UM Mathletics 2005/06
Problem Set #3

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1. Prove that there exists a positive integer n such that for every $x \geq 0$ the inequality
$$(x-1)(x^{2005} - 2005x^{n+1} + 2005x^n - 1) \geq 0$$
holds.

2. (a) Find the formula for $\sum_{i=1}^{n} i^4$. Note that we assume that all of us know how to prove this type of formulas using Math Induction, so you need to show how this formula can be derived.

   (b) Prove
$$\lim_{n\to\infty} \frac{1^k + 2^k + \cdots + n^k}{n^{k+1}} = \frac{1}{k+1}.$$

3. Prove that if the polynomial
$$P(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$
with integral coefficients is such that $P(0)P(1)$ is odd, then the equation $P(x) = 0$ cannot have integral roots.

4. Prove that if p and q are relatively prime natural numbers, then
$$\sum_{k=1}^{q-1} \left[\frac{kp}{q}\right] = \frac{(p-1)(q-1)}{2},$$
where $[x]$ is the integer part of x.

5. Find all integers n which are divisible by all integers not exceeding $\sqrt{n}$.