University of Manitoba Mathletics

Miscellaneous problems with solutions, 24 November 2009

Many Putnam exams have at least one or two fairly straightforward questions. Here are a few that might bring together some of the concepts looked at this year. Almost all of the problems below were from a Putnam given in a year ending with a 9, from 1939–1999.

1 Examples

Example 1: (Question A1, March 1939 Putnam) Find the length of the curve $y^2 = x^3$ from the origin to the point when the tangent makes an angle of 45 degrees with the positive x-axis.

Solution: In the first quadrant, $y = x^{3/2}$, and so the slope is $\frac{dy}{dx} = \frac{3}{2}x^{1/2}$. Let $P(x_0, y_0)$ be the point at which the tangent is at 45 degrees; since $\tan(45) = 1$, set $1 = \frac{3}{2}x^{1/2}$, giving $x_0 = \frac{4}{9}$. Thus the length is

$$\begin{aligned}\int_0^{4/9} \sqrt{1 + (\frac{dy}{dx})^2}\, dx &= \int_0^{4/9} \sqrt{1 + (\frac{9x}{4})^2}\, dx \\ &= \int_1^2 \sqrt{u}\, du = \frac{8}{27}(2\sqrt{2} - 1).\end{aligned}$$

□

Example 2: (1949 Putnam, 1st question in afternoon) Each rational p/q (p, q relatively prime positive integers) in the open interval $(0, 1)$ is covered by a closed interval of length $\frac{1}{2q^2}$, whose centre is at p/q. Prove that $\sqrt{2}/2$ is not covered by any of the above closed intervals.

Solution: The problem may be restated as follows: Show that

$$\left|\frac{\sqrt{2}}{2} - \frac{p}{q}\right| \leq \frac{1}{4q^2} \tag{1}$$

is impossible if $0 < p < q$ are integers. Suppose that (1) holds. Then

$$\left|\frac{1}{2} - \frac{p^2}{q^2}\right| = \left|\frac{\sqrt{2}}{2} - \frac{p}{q}\right| \cdot \left|\frac{\sqrt{2}}{2} + \frac{p}{q}\right| < \frac{1}{4q^2} \cdot 2 = \frac{1}{2q^2},$$

and so $|q^2 - 2p^2| < 1$. But $q^2 - 2p^2 \in \mathbb{Z}$, so $q^2 = 2p^2$, which is impossible since $\sqrt{2}$ is irrational.

Also note that (1) is impossible when $p \geq q > 0$ are integers. (Why?). □

Comment: The above problem is reminiscent of the following:

Theorem 1.1 (Dirichlet, 1879)**.** *Let $x \in \mathbb{R}$. For any $n \in \mathbb{Z}^+$, there is a rational number p/q (where p and q are integers, $q \neq 0$) such that $1 \leq q \leq n$ and*

$$\left| x - \frac{p}{q} \right| < \frac{1}{nq}.$$

Proof: Let $\{x\}$ denote the fractional part of x; for example, if $x = 3.12$, then $\{x\} = .12$.

If x is already rational, then there is nothing to prove. So suppose that x is irrational and consider the $n+1$ numbers

$$\{x\}, \{2x\}, \{3x\}, \ldots, \{nx\}, \{(n+1)x\}.$$

Putting these numbers in the pigeonholes

$$\left(0, \frac{1}{n}\right), \left(\frac{1}{n}, \frac{2}{n}\right), \ldots, \left(\frac{n-1}{n}, 1\right),$$

two must be in the same pigeonhole, say $\{ax\}$ and $\{bx\}$ with $a < b$, and so differ by at most $1/n$. Put $q = b - a$; since q is the difference between two different numbers in $1, 2, \ldots, n+1$, it follows that $1 \leq q \leq n$.

It remains only to see that there is an integer p so that $|qx - p| < 1/n$, and then division by q finishes the theorem. □

Another comment: In 1891, Hurwitz showed that for any irrational α, there are infinitely many pairs p, q of relatively prime integers so that

$$\left| \alpha - \frac{p}{q} \right| < \frac{1}{\sqrt{5}q^2}.$$

If the $\sqrt{5}$ is increased, there are only finitely many solutions.

Example 3: (1959 Putnam, 1st afternoon) Let each of m distinct points on the positive x-axis be joined to n distinct points on the positive y-axis by straight line segments. Obtain a formula for the number of intersection points (excluding endpoints), assuming that no three of the segments are concurrent.

Solution: Every intersection point is determined by two points on the x-axis and two points on the y-axis; further more, every such four points determine an intersection. Hence there are

$$\binom{m}{2}\binom{n}{2} = \frac{mn(m-1)(n-1)}{4}.$$

□

Comment: This problem is similar to one that Lovász solved on a publicly aired television contest show in Hungary (forget the year, perhaps 1974), which won him the contest! The question was:

Question: If n points are placed around a circle and all possible chords are drawn between these points, how many points of intersection are there? The solution was simple: notice that each intersection point is determined by 4 points (two chords, each determined by two points). If three chords are concurrent, some intersection points are determined by more than one set of 4 points. Hence, the maximum number of intersection points is $\binom{n}{4}$.

Example: (A1 1969 Putnam) Let $f(x, y)$ be a polynomial with real coefficients in the real variables x and y defined over the entire x-y plane. What are the possibilities for the range of $f(x, y)$?

Solution: The continuity of $f(x, y)$ implies that the range of f is connected (*i.e.*, if a, b are in the range and c is between a and b, then c is in the range). If the range is bounded above and below, then so if the domain is restricted to, say, for some constant $x = ky$, this gives a single variable polynomial, and so if bounded, is constant. In this case, then for every x, y, $f(x, y) = f(0, 0)$, a range having a single point. Thus the only possibilities are the following types: (i) single point, (ii) intervals of the form $[a, \infty)$, or $(-\infty, b]$, (iii) intervals of the form (a, ∞) or $(-\infty, a)$, or (iv) all real numbers.

Examples for (i), (ii), and (iv) are easy (can you find some?) but (iii) is harder to find. Imagine the graph of the function as a trough, one way cross sections are parabolas, and the other way, the parabolas have lower vertices than approach a limit (going off in either direction). For example, $f(x, y) = (xy - 1)^2 + x^2$ does the trick. □

Example: (B1 1969 Putnam) The positive integer $n + 1$ is divisible by 24. Show that the sum of all the positive divisors n (including 1 and n) is also divisible by 24.

Solution: The condition $24 \mid (n + 1)$ says $n + 1 \equiv 0 \pmod{24}$, which is equivalent to $n \equiv -1 \pmod 3$ and $n \equiv -1 \pmod 8$. Let d be a divisor of n; then $d \equiv 1$ or $2 \pmod 3$ and $d \equiv 1, 3, 5, 7 \pmod 8$. Since $d(\frac{n}{d}) \equiv -1 \pmod 3$ or $\pmod 8$, the only possibilities are (and their opposites)

$$\begin{aligned} d \equiv 1,\ \frac{n}{d} &\equiv 2 \pmod 3 \\ d \equiv 1,\ \frac{n}{d} &\equiv 7 \pmod 8 \\ d \equiv 3,\ \frac{n}{d} &\equiv 5 \pmod 8. \end{aligned}$$

In each case above, $d + \frac{n}{d} \equiv 0 \pmod 3$ and $\pmod 8$. Thus $n + \frac{n}{d}$ is a multiple of 24. It remains to show that no divisor is used twice in any pairing above, that is, that $d \neq \frac{n}{d}$, however this follows because d and $\frac{n}{d}$ are different modulo 3 or 8. Hence the sum of all divisors is divisible by 24. □

Example: (1979 Putnam A1) Find the set of positive integers with sum 1979 and maximum possible product.

Note: Here, the term "set" is used loosely—numbers are allowed to be repeated! (I have not solved the problem when "set" means distinct elements—can you?)

Solution: The desired set cannot contain any 1's, because $n+1 > 1 \times n$, and so one can replace any pair $1, n$ with $n + 1$ and get a larger product. For $n > 4$, $2(n - 2) > n$, and so if n were to be included in the desired set, one increases the product by replacing it with 2 and $n - 2$, maintaining the sum. So, no integer greater than 4 is included. Also, $6 = 2 + 2 + 2 = 3 + 3$, and $2^3 < 3^2$, so the desired set can not contain more than two 2's. Since $4 = 2 + 2 = 2 \times 2$, if one had both 4 and a 2 in the set, replacing the 4 with two 2's gives the same product, but with three 2's, so the set cannot contain both a 4 and a 2.

Hence, the maximum product is achieved with mostly 3's, with either no, one or two 2's (or one 4). Since $1979 = 3 \times 659 + 2$, the maximum occurs with 659 many 3's and one 2 (with product $2 \cdot 3^{659}$). $\square$

Example: (1989 Putnam, A1) Which members of the sequence 101, 10101, 1010101, ... are prime?

Solution: Answer: 101. For $i = 2, 3, \ldots$, let s_i be the term in the sequence with i ones (so the sequence begins $s_2, s_3, \ldots$).

Observe that for each even i, 101 divides s_i. So consider when i is odd, say $i = 2n + 1$ ($n \geq 1$). Then

$$s_{2n+1} = 1 + 10^2 + 10^4 + \cdots + 10^4n = \frac{10^{4n+2} - 1}{99} = \left(\frac{10^{2n+1} + 1}{11}\right)\left(\frac{10^{2n+1} - 1}{9}\right).$$

In this last expression, the first factor is $1 - 10 + 10^2 - \cdots + 10^{2n}$, and the second is a string of $2n + 1$ ones. $\square$

Example: (1999 Putnam) Find polynomials $f(x)$, $g(x)$ and $h(x)$, if they exist, such that for all x,

$$|f(x)| - |g(x)| + h(x) = \begin{cases} -1 & \text{if } x < -1, \\ 3x + 2 & \text{if } -1 \leq x \leq 0, \text{ or} \\ -2x + 2 & \text{if } x > 0. \end{cases}$$

Solution outline: After considering some cases, solve the three linear equations

$$-f + g + h = -1, \quad f + g + h = 3x + 2, \quad f - g + h = -2x + 2.$$

Find that $f(x) = \frac{3}{2}x + \frac{3}{2}$, $g(x) = \frac{5}{2}x$ and $h(x) = -x + \frac{1}{2}$ work. (See the October issue 1970 of the American Math. Monthly for more details.) $\square$

2 Problems for discussion

Discussion problem 1: For $t \geq 0$, define the *Fermat numbers* $F_t = 2^{2^t} + 1$. Then $F_0 = 3$, $F_1 = 5$, $F_2 = 17$, $F_3 = 257$, and $F_4 = 65537$ (which all happen to be prime, but F_5 is not.
Problem: Prove that distinct Fermat numbers are relatively prime.

Solution: First prove that for $n = 0, 1, 2, \ldots$,

$$F_n = \left(\prod_{i=0}^{n-1} F_i \right) + 2, \tag{2}$$

from which the result follows because if any two had a common divisor, it would also have to divide 2, but Fermat numbers are odd.

For $n = 0, 1, 2, \ldots$, let $A(n)$ be the assertion in equation (2).
BASE STEP: Considering an empty product to be 1, $F_0 = 3 = 1 + 2$, and so $A(0)$ is true. To be sure, however, also check that since $F_1 = 2^2 + 1 = 5 = 3 + 2 = F_0 + 2$, $A(1)$ also holds.

INDUCTION STEP: Fix $k \geq 1$, and suppose that $A(k)$ holds. It remains to show

$$A(k+1): \qquad F_{k+1} = \left(\prod_{i=0}^{k} F_i \right) + 2.$$

Beginning with the left side of $A(k+1)$,

$$\begin{aligned}
F_{k+1} &= 2^{2^{k+1}} + 1 \\
&= 2^{2^k} \cdot 2^{2^k} + 1 \\
&= (F_k - 1)(F_k - 1) + 1 \\
&= F_k(F_k - 1) - F_k + 2 \\
&= F_k(\prod_{i=0}^{k-1} F_i + 2 - 1) - F_k + 2 && \text{(by } A(k)\text{)} \\
&= \prod_{i=0}^{k} F_i + F_k - F_k + 2 \\
&= \prod_{i=0}^{k} F_i + 2,
\end{aligned}$$

which finishes the proof of $A(k+1)$ and hence the inductive step $A(k) \to A(k+1)$.

By MI, for each $n \geq 0$, the expression $A(n)$ holds. □

Comment: Note that the above exercise gives a slick proof that the number of primes is infinite (since there are infinitely many Fermat numbers)!

Discussion problem 2: Here is a similar question to the 1959 problem.
Question: (1969 IMO) Given $n > 4$ points in the plane, no three collinear, prove that there are at least $\binom{n-3}{2}$ convex quadrilaterals whose vertices are four of the given points.

Solution: The key here is to fix three of the points. Let A, B, C be three points that lie on the boundary of the convex hull of the n given points. There are $\binom{n-3}{2}$ ways to select two additional points, call them D and E. If D lies outside $\triangle ABC$, then $ABCD$ forms a quadrilateral; similarly if E lies outside $\triangle ABC$. So suppose that D and E lie in the triangle. Since no three points are collinear, neither of D or E lies on any edge of $\triangle ABC$, so suppose that D and E lie strictly inside $\triangle ABC$. The line formed by DE intersects two of the sides of $\triangle ABC$. If the line formed by DE intersects both AB and AC, then $BCDE$ forms a convex quadrilateral. If the line formed by DE intersects both AB and AC, then $ADEC$ forms a convex quadrilateral. The remaining case where the line formed by DE intersects BC and AC is the same as the first case.

So, for all possible choices for D and E, at least $\binom{n-3}{2}$ convex quadrilaterals are formed. $\square$