Analysis

University of Manitoba, Mathletics 2009

1 Some Facts and Examples

- Need to know definitions and properties of continuous/differentiable functions, sequences, infinite series, etc. (everything that is studied in our Calculus sequence) - those concepts are not discussed here
- **Equidistribution criterion** (useful for convergence of infinite series): If $f : \mathbb{R} \to \mathbb{R}$ is a continuous periodic function with irrational period and if $\sum_n \frac{|f(n)|}{n} < \infty$, then f is identically zero.
  Example 1: does the series $\sum_{n=1}^{\infty} \frac{|\sin n|}{n}$ converge? (Take home problem: provide a direct proof that this series diverges.)
- Definite integrals (pay attention to limits of integration!).
  Example 2: Let $f : [0,1] \to \mathbb{R}$ be a continuous function. Prove that

$$\int_0^{\pi} x f(\sin x)\, dx = \pi \int_0^{\pi/2} f(\sin x)\, dx$$

  Solution: $\int_0^{\pi} x f(\sin x)\, dx = \int_0^{\pi/2} x f(\sin x)\, dx + \int_{\pi/2}^{\pi} x f(\sin x)\, dx$ and using the substitution $y = \pi - x$ in the second integral we get

$$\int_{\pi/2}^{\pi} x f(\sin x)\, dx = \int_0^{\pi/2} (\pi - y) f(\sin y)\, dy\,.$$

  Now, adding the two integrals we get $\pi \int_0^{\pi/2} f(\sin x)\, dx$ as desired.
- **Riemann Sums:**

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(\xi_i) \frac{b-a}{n} = \int_a^b f(x)\, dx\,,$$

  where each ξ_i is a number in $\left[a + (i-1)\frac{b-a}{n}, a + i\frac{b-a}{n}\right]$.
  Example 3: Compute the limit

$$\lim_{n\to\infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \right)$$

  Solution: Rewrite the sum inside the limit as:

$$\frac{1}{n} \left(\frac{1}{1+1/n} + \frac{1}{1+2/n} + \cdots + \frac{1}{1+n/n} \right) = \sum_{i=1}^{n} \frac{1}{1+\xi_i} \frac{1}{n}\,,$$

where $\xi_i = i/n$, $i = 1, \ldots, n$. Hence, $a = 0$, $b = 1$, $f(x) = \frac{1}{1+x}$, and ξ_i is the right endpoint of $\left[(i-1)\dfrac{1}{n}, i\dfrac{1}{n}\right]$, $i = 1, \ldots, n$. Hence, this limit is equal to

$$\int_0^1 \frac{1}{1+x}\, dx = \ln 2\,.$$

- **Cauchy-Schwarz inequality:**

$$\left(\int_D f(x)g(x)\, dx\right)^2 \leq \left(\int_D f^2(x)\, dx\right)^2 \left(\int_D g^2(x)\, dx\right)^2$$

- **Hölder's inequality:** If $p, q > 1$ are such that $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\int_D |f(x)g(x)|\, dx \leq \left(\int_D |f(x)|^p\, dx\right)^{1/p} \left(\int_D |g(x)|^q\, dx\right)^{1/q}\,.$$

  Equality holds if and only if there are constants α and β, not both 0, such that $\alpha|f|^p = \beta|g|^q$ a.e.

- **Minkowski's inequality:** If $p \geq 1$, then

$$\left(\int_D |f(x) + g(x)|^p\, dx\right)^{1/p} \leq \left(\int_D |f(x)|^p\, dx\right)^{1/p} + \left(\int_D |g(x)|^p\, dx\right)^{1/p}\,.$$

- **Chebyshev's inequality:** Let f and g be two increasing functions on $\mathbb{R}$. Then, for any real numbers $a < b$,

$$(b-a)\int_a^b f(x)g(x)\, dx \geq \left(\int_a^b f(x)\, dx\right)\left(\int_a^b g(x)\, dx\right)\,.$$

  Proof: Because f and g are increasing, $[f(x) - f(y)][g(x) - g(y)] \geq 0$. Integrating this over $[a, b] \times [a, b]$ and expanding, we obtain

$$\int_a^b \int_a^b f(x)g(x)\, dx\, dy + \int_a^b \int_a^b f(y)g(y)\, dx\, dy - \int_a^b \int_a^b f(x)g(y)\, dx\, dy - \int_a^b \int_a^b f(y)g(x)\, dx\, dy \geq 0\,,$$

  which implies what we need.
  (To think at home: what if f and g are both decreasing? what if f is increasing and g is decreasing?)

- **Fubini's theorem:** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a piecewise continuous function such that $\int_c^d \int_a^b |f(x, y)|\, dx\, dy < \infty$. Then

$$\int_c^d \int_a^b f(x, y)\, dx\, dy = \int_a^b \int_c^d f(x, y)\, dy\, dx\,.$$

- **Tonelli's theorem:** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a positive piecewise continuous function. Then

$$\int_c^d \int_a^b f(x,y)\, dx\, dy = \int_a^b \int_c^d f(x,y)\, dy\, dx\,.$$

  Remark: Let $f(x,y) = (x^2 - y^2)/(x^2 + y^2)^2$. Then $\int_0^1 \int_0^1 f(x,y)\, dx\, dy = -\pi/4$ and $\int_0^1 \int_0^1 f(x,y)\, dy\, dx = \pi/4$. In particular, by Fubini's theorem, one can conclude that $\int_0^1 \int_0^1 |f(x,y)|\, dx\, dy = \infty$.

- **Gaussian integral formula:** $\displaystyle\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$

  (Take home problem: prove this formula. Hint: consider $\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\, dx\, dy$ and use polar coordinates.)

- **Cauchy's equation:** $f(x+y) = f(x) + f(y)$. If f is assumed to be continuous, then the linear functions $f(x) = cx$, $c \in \mathbb{R}$, are the only solutions. (Idea of the proof: show that $f(x) = f(1)x$ for rational x and then use continuity.)

2 Problems for discussion

Discussion problem 1: Compute $\displaystyle\int_0^{\pi} \frac{x\sin x}{1+\sin^2 x}\,dx$

Discussion problem 2: Compute

$$\lim_{n\to\infty}\left(\frac{1}{\sqrt{4n^2-1^2}}+\frac{1}{\sqrt{4n^2-2^2}}+\cdots+\frac{1}{\sqrt{4n^2-n^2}}\right)$$

Discussion problem 3: Let $f:[a,b]\to[a,b]$ be a continuous function. Prove that f has a fixed point (i.e., there is a point $\xi\in[a,b]$ such that $f(\xi)=\xi$).

Discussion problem 4: Let f be a function having a continuous derivative on $[0,1]$ and such that $0<f'(x)\le 1$ and $f(0)=0$. Prove that

$$\left(\int_0^1 f(x)\,dx\right)^2 \ge \int_0^1 (f(x))^3\,dx\,.$$

Give an example in which equality occurs.

Discussion problem 5: Find the maximal value of the ratio

$$\left(\int_0^3 f(x)\,dx\right)^3 \Big/ \int_0^3 f^3(x)\,dx\,,$$

as f ranges over all positive continuous functions on $[0,1]$.

Discussion problem 6: Show that for $a,b>0$,

$$\int_0^{\infty}\frac{e^{-ax}-e^{-bx}}{x}\,dx=\ln(b/a)\,.$$

Discussion problem 7: Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous nonzero function satisfying the equation

$$f(x+y)=f(x)f(y)\,,\quad \text{for all } x,y\in\mathbb{R}\,.$$

Prove that there exists $c>0$ such that $f(x)=c^x$ for all $x\in\mathbb{R}$.

3 Solutions for discussion problems

Discussion problem 1: Compute $\displaystyle\int_0^{\pi} \frac{x\sin x}{1+\sin^2 x}\,dx$

Solution: Use Example 2 with $f(x) = \frac{x}{1+x^2}$ to transform this integral into

$$\pi \int_0^{\pi/2} \frac{\sin x}{1+\sin^2 x}\,dx$$

Now, use the substitution $u = \cos x$ to show that this integral is equal to

$$\pi \int_0^1 \frac{1}{2-u^2}\,du = \frac{\pi}{2\sqrt{2}} \ln\left(\frac{\sqrt{2}+u}{\sqrt{2}-u}\right)\Bigg|_0^1 = \frac{\pi}{2\sqrt{2}} \ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right).$$

Discussion problem 2: Compute

$$\lim_{n\to\infty}\left(\frac{1}{\sqrt{4n^2-1^2}} + \frac{1}{\sqrt{4n^2-2^2}} + \cdots + \frac{1}{\sqrt{4n^2-n^2}}\right)$$

Solution: We have

$$s_n = \frac{1}{n}\left(\frac{1}{\sqrt{4-(1/n)^2}} + \frac{1}{\sqrt{4-(2/n)^2}} + \cdots + \frac{1}{\sqrt{4-(n/n)^2}}\right),$$

and so s_n is the Riemann sum of the function $f(x) = \frac{1}{\sqrt{4-x^2}}$ associated with the subdivision of $[0,1]$ into n intervals of equal length with $\xi_i = i/n$. Therefore,

$$\lim_{n\to\infty} s_n = \int_0^1 \frac{1}{\sqrt{4-x^2}}\,dx = \sin^{-1}(x/2)\big|_0^1 = \pi/6\,.$$

Discussion problem 3: Let $f : [a,b] \to [a,b]$ be a continuous function. Prove that f has a fixed point (i.e., there is a point $\xi \in [a,b]$ such that $f(\xi) = \xi$).

Solution: Apply the Intermediate Value Theorem to $g(x) = f(x) - x$. Because $f(a) \geq a$ and $f(b) \leq b$, it follows that $g(a) \leq 0$ and $g(b) \geq b$. Hence, there is a point $\xi \in [a,b]$ such that $g(\xi) = 0$.

Discussion problem 4: Let f be a function having a continuous derivative on $[0,1]$ and such that $0 < f'(x) \leq 1$ and $f(0) = 0$. Prove that

$$\left(\int_0^1 f(x)\,dx\right)^2 \geq \int_0^1 (f(x))^3\,dx\,.$$

Give an example in which equality occurs.

Solution: First of all, note that f is nonnegative on $[0,1]$ since $f(0)=0$ and $f'(x)>0$ (i.e., f is increasing). Consider

$$F(t)=\left(\int_0^t f(x)\,dx\right)^2-\int_0^t (f(x))^3\,dx\,,\quad t\in[0,1]\,.$$

We will show that $F(t)\geq 0$, for all $t\in[0,1]$, which implies that $F(1)\geq 0$ as needed. Since $F(0)=0$, it suffices to show that F is increasing. To prove this, we need to show that $F'(t)\geq 0$.

$$F'(t)=f(t)\left(2\int_0^t f(x)\,dx-f^2(t)\right)\,.$$

So it remains to show that

$$G(t):=2\int_0^t f(x)\,dx-f^2(t)$$

is nonnegative on $[0,1]$. Since $G(0)=0$, it suffices to prove that G is increasing on $[0,1]$ (i.e., $G'(t)\geq 0$).

$$G'(t)=2f(t)(1-f'(t))\geq 0\,,$$

since $f'(t)\leq 1$ and f is nonnegative. An example in which equality holds is $f(x)=x$.

Discussion problem 5: Find the maximal value of the ratio

$$\left(\int_0^3 f(x)\,dx\right)^3\Big/\int_0^3 f^3(x)\,dx\,,$$

as f ranges over all positive continuous functions on $[0,1]$.

Solution: By Hölder's inequality,

$$\int_0^3 f(x)\cdot 1\,dx\leq\left(\int_0^3 |f(x)|^3\,dx\right)^{1/3}\left(\int_0^3 1^{3/2}\,dx\right)^{2/3}=3^{2/3}\left(\int_0^3 f^3(x)\,dx\right)^{1/3}\,,$$

which implies that

$$\left(\int_0^3 f(x)\,dx\right)^3\Big/\int_0^3 f^3(x)\,dx\leq 9\,.$$

To see that the maximum 9 can be achieved, choose $f\equiv 1$.

Discussion problem 6: Show that for $a,b>0$,

$$\int_0^\infty \frac{e^{-ax}-e^{-bx}}{x}\,dx=\ln(b/a)\,.$$

Solution: Applying Tonelli's theorem to the function $f(x,y)=e^{-xy}$, we write

$$\int_0^\infty \frac{e^{-ax}-e^{-bx}}{x}\,dx=\int_0^\infty\int_a^b e^{-xy}\,dy\,dx=\int_a^b\int_0^\infty e^{-xy}\,dx\,dy=\int_a^b (1/y)\,dy=\ln(b/a)\,.$$

Discussion problem 7: Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous nonzero function satisfying the equation

$$f(x+y) = f(x)f(y)\,, \quad \text{for all } x, y \in \mathbb{R}\,.$$

Prove that there exists $c > 0$ such that $f(x) = c^x$ for all $x \in \mathbb{R}$.

Solution: Because $f(x) = f^2(x/2) > 0$, the function $g(x) = \ln f(x)$ is well defined. It satisfies Cauchy's equation and is continuous. Hence, $g(x) = \alpha x$ for some $\alpha \in \mathbb{R}$. We obtain $f(x) = c^x$ with $c = e^\alpha$.

References

[1] Răzvan Gelca and Titu Andreescu, *Putnam and beyond*, Springer, New York, 2007. MR2334764 (2008c:00002)