University of Manitoba, Mathletics 2009

Seesion 1: Mathematical induction, 15 September 2009

1 Facts and definitions

1.1 The two main principles of mathematical induction

"Mathematical induction" (abbreviated "MI") is a proof technique that applies to many mathematical statements about integers. There are many forms an "inductive" proof can take, the first of which has a very simple form:

Theorem 1.1 (Principle of Mathematical Induction (MI))**.**
Let $S(n)$ denote a statement involving an integer n. Let $b \in \mathbb{Z}$ be fixed. If
(i) $S(b)$ holds, and
(ii) for any $k \geq b$, $S(k) \to S(k+1)$,
then for every $n \geq b$, $S(n)$ holds.

Above, (i) is called the "base step", and (ii) is called the "induction step" (or "inductive step"). The assumption of $S(k)$ in the induction step is called the "inductive hypothesis".

Another form for an inductive argument uses a stronger inductive hypothesis. In the next statement, the symbol "$\wedge$" is short for "and".

Theorem 1.2 (Strong Mathematical Induction (SMI))**.** *Let $S(n)$ denote a statement involving an integer n. Fix some $b \in \mathbb{Z}$. If*
(i) $S(b)$ is true and
(ii) for every $k \geq b$, $[S(b) \wedge S(b+1) \wedge \cdots \wedge S(k)] \to S(k+1)$
then for every $n \geq b$, the statement $S(n)$ is true.

Both MI and SMI are logically equivalent (each implies the other).

When writing a MI proof, four steps are usually required.

1. Identify for which n a statement $S(n)$ is to be proved, and write out what $S(n)$ says.
2. Prove the base case(s).
3. Prove the inductive step.
4. State the conclusion.

One common failure in a MI proof attempt is to be unclear about what variables are fixed. For example, (with $b = 0$) some authors say something like "For $n = 0$, the result is trivial, so asssume it to be true for $n - 1$ and prove it for n." Perhaps a better format is to state why the base case is trivial, and then begin the inductive step with something like "Fix $k \geq 0$ and assume that $S(k)$ holds. It remains to show that $S(k+1)$ follows." Note that a new variable in the inductive step often helps to avoid confusion.

There are many variants of MI. Some variants require two base cases; for example, if one can only see how to prove $S(k) \to S(k+2)$, then two base cases, one for the even values and one for the odd values, are required. Sometimes an inductive step requires two (or more) inductive hypotheses, a variant of strong induction. Sometimes two variables need to be inducted on; to see that your proof works for two variables, it is often convenient to plot the values (m, n) for which you wish $S(m, n)$ to hold on a grid and see that together with your base steps and inductive steps, that any point can be reached. For example, one might begin by proving that all cases of the form $S(m, 0)$ hold, and then by proving $S(m, k) \rightarrow S(m, k+1)$. Often, these arguments are nested inside one another.

Another variant of MI is called "proof by infinite descent", and often relies on a set being "well-ordered"—any subset of the set has a least element. For example, among any collection of positive integers is well-ordered, and so does not contain any infinite descending sequence. The idea behind infinite descent is to assume that a solution to a problem exists, and use this solution to find an even "smaller" solution. Continuing this process gives an infinite "descending" sequence that is impossible.

MI is also implicitly used in algorithms or recursion, often without explicit mention. One last comment: often the appearance of a variable n indicates that an MI proof is available, and since MI proofs are usually fairly easy, an attempt using MI is usually "cost-effective".

2 Examples

For the next problem, the notation $a \mid b$ means that a divides b (so b/a is an integer).

Example 1.1: Prove that if $n \geq 1$, then $80 \mid (3^{4n} - 1)$.

Solution: For each $n \geq 1$, let $P(n)$ denote the proposition that $80 \mid (3^{4n} - 1)$.

BASE STEP: Since $3^4 - 1 = 80$, $P(1)$ is true.

INDUCTIVE STEP: Fix some integer $k \geq 1$, and let the inductive hypothesis be that $P(k)$ is true; to be precise, assume that ℓ is an integer so that $3^{4k} - 1 = 80\ell$. Our goal is to show that $P(k+1)$ is true, that is, that 80 divides $3^{4(k+1)} - 1$. Toward this goal,

$$\begin{aligned} 3^{4k+4} - 1 &= 3^4 3^{4k} - 1 \\ &= 81(3^{4k} - 1) + 81 - 1 \\ &= 81 \cdot 80\ell + 80 \quad (\text{by } P(k)) \\ &= 80(81\ell + 1) \end{aligned}$$

shows that $P(k+1)$ indeed follows from $P(k)$, completing the induction step.

By MI, for each $n \geq 1$, 80 divides $3^{4n} - 1$. □

Before giving the next example, examine the following (recall that the integer lattice $\mathbb{Z}^2$ is the set of points in the plane with integer coordinates):

Lemma 2.1. *No three points in the integer lattice $\mathbb{Z}^2$ form an equilateral triangle.*

Proof: Let T be an equilateral triangle with side length c, and suppose that the corners of T are lattice points. If two of these points have (integer) coordinates (x_1, y_1) and (x_2, y_2), then by the distance formula, $c^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$ is an integer. Hence the area $\frac{\sqrt{3}}{4}c^2$ is irrational. However, the area of any polygon with vertices on the integer lattice is rational (see Pick's theorem, if necessary). □

For each $k \in \mathbb{Z}^+$, then no regular $3k$-gon can have all vertices as integer lattice points (because such a polygon has vertices that determine an equilateral triangle). For example, no regular hexagon can have all integer lattice points for vertices. Of course, it is easy to find a square with integer lattice points. The result in the next exercise might seem rather strong, but one proof is surprisingly simple.

Example 1.2: Show that for each positive integer $n \geq 5$, no regular n-gon exists whose vertices are integer lattice points.

Solution: One proof is by infinite descent (but not on n). [The following proof idea occurs in [4].] Let $n \geq 5$ and suppose that lattice points $P_1, \ldots, P_n$ form the vertices of a regular n-gon (in that order). All vectors $\overrightarrow{P_1P_2}$, $\overrightarrow{P_2P_3}$, ..., $\overrightarrow{P_{n-1}P_n}$, and $\overrightarrow{P_nP_1}$ also have integer coordinates.

As in Figure 1, attach the vector $\overrightarrow{P_2P_3}$ to P_1, $\overrightarrow{P_3P_4}$ to P_2, ..., and vector $\overrightarrow{P_1P_2}$ to P_n. These new segments do not overlap unless $n = 5$, in which case they just touch (try it!). All new endpoints thereby formed also have integer coordinates, yet form another n-gon with smaller side length (whose square is an integer, as in the proof of Lemma 2.1). Repeating this process of getting a smaller n-gon on gives an infinite sequence of n-gons on lattice points whose squares of the respective side lengths squared is a decreasing sequence of positive integers. This sequence violates the well-ordering of the positive integers, and so the original polygon does not exist. □

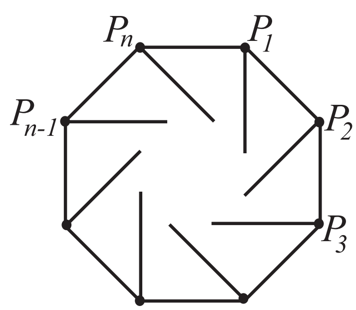

Figure 1: Making a smaller n-gon

Example 1.3: Show that for any integer $n \geq 14$, n is expressible as a sum of 3's and/or 8's.

Solution: Let $S(n)$ be the statement that n is expressible as a sum of 3's and/or 8's.

BASE CASES ($S(14), S(15), S(16)$): Since $14 = 3 + 3 + 8$, $15 = 3 + 3 + 3 + 3 + 3$, and $16 = 8 + 8$, the base steps are shown.

INDUCTIVE STEP ($S(k) \rightarrow S(k + 3)$): Fix $k \geq 14$, and assume that $S(k)$ holds, that is, there exist $\alpha, \beta \in \mathbb{Z}$ so that $k = \alpha \cdot 3 + \beta \cdot 8$. Then $k + 3 = \alpha \cdot 3 + \beta \cdot 8 + 3 = (\alpha + 1) \cdot 3 + \beta \cdot 8$, that is, $k + 3$ is expressible as a sum of 3's and/or 8's, showing $S(k + 3)$ holds, completing the inductive step.

By MI, for all $n \geq 14$, the statement $S(n)$ is true. (Actually, there are three separate proofs by MI rolled into one, one proving the statement for the sequence $n = 14, 17, 20, \ldots$, one for $n = 15, 18, 21, \ldots$, and another for $n = 16, 19, 22, \ldots$.) □

3 Problems for discussion

Discussion problem 1.1: Prove that for each $n \geq 1$,

$$2^2 + 4^2 + 6^2 + \cdots + (2n)^2 = \frac{2n(n+1)(2n+1)}{3}.$$

Discussion problem 1.2: Prove that for every $n \in \mathbb{Z}^+$ and every non-negative real number a,

$$\sqrt{a + 1 + \sqrt{a + 2 + \cdots + \sqrt{a+n}}} < a + 3.$$

Discussion problem 1.3: Prove that the equation

$$a^2 + b^2 + c^2 + d^2 = 2abcd \tag{1}$$

has no solutions in positive integers.

Discussion problem 1.4: For any given $n \geq 1$, consider all the subsets of $\{1, 2, \ldots, n\}$ that do not contain two consecutive numbers. For example, when $n = 4$, the sets are $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{1, 3\}$, $\{1, 4\}$, and $\{2, 4\}$. Prove that sum of the squares of the products in each set is $(n+1)! - 1$. (For example, when $n = 4$, the number is $1^2 + 2^2 + 3^2 + 4^2 + 3^2 + 4^2 + 8^2 = 119 = 5! - 1$.)

4 Solutions for discussion problems

Solution Discussion problem 1.1: For $n \geq 1$, denote the statement in the exercise by

$$S(n): \quad 2^2 + 4^2 + 6^2 + \cdots + (2n)^2 = \frac{2n(n+1)(2n+1)}{3}.$$

BASE STEP ($n = 1$): Since $2^2 = 4 = \frac{2(1+1)(2+1)}{3}$, the statement $S(1)$ holds.

INDUCTIVE STEP: For some fixed $k \geq 1$, assume the inductive hypothesis

$$S(k): \quad 2^2 + 4^2 + 6^2 + \cdots + (2k)^2 = \frac{2k(k+1)(2k+1)}{3}.$$

to be true. It remains to show that

$$S(k+1): \quad 2^2 + 4^2 + 6^2 + \cdots + (2(k+1))^2 = \frac{2(k+1)(k+2)(2(k+1)+1)}{3}$$

follows from $S(k)$. Starting with the left side of $S(k+1)$, derive the right side:

$$\begin{aligned}
&2^2 + 4^2 + 6^2 + \cdots + (2k)^2 + (2(k+1))^2 \\
&= \frac{2k(k+1)(2k+1)}{3} + (2(k+1))^2 \quad \text{(by ind. hyp. } S(k)\text{)} \\
&= (k+1)\frac{2k(2k+1)}{3} + \frac{(3(4(k+1)^2}{3} \\
&= 2(k+1)\left[\frac{k(2k+1)}{3} + \frac{(3(2(k+1)}{3}\right] \\
&= 2(k+1)\frac{2k^2 + k + 6k + 6}{3} \\
&= 2(k+1)\frac{2k^2 + 7k + 6}{3} \\
&= 2(k+1)\frac{(k+2)(2k+3)}{3},
\end{aligned}$$

which agrees with the right side of $S(k+1)$. This completes the inductive step $S(k) \to S(k+1)$.

Therefore, by the principle of mathematical induction, $S(n)$ is true for all $n \geq 1$. □

Solution Discussion problem 1.2: Let $S(n)$ be the statement that for any non-negative real a,

$$\sqrt{a + 1 + \sqrt{a + 2 + \cdots + \sqrt{a+n}}} < 3.$$

BASE STEP: $S(1)$ says $\sqrt{a+1} < a+3$, which is verifiable since $a+1 < (a+3)^2 \Leftrightarrow 0 < a^2+5a+8$, which is true for $a \geq 0$.

Inductive step: Fix some $k \geq 0$, and suppose that $S(k)$ is true, that is, for any non-negative a,

$$\sqrt{a+1+\sqrt{a+2+\cdots+\sqrt{a+k}}} < a+3$$

is true. It remains to prove $S(k+1)$, namely that for every non-negative b,

$$\sqrt{b+1+\sqrt{b+2+\cdots+\sqrt{b+k+\sqrt{b+k+1}}}} < b+3.$$

Indeed, using $a = b+1$,

$$\begin{aligned}
&\sqrt{b+1+\sqrt{b+2+\cdots+\sqrt{b+k+\sqrt{b+k+1}}}} \\
&= \sqrt{b+1+\sqrt{a+1+\cdots+\sqrt{a+k-1+\sqrt{a+k}}}} \\
&< \sqrt{b+1+a+3} \quad \text{(by } S(k)\text{)} \\
&= \sqrt{2b+5} \\
&< b+3,
\end{aligned}$$

where the last inequality follows since $2b+5 < (b+3)^2 = b^2+6b+9$ and $b^2 \geq 0$. This proves $S(k+1)$, concluding the inductive step.

Hence, by MI, $S(n)$ is true for all $n \geq 1$. □

Solution for Discussion problem 1.3: Suppose that positive integers a, b, c, d satisfy equation (1). Since the right side (1) is even, so is the left side. Hence among a, b, c, d, the number of odd numbers is even. If all four are odd, the left side is divisible by 4 but the right is divisible by only two. If exactly two are odd, the left is divisible by only 2 (not 4) and the right is divisible by 8. Hence all of a, b, c, d are even, say $a = 2a_1$, $b = 2b_1$, $c = 2c_1$, $d = 2d_1$. Replacing these values into (1) gives (after a bit of simplification)

$$a_1^2 + b_1^2 + c_1^2 + d_1^2 = 8a_1b_1c_1d_1. \tag{2}$$

Arguing as before, the left side of (2) is even; if all of a_1, b_1, c_1, d_1 are odd, the left is divisible by only 4, whereas the right is divisible by 8; if just two are odd, the left is divisible by only 2; so all are even, say $a_1 = 2a_2$, $b_1 = 2b_2$, $c_1 = 2c_1$, and $d_1 = 2d_2$. Replacing these values in (2),

$$a_2^2 + b_2^2 + c_2^2 + d_2^2 = 32a_2b_2c_2d_2. \tag{3}$$

Continuing inductively, for each $t \in \mathbb{Z}^+$, the numbers $a_t = a/2^t$, ,$b_t = b/2^t$, $c_t = c/2^t$, and $d_t = d/2^t$ are positive integers so that

$$a_t^2 + b_t^2 + c_t^2 + d_t^2 = 2^{2t+1}a_tb_tc_td_t. \tag{4}$$

However, this gives an infinite decreasing sequence of positive integers a_i's (for example), contrary to the well-ordering property of $\mathbb{Z}^+$. Hence no such solution a, b, c, d exists. □

Brief solution for Discussion problem 1.4: If $n = 1$, then the sum of the products is $1 = 2! - 1$. If $n = 2$, the sum of the products is $1^2 + 2^2 = 5 = 3! - 1$. So assume the statement is true for $n = k - 1$ and $n = k$, and examine all subsets of $\{1, 2, \ldots, k + 1\}$ that contain no two consecutive numbers. Of these, there are two kinds of subsets, those containing $k+1$, and those which don't. Of those which contain $k + 1$, they can not contain k, and so by the inductive hypothesis with $n = k - 1$, the contribution to the sum of the squares of products for sets in $\{1, 2, \ldots, k - 1\}$ is $k! - 1$. Since the product of numbers in the sets containing both $k + 1$ and elements from $\{1, 2, \ldots, k - 1\}$ have an additional factor of $(k + 1)^2$, the total for such sets is $(k+1)^2[k! - 1]$. Together with the set $\{k+1\}$, the total for all suitable subsets containing $k+1$ is $(k + 1)^2[k! - 1] + (k + 1)^2$. For those sets not containing $k + 1$, this reduces precisely to the case when $n = k$, and so by the inductive hypothesis, the sum of the squares of the products of subsets not containing $k + 1$ is $(k + 1)! - 1$.

Therefore, in all, the sum of the squares is

$$
\begin{aligned}
(k + 1)^2[k! - 1] + (k + 1)^2 + (k + 1)! - 1 &= (k + 1)^2 k! + (k + 1)! - 1 \\
&= (k + 2)! - 1.
\end{aligned}
$$

This finishes the inductive step, and hence by MI, the solution. □

5 Take home problems

The first problem is fairly straightforward, and is intended to help writing skills; please feel free to submit a solution for comments. The next two problems are somewhat more challenging; we look forward to seeing solutions presented.

Take home problem 1.1: Prove that if $n \geq 1$, then $3 \mid (2^{2n} - 1)$.

Take home problem 1.2: (Difficult) Show that if a, b are non-negative integers, one of which is non-zero, and $q = (a^2 + b^2)/(ab + 1)$ is a positive integer, then $q = [\gcd(a, b)]^2$.

Take home problem 1.3: Planes in three dimensional space are said to be in *general position* if no three planes share a common line and no two planes are parallel. Prove that the maximum number of regions three dimensional space is divided into by n planes in general position is

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3},$$

and the number of infinite (unbounded) regions is

$$2\binom{n}{0} + 2\binom{n}{2}.$$

6 Take home problem solutions

Take home problem 1.1 solution: For $n \geq 1$, let $S(n)$ be the statement that $3 \mid (2^{2n} - 1)$.

BASE STEP: Since $2^{2\cdot 1} - 1 = 3$, the statement $S(1)$ is true.

INDUCTION STEP: For some fixed $k \geq 1$, suppose that $S(k)$ holds, that is, the induction hypothesis (IH) is that there is some integer m so that $2^{2k} - 1 = 3m$. Then

$$2^{2(k+1)} - 1 = 4 \cdot 2^{2k} - 1 = 4(2^{2k} - 1) + 3 \stackrel{\text{IH}}{=} 4(3m) + 3 = 3(4m + 1)$$

shows that $2^{2(k+1)} - 1$ is also divisible by 3, proving $S(k+1)$. [Note: the notation $\stackrel{\text{IH}}{=}$ indicates where the induction hypothesis is used. This solution could have been written in "vertical form" (one inequality per line) with "(by $S(k)$)" on the side.]

Since $S(1)$ is true and $S(k) \to S(k+1)$, by mathematical induction, for any $n \geq 1$, the statement $S(n)$ holds. $\square$

Take home problem 1.2 solution: This problem and solution appears in [4, 8.1, pp. 207,211], and is referred to as "this famous IMO 1988 problem". In fact, the original IMO problem ([7, q. 6, p.216]) asks only to prove that when a and b are positive, then q is a square; the IMO solution given is by infinite descent and is given here following the solution to the present exercise (an "upward" induction adapted from a solution credited to J. Campbell in [4]).

Solution to problem 1.2: For non-negative integers a and b, let $S(a, b)$ be the statement that if $q = \frac{a^2+b^2}{ab+1}$ is an integer, then $q = (\gcd(a, b))^2$.
BASE STEP: When one of $a = 0$ or $b = 0$ holds, either $q = a^2$ or $q = b^2$. In either case, the result is true using $\gcd(a, 0) = a$ or $\gcd(0, b) = b$.

INDUCTIVE STEP: Fix positive integers a, b, and without loss of generality, assume that $0 < a \leq b$. [Note that if $S(a, b)$ is true, then trivially, so is $S(b, a)$.] For every c with $0 \leq c < b$, assume that $S(a, c)$ holds. Put $q = \frac{a^2+b^2}{ab+1}$; it remains to show that $q = \gcd(a, b)^2$. First seek a c giving the same q, that is, look for c so that

$$q = \frac{a^2 + c^2}{ac + 1}.$$

To accomplish this, one can use an old trick: if $\frac{A}{B} = \frac{C}{D} = q$, then $\frac{A-C}{B-D} = \frac{Bq-Dq}{B-D} = q$ as well. So putting

$$q = \frac{a^2 + b^2}{ab + 1} = \frac{a^2 + c^2}{ac + 1},$$

and subtracting both numerators and denominators as in the trick, get

$$q = \frac{b^2 - c^2}{ab - ac} = \frac{b + c}{a},$$

and so $c = aq - b$ works.

Claim: $0 \leq c < b$. To see the upper bound,

$$q = \frac{a^2 + b^2}{ab + 1} < \frac{a^2 + b^2}{ab} = \frac{a}{b} + \frac{b}{a},$$

which gives $aq < \frac{a^2}{b} + b \le \frac{b^2}{b} + b = 2b$, and so $c = aq - b < b$. To see the lower bound, $q = \frac{a^2+c^2}{ac+1}$ implies that $ac + 1 > 0$ and so $c \ge 0$. Thus, the claim is proved.

Since a suitable c less than b has been found, it remains to observe that

$$\gcd(a, c) = \gcd(a, aq - b) = \gcd(a, b),$$

and so by the induction hypothesis $S(a, c)$, $q = \gcd(a, c)^2 = \gcd(a, b)^2$, completing the proof of $S(a, b)$.

Thus, by induction, for all non-negative integers a, b, the statement $S(a, b)$ holds. □

Original IMO problem solution: In this solution, one has to prove only that q is a perfect square. The idea is similar to that above, however uses infinite descent and assumes that both a and b are positive. The technique is as follows: If some pair (a, b) violates the result, one exhibits another "smaller pair" (a', b') for which the result is again violated; continuing this process, obtain an infinite sequence of smaller and smaller pairs. Since there are only finitely many pairs of positive integers smaller than any given (a, b), this process of finding smaller pairs can not be infinite, so no pair (a, b) violates the result. Such a proof by infinite descent is often employed by assuming that one has found a "minimal" case for which the result fails, and then showing that one can find an even smaller case for which the result fails.

Fix positive integers a, b, and suppose that

$$k = \frac{a^2 + b^2}{ab + 1}$$

is an integer, but not a perfect square. Then $abk + k = a^2 + b^2$ implies

$$a^2 - kab + b^2 = k. \tag{5}$$

Since k is not a perfect square, $k \ge 2$. Let (a, b) be a minimal pair satisfying (5). Assume w.l.o.g., that $a \ge b$.

The case $a = b$ is impossible because (5) implies $a^2(2 - k) = k$, and $2 - k \le 0$ gives $k \le 0$, a contradiction. So assume that $a < b$. The polynomial

$$p(x) = x^2 - kbx + b^2 - k$$

has two roots, one of which is a; let a_1 be the other root. By Viéte's relations (see handout for polynomials for details, but this is just a simple fact for quadratic equations and their roots), $a + a_1 = kb$, and so $a_1 \in \mathbb{Z}$.

Claim: $a < kb$.

Proof of claim: If $a = kb$, then (5) gives $b^2 = k$, contrary to the assumption that k is not a perfect square, so $a \ne kb$. If $a > kb$, then again from (5), $k > (kb)^2$, and

$$k = a^2 - kab + b^2 = a(a - kb) + b^2 > a + b^2 > a > kb$$

gives a contradiction (the second last inequality is because $k \ge 2$). The claim is proved.

Claim: $a_1 < a$, and a_1 is a positive integer.

Proof of Claim: The condition $a < kb$ implies that $a^2 < kab$; thus $a^2 - kab < 0$, and again by (5), $b^2 > k$. Again by Viéte's relations, and the fact that $a > 0$, then $aa_1 = b^2 - k$, which is positive, and so a_1 is a positive integer. Since $b^2 \geq a^2$ and $k > 1$,

$$a_1 = \frac{b^2 - k}{a} < \frac{a^2 - 1}{a} < a,$$

and so the claim is proved.

Thus, (a_1, b) is a smaller solution to (5) with k not a square; so (a, b) is not a mimimal counterexample to the result. □

Take home problem 1.3 solution outline: For each $n \geq 0$, let r_n be the maximum number of regions in the plane determined by n lines in general position (no two parallel, no three concurrent). By adding a line to a configuration of n lines, obtain the recurrence

$$r_{n+1} = r_n + n + 1,$$

and use this (with induction) to show that $r_n = 1 + \binom{n+1}{2}$.

Define s_n to be the maximum number of regions into which n planes separate 3-space. By adding a plane to an existing configuration, show the recurrence

$$s_{n+1} = s_n + r_n.$$

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