UM Mathletics 2005/06
Problem set #4

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1. Let A be a matrix obtained from the numbers in a standard calendar page (one month view in 7 column format, with rows representing weeks), omitting the first week (row) if the month does not begin on a Sunday and the last week (row) if the month does not end on a Saturday. For example, the matrix obtained from Sept. 2005 is $A = \begin{pmatrix} 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ 16 & 17 & 18 & 19 & 20 & 21 & 22 \\ 23 & 24 & 25 & 26 & 27 & 28 & 29 \end{pmatrix}$. What are the possible values for the determinant $|AA^T|$?

2. How many ways can the numbers $1, ..., n$ be arranged in a line so that each number except the leftmost differs by 1 from some preceding element?

3. Evaluate $\lim_{n\to\infty} \sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + \cdots + (n-1)\sqrt{1+n}}}}$.

4. Evaluate (a) $\int_{-\infty}^{\infty} (1+x^2)^{-3} dx$; (b) $\int_{-\infty}^{\infty} (k+x^2)^{-3} dx$, $k \in \mathbb{R}$.

5. Prove that for any $m \times n$ matrix A and any $n \times m$ matrix B, $\det(I_m + AB) = \det(I_n + BA)$.

6. Prove that there exists an $n \in \mathbb{Z}^+$ such that the distance from $(0,1)$ to $(\sin(n), \cos(\sqrt{n}))$ is less than $\frac{1}{2005}$.

7. A sadistic electrician installs the wiring in a certain building as follows: each room is connected to certain others in such a way that when the light in that room changes states (i.e. switches from on to off or off to on), the same happens in all the rooms *directly* connected to it in this way, while all others remain the same (the relation"is connected to" is symmetric but not necessarily transitive). Initially all rooms have their lights off. Prove that it is possible, by toggling some sequence of switches, so turn on the lights in all rooms.

8. (a) Prove that the 300th digit after the decimal point in $x = (\sqrt{2}+\sqrt{3})^{2006}$ is a 9. (b) What is the ones digit of x? (c) What is the tens digit?

9. Let $\{a_n\}_{n=0}^{\infty}$ be any sequence of positive real numbers. Prove that $\left(\frac{a_0+a_{n+1}}{a_n}\right)^n \geq e$ for infinitely many values of n.

Solutions

1. Clearly, $A = [a_{ij}]$ is a 3×7 or 4×7 matrix whose entries are of the form $a_{ij} = c + 7i + j$. Thus,

$$A = [7i]+[c+j] = \begin{pmatrix} 7 & 7 & \cdots & 7 \\ 14 & 14 & \cdots & 14 \\ \vdots & & \vdots & \end{pmatrix} + \begin{pmatrix} c+1 & c+2 & \cdots & c+7 \\ c+1 & c+2 & \cdots & c+7 \\ \vdots & & & \vdots \end{pmatrix},$$

a sum of rank one matrices. Accordingly, A and so also AA^t have rank at most 2. It follows that $|AA^T| = 0$ for all such matrices, and so the required maximum value is 0.