2009 University of Manitoba Mathletics Training

WEEK 6: Sequences and Recursion (Draft: October 21, 2009)

A **sequence** is a finite or infinite list of numbers, often denoted formally with notations like

$$\{a_n\}_{n=1}^{\infty};\ \{a_n\}_{n=1}^{N};\ a_1, a_2, a_3, \ldots, a_n, \ldots;\quad \text{or } a_1, a_2, a_3, \ldots, a_N.$$

a_n is the nth **element**, or **entry**, of the sequence; the subscript n used to enumerate the elements is the **index**. It should be obvious that the index, may start at 0, 2, or any integer; sequences can be finite, singly or doubly infinite ($\{a_n\}_{n=-\infty}^{\infty} = \{\ldots, a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots\}$). Also note that a sequences is NOT a set—in spite of the traditional curly brace notation, which is usually reserved for sets (for this reason you'll often see other notations, such as round or square parentheses).

Sequences are generally specified in one of two ways (there are, of course, many variants):

1. By expressing each element a_n **explicitly**, or **in closed form** as an elementary function, $f(n)$, of the index, n. For example, the sequence of perfect squares is $\{n^2\}_{n=0}^{\infty}$.
2. By expressing each element a_n (after the first, or the first few) by its relationship to its predecessor. A sequence given in this way is said to be **recursively defined**. (Notice the obvious echo of mathematical induction here!)

Often a problem simply requires a closed form expression (i.e., without infinite sums or indirect ways of specifying values) for a sequence given recursively. This is called **solving** the sequence.

For example, let $a_0 = 0$ and for $n > 0$, define $a_n := a_{n-1} + n$. (This is called the sequence of **triangular numbers**. Can you see why?) Thus, $\{a_n\} = \{0, 1, 3, 6, 10, 15, \ldots\}$. There are many ways "solve" (express in closed form) this sequence. For example, you may observe that

$$2a_n = (1 + 2 + \cdots + n) + (n + \cdots + 2 + 1) = (1 + n) + (2 + (n - 1)) + \cdots = n(n + 1),$$

so $a_n = \frac{n(n+1)}{2}$.

Important common sequences (Not exhaustive!)

Arithmetic: $a_n = \begin{cases} a & n = 0 \\ a_{n-1} + d & n > 0 \end{cases}$. That is, $\{a, a + d, a + 2d, \ldots\} = \{a + nd\}$

Geometric: $a_n = \begin{cases} a & n = 0 \\ ra_{n-1} & n > 0 \end{cases}$. That is, $\{a, ar, ar^2, \ldots\} = \{ar^n\}$

Harmonic: $a_n = \frac{1}{n}$. That is, $\{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$

Fibonacci: $a_n = \begin{cases} n & n \in \{0, 1\} \\ a_{n-1} + a_{n-2} & n > 1 \end{cases}$. That is, $\{0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots\}$

Linear recurrences

A recurrence is **linear** of **order** k if each term after the first k is a linear combination of the immediately preceding k terms. The coefficients are—in the most general setting—functions of n, but here we consider the case where they are all constant. For example, the Fibonacci sequence is a second order recurrence and arithmetic and geometric sequences are linear of first order. The harmonic sequence, however, is not linear in the sense we are using here.

The first k terms, $a_1, a_2, \ldots, a_k$, are called the **initial conditions** while the specified condition for larger terms, $a_n = c_1a_{n-1}+\cdots+c_ka_{n-k} = \sum_{i=1}^{k} c_ia_{n-i}$, $n > k$, is called the **recurrence relation**. It is easier, in general, to solve the general recurrence (that is, for *all possible initial conditions*) than for one specified set of initial conditions.

The **characteristic polynomial** of the recurrence relation $a_n = \sum_{i=1}^{k} c_ia_{n-i}$ is the polynomial

$$f(x) = x^n - \sum_{i=1}^{k} c_ia_{n-i}.$$

To solve linear recurrences we need only a few simple rules, which are easy to use:

1. If a_n, b_n, both solve a recurrence and c is a constant, then $a_n + cb_n$ also solves the recurrence. (Thus, any linear combination of solutions is a solution—or the solutions form a subspace of the vector space of all sequences!)
2. There exist k solutions $f_1(n), f_2(n), \ldots, f_k(n)$, such that every solution is of the form

   $$a_n = c_1f_n + \cdots + c_kf_k(n).$$

   Further, there are no fewer than k such solutions. (That is, the f_i's form a basis of the subspace of solutions.) Such sequences are called **fundamental solutions**.
3. If r is a zero of the characteristic polynomial of a linear recurrence, then $a_n = r^n$ is a solution.
4. Further, if the multiplicity of r as a root is m and $0 \leq e < m$, then $a_n = n^er^n$ is a solution.
5. Further, the solutions of this type form a set of fundamental solutions.

Example. Solve the recurrence $a_n = \begin{cases} 1 & n = 0 \\ 7 & n = 1 \\ 3a_{n-1} + 10a_{n-2} & n > 1 \end{cases}.$

Solution. *First solve the general recurrence, whose characteristic polynomial is*

$$x^2 - 3x - 10 = (x + 2)(x - 5).$$

Thus $(-2)^n$ and 5^n are fundamental solutions, so every solution is of the form $a_n = a(-2)^n + b \cdot 5^n$. The initial conditions give $a_0 = a + b = 2$ and $a_1 = -2a + 5b = 7$, a linear system easily solved: $(a, b) = \left(\frac{-2}{7}, \frac{9}{7}\right)$. The required sequence is $a_n = \frac{(-2)^{n+1}+9\cdot5^n}{7}$.

Example. Solve $a_n = \begin{cases} n & n \in \{0, 1, 2\} \\ 6a_{n-1} - 12a_{n-2} + 8a_{n-3} & n > 2 \end{cases}$

Solution. *The characteristic polynomial is $x^3 - 6x^2 + 12x - 8 = (x - 2)^3$, so $2^n, n2^n$ and n^22^n are fundamental solutions; the general solution is $a_n = 2^n(a+bn+cn^2)$; the initial conditions give $a = 0$, $2(a+b+c) = 1$, $4(a+2b+4c) = 2$, so that $(a, b, c) = (0, \frac{3}{4}, \frac{-1}{4})$, and the solution is $a_n = n(3-n)2^{n-2}$.*

(Verify a few terms for both examples!)

Some exercises. Solve:

1. $a_n = \begin{cases} 3 & n=0 \\ 4 & n=1 \\ 21 & n=11 \\ -a_{n-1}+5a_{n-2}-3a_{n-3} & n>2 \end{cases}$

2. The Fibonacci sequence.

3. $a_n = \begin{cases} 2 & n=0 \\ 8 & n=1 \\ \frac{1}{2048} & n=3 \\ \\ \frac{(a_{n-2})^5}{a_{n-1}(a_{n-3})^3} & n>2 \end{cases}$

4. The sequence $\{a_n\}n = 0^\infty$, where for $n \geq 5$, $a_n = 5a_{n-1} - 10a_{n-2} + 10a_{n-3} - 5a_{n-4} + a_{n-5}$, and such that $a_1 = -1$, $a_3 = 21$, $a_5 = 111$ and $a_{10} = 971$.

Some hints.

1. Straightforward.

2. Because of its form you'll find it easier to represent one of the roots of the characteristic polynomial as γ (express the other root in terms of γ). You can eliminate γ after simplifying the answer.

3. Make it linear first!
   Further hint: logarithms.

4. Repeated roots. Does it matter that the "initial conditions" aren't the "initial" elements of the sequence?

Series

A **series** is the sum of a sequence, whether finite ($s_n = \sum_{i=0}^{n} a_i$) or infinite ($s = \sum_{i=0}^{\infty} a_i$).

Some well-known series facts/formulas:

Arithmetic: $s_n = \sum_{i=0}^{n} a + di = n\left(\frac{2a+nd}{2}\right)$. That is, (number of terms)$\cdot$ $\left(\frac{\text{first+last}}{2}\right)$.

Geometric: $s_n = \sum_{i=0}^{n} r^i = \frac{1-r^i}{1-r}$ and (for $|r| < 1$) $s = \sum_{i=0}^{\infty} r^i = \frac{1}{1-r}$. Notice that $s = \lim_{n\to\infty} s_n$.

Harmonic: $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (but *very* slowly).

Alternating Harmonic: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = \ln 2$ (i.e., converges)

Things to know/recall:

- If $f(x)$ is a decreasing function, $x \geq 0$, then $\int_0^{n+1} f(x) \leq \sum_{i=0}^{n} f(n) \leq f(0) + \int_0^n f(x)$. Endpoints can be adjusted as needed. Similarly for increasing functions. Infinite version: $\int_0^\infty f(x) \leq \sum_{i=0}^{\infty} f(n) \leq f(0) + \int_0^\infty f(x)$.

- If a_n decreases monotonically to zero then $\sum_{i=0}^{\infty} (-1)^n a_n$ converges.

- Quite often elements of an infinite series can be recognized as having the form $c_n a^n$, where a function $f(x)$ is known whose Maclauren (or taylor) series has the same coefficients, $f(x) = \sum_{i=0}^{\infty} c_i x^i$. Then, obviously, $f(a)$ is the sum of the series, if a is on the interval of convergence. (In some problems this situation is cleverly disguised, and the main trick is to recognize it. *Know your power series expansions for common functions!*)

- All the standard tests for convergence of series from your calculus and analysis courses.

- Standard Taylor/Maclauren series and their intervals of convergence, including:

$$\sin x = \sum_{n\geq 0} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \text{ all } x \in \mathbb{R}$$

$$\cos x = \sum_{n\geq 0} \frac{(-1)^n x^{2n}}{(2n)!}, \text{ all } x \in \mathbb{R}$$

$$e^x = \sum_{n\geq 0} \frac{x^n}{n!}, \text{ all } x \in \mathbb{R} \text{ (and the corollary, } e^{i\theta} = \cos\theta + i\sin\theta)$$

$$\tfrac{1}{1-x} = \sum_{n\geq 0} x^n, \ |x| < 1 \text{ (Geometric series)}$$

$$\ln(1+x) = \sum_{n\geq 1} \frac{(-x)^{n+1}}{n}, \ -1 < x \leq 1.$$

Some exercises.

1. Suppose $a_n > 0$, $a_n \to 0$ monotonically as $n \to \infty$, and $\sum_{i=0}^{\infty} a_i$ diverges. Prove that, for every $\lambda > 0$, there exists an infinite subsequence, $\{c_n\}_{n=0}^{\infty}$ of $\{a_n\}_{n=0}^{\infty}$, such that $\sum_{n=0}^{\infty} c_n = \lambda$.

2. Suppose $\{a_n\}_{n\geq 0}$ is the sequence above and we define a new sequence by $b_n = (-1)^n a_n$, $n \geq 0$. Prove that, for every real number λ, there exists a rearrangement of this sequence, $c_n = b_{p(n)}$ (where p is some permutation of the nonnegative integers) that converges to λ (Note this means that every term b_k appears in exactly one position in $\{c_n\}_{n\geq 0}$, so the two sequences are identical as multisets).

3. Can you find a function whose Maclauren series (on some interval) is $\sum_{n\geq 0} nx^n$?

   How about $\sum_{n\geq 0} n^2 x^n$?

4. If $f(x) = \sum_{n\geq 0} a_n x^n$ and, for each n, $b_n = a_0 + a_1 + \cdots + a_n$, can you find, in terms of $f(x)$, an expression for the function $g(x) = \sum_{n\geq 0} b_n x^n$

(We will solve the last two questions and show how to do more general manipulations of power series in our later session Combinatorics II, which will cover "generating functions".)

Some general things to keep in mind in sequence/recursion problems

- They don't always *look* like sequence/recursion problems. Watch for ways to transform a problem into these types.

- Problems involving sums of powers of numbers generally *can* be transformed into recursion problems by treating the numbers as roots of a polynomial. The following example illustrates:

  Let a, b, c be the (unknown) roots of the polynomial $x^3 - 7x^2 + 2x + 5$. Find the sums of the 7th powers of a, b, c. To solve this we find a recurrence between the sums of the nth powers $s_n := a^n + b^n + c^n$, for all n, as follows:

  $$a^3 - 7a^2 + 2a + 5 = b^3 - 7b^2 + 2b + 5 = c^3 - 7c^2 + 2c + 5 = 0.$$

  Multiplying by $a^{n-3}, b^{n-3}, c^{n-3}$ and solving for the highest power term in each case we obtain $a^n = 7a^{n-1} - 2a^{n-2} - 5a^{n-3}$, $b^n = 7b^{n-1} - 2b^{n-2} - 5b^{n-3}$, $c^n = 7c^{n-1} - 2c^{n-2} - 5c^{n-3}$. Adding we obtain

  $$s_n = a^n + b^n + c^n = 7s_{n-1} - 2s_{n-2} - 5s_{n-3}.$$

  It is easy enough to find three initial conditions s_0, s_1, s_2, for the sequences $\{s_n\}_{n=0}^{\infty}$, from Viete's relations: From there we may iterate the recurrence and find the required sum, s_7.

- Recurrences—even non-linear ones—can generally be run forwards *and* backwards. That is, from any k consecutive terms it is usually possible to fine the term just prior, then the one just prior to that, etc. Often this means the recursively defined sequence $\{a_n\}_{n=0}^{\infty}$ can be treated as a subsequence of a doubly-infinite sequence $\{a_n\}_{n=-\infty}^{\infty}$. In many problems, this is precisely the key step to a solution.

- In many problems looking at a recurrence modulo some fixed integer is the right way to knock it apart. The pigeon-hole principle (PHP) often lies at the heart of problems involving integer sequences—looking at the sequence modulo n, that there are only n possible values, so it must eventually repeat (modulo n). The sequence, reduced modulo n, becomes periodic. This often gives us the desired result very easily!

Problems for discussion and homework

1. (1999 NCS) Let $f_1(x) = f(x) = \frac{1}{1-x}$, and for $n > 1$, define $f_n(x) = f(f_{n-1}(x))$. Evaluate $f_{2000}(1999)$.

2. (2005 CMO) Let us say that the triple of positive integers (a, b, c) is *n-powerful* if $a \le b \le c$, $\gcd(a, b, c) = 1$ and $(a + b + c)|(a^n + b^n + c^n)$. For example, $(1, 2, 2)$ is 5-powerful.

   (a) Find all n-powerful triples (if any), for all $n \ge 1$.

   (b) Find all ordered triples (if any) which are 2004-powerful and 2005-powerful but not 2007-powerful.

3. (1997 NCS) Recursively define a sequence by

   $$a_n := \begin{cases} 1 & n = 1, 2, 3 \\ a_{n-3} + a_{n-2}a_{n-1} & n > 3 \end{cases}.$$

   Prove that for every positive integer r, there is a term a_s that is divisible by r.

4. (1998 NCS) Let $a_1 = 3$ and, for $n \ge 1$, $a_{n+1} = a_n^2 - 2$. Prove that, if $m \ne n$, then a_m and a_n are relatively prime.

5. (1998 NCS) Find, with justification, the sum of the series $\displaystyle\sum_{n=2}^{\infty} \ln\left(\frac{n^3}{(n+2)(n-1)^2}\right)$.

6. (1999 NCS) Evaluate in closed form: $\displaystyle\sum_{n=0}^{\infty} \frac{\cos 3n}{n!}$.

7. (2000 NCS) Find a closed form expression for $\displaystyle\sum_{k=1}^{n^2} \frac{n - \lfloor \sqrt{k-1} \rfloor}{\sqrt{k} + \sqrt{k-1}}$.

8. (2001 NCS) If $a_1, a_2, \ldots, a_n$ is a finite sequence of numbers, it's *Cesáro sum* is defined to be

$$\frac{s_1 + s_2 + \cdots + s_n}{n},$$

where $s_k = a_1 + a_2 + \cdots + a_k$ for each k, $1 \leq k \leq n$. Suppose that the Cesáro sum of the 100-term sequence $a_1, a_2, \ldots, a_{99}$ is 100. find the Cesáro sum of the 100-term sequence

$$1, a_1, a_2, \ldots, a_{99}.$$

9. (2001 NCS) Find the sum of the series $\displaystyle\sum_{n=0}^{\infty} \frac{1}{4n!}$.

10. (2001 NCS) If a sequence $a_0, a_1, a_2, \ldots$ satisfies $a_1 = 1$ and

$$a_{2m} + a_{2n} = 2(a_{m+n} + a_{m-n})$$

for all integers m and n with $m \geq n \geq 0$, determine, with proof, a_{2001}.

Solutions on last page. I'll add more problems and/or solutions, sometime later. If you desperately want to see a solution let me know and I'll try to get it online for you.

Some solutions (more to be posted on the web page)

1. HINT: Look at the first few functions. The answer is $\frac{1998}{1999}$.

3. Modulo r there are at most r^3 distinct triples (a_{n-2}, a_{n-1}, a_n) so, by PHP (and the fact that three consecutive elements determine the subsequent sequence), the sequences is *eventually* periodic $\pmod r$.

   Since $a_{n-3} = a_n - a_{n-1}a_{n-2}$, the sequence is recursively defined in both directions and we may extend it to a doubly-infinite sequence. Further, it is periodic over the entire doubly-infinite sequence by the above argument. Thus, it must eventually return to its initial state modulo r, that is, there exists some $s \geq 1$ such that $a_{s-3} \equiv a_{s-2} \equiv a_{s-1} \pmod r$, from which it follows that $a_s \equiv a_{s-1} - a_{s-2}a_{s-3} \equiv 0 \pmod r$.

4. By an easy induction, $a_n \equiv 1 \pmod 2$ for all n. Let $m < n$. Then

$$a_{m+1} = a_m^2 - 2 \equiv -2 \pmod{a_m}$$
$$a_{m+2} \equiv (-2)^2 - 2 \equiv 2 \pmod{a_m}$$

   By induction, $a_{m+k} \equiv 2 \pmod{a_m}$ for all $k \geq 2$. Thus $a_n = ra_m \pm 2$ for some integer r. So every common factor of a_m, a_n is a divisor of 2. Since a_m, a_n are both odd they are relatively prime.

5. HINT: use the properties of logs and look at partial sums (necessary, for proper justification!)

6. HINT: This is the value of a certain power series in x, evaluated for a certain value of x. Remember your standard functions! In this problem, think of a natural *partner* for this function, and consider the two functions together! The answer is $e^{\cos 3}\cos(\sin 3)$.

7. HINT: group terms so as to get rid of the most troublesome part of this problem.

8. HINT: Not as hard as it looks. The answer will drop out if you manage to interpret the question properly and write down what it's asking for, with only a small amount of algebra.

9. HINT: hyperbolic functions.

10. HINT: You'll find this surprisingly easy if you use the following strategy: First show there is only one solution. Then find a sequence that satisfies the conditions. (On the other hand if you try deriving the solution directly you may get frustrated from going around in circles.)