UM Mathletics 2005/06
Problem Set #5

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1. Is it possible that both numbers $2^n - 1$ and $2^n + 1$ are prime for some $n > 2$?

2. Prove that the sum
$$1^k + 2^k + \cdots + n^k \,,$$
where $n \in \mathbb{N}$ and k is odd, is divisible by
$$1 + 2 + \cdots + n \,.$$

3. Prove that among the numbers of the arithmetic progression
$$3, 7, 11, 15, 19, 23, \ldots$$
there are infinitely many primes.

4. Prove that for any odd integer $p > 1$ the angle $\cos^{-1}(1/p)$ measured in degrees is not a rational number.

5. Prove that there does not exist a polynomial
$$P(x) = a_0 x^n + a_1 x^{n-1} + \ldots a_{n-1} x + a_n$$
such that $P(0)$, $P(1)$, $P(2)$, $\ldots$ are all prime numbers.

6. Prove that for any integer $n > 6$
$$(n/2)^n > n! > (n/3)^n \,.$$

7. (a) Prove that, for any $x \in \mathbb{R}$ and any $\epsilon > 0$, there exist integers m and n such that
$$|xm - n| < \epsilon \,.$$

   (b) Suppose that $f : \mathbb{R} \mapsto \mathbb{R}$ is a continuous function such that, for any $x \in \mathbb{R}$, $f(x + f(x)) = f(x)$. Prove that f is a constant function.