UM Mathletics 2005/06
Problem set #2

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1. Suppose that every point in the plane is coloured either red, white, or blue. Show that there exist two points of the same colour exactly one unit apart.

2. How many positive integers have decimal representations consisting of distinct digits? For each N, how many positive integers have base N representations consisting of distinct digits?

3. Observe that, for the rational polynomial $p(x) = \frac{1}{2}x^2 - \frac{1}{2}x + 2$, $p(1) = 2$, $p(2) = 3$ and $p(3) = 5$. Does there exist a polynomial, $q(x)$, with integer coefficients, such that $q(1) = 2$, $q(2) = 3$ and $q(3) = 5$?

4. **Generalized Pythagoras' Theorem.** Define a *right tetrahedron* to be a tetrahedron, three of whose faces are right triangles, meeting at their right-angle vertices. Call its fourth face the *hypotenuse.*

   Prove that the square of the area of the hypotenuse of a right tetrahedron equals the sum of the squares of the areas of the other three sides.

5. Show that, in any set of five integers, there are three whose sum is a multiple of 3.

6. For which values of n is $n!$ divisible by 2^n? Prove your answer.

7. Find the last digit of $2^{3^{4^5}}$.

8. Observe that $126^2 = 15876$ and $116^2 = 13456$ are perfect squares, the last few digits of which are consecutive. What is the greatest length of a string of consecutive digits that can end a perfect square?