $F_3 = 257$, and $F_4 = 65537$ (which all happen to be prime, but F_5 is not.
Problem: Prove that distinct Fermat numbers are relatively prime.

Discussion problem 2: Here is a similar question to the 1959 problem.
Question: (1969 IMO) Given $n > 4$ points in the plane, no three collinear, prove that there are at least $\binom{n-3}{2}$ convex quadrilaterals whose vertices are four of the given points.