University of Manitoba Mathletics
All Day Practise #1 Nov. 5, 2005, 9–12

INSTRUCTIONS

These problems are designed to be fun as well as challenging. Partial credit will be given for *significant* progress, but a thorough job on a few problems is worth more than exploratory work on all. Think of a solution as an essay—a logical argument which makes clear why your answer to the question is correct, or why the assertion whose proof is called for in the problem is true. The test is designed so that it is unlikely that anyone, working alone, could finish in 3 hours—so part of your challenge is to work out an effective division of labor.

Team identification and problem number should be clearly given at the top of each sheet of paper submitted. Submit only one set of solutions; all writing must be completed during the allotted time. **Please begin the solution to each problem on a new sheet of paper. Do not consult Books, Notes, Calculators, Computers OR Non-Team-Members.**

1. A triangle has an angle of measure $\frac{2003\pi}{2005}$, subtended by sides whose lengths are a and b, and opposite side has length c. Find, with proof, the value of K (which does not depend on on a, b and c) such that the area of the triangle is
$$A = K(a+b+c)(a+b-c)\cot\frac{\pi}{2005}.$$
2. Which is larger, 2005!, or 2^{18000}?
3. Here is a well-known characterization for all pythagorean triples (i.e., $(a, b, c) \in (\mathbb{Z}^+)^3$ such that $a^2 + b^2 = c^2$): $(a, b, c) = (m^2 - n^2, 2mn, m^2 + n^2)$, $m, n \in \mathbb{Z}^+$, $m > n$.

   Give a similar characterization for all *inverse Pythagorean triples*, which are triples $(a, b, c) \in (\mathbb{Z}^+)^3$ such that $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c^2}$.
4. Prove that the product of any 2005 consecutive integers is divisible by 2005!.
5. Prove that there is no polynomial $p(x)$ such that, for all positive integers n, $p(n)$ is a prime number.
6. Let $a_1 = 3, a_2 = 9, a_3 = 6, a_4 = 4$, and define a_{n+4} to be the last decimal digit of $a_n + a_{n+1}$, $n \in \mathbb{Z}^+$. Prove that the sequence $\{a_n\}_{n=1}^{\infty}$ is periodic. What is its period?
7. Let $p(x)$ be a polynomial of degree 2005 with integer coefficients. Let $n_1, n_2, \ldots, n_{2005}$ be distinct integers such that $f(n_i) = \pm 1$, $i = 1, 2, \ldots, 2005$. Prove that $p(x)$ cannot be factored as the product of polynomials with of lower degree, having integer coefficients.
8. Let $f(x)$ be a polynomial with real coefficients such that, for all irrational numbers $\gamma \in \mathbb{R}$, $f(\gamma)$ is irrational. Prove that the degree of $f(x)$ is at most 1.
9. Find 31 (not necessarily distinct) positive integers, at least one of which is greater than 2005, the sum of whose squares is 31 times their product.
10. In triangle ABC, X, Y, Z are points on the interior of AB, BC and AC respectively. If $|XY| = |YZ| = |ZX|$ and $|XB| = |YC| = |ZA|$, prove that ABC is equilateral.

SOLUTIONS

1. $\theta = \frac{2003\pi}{2005}$ is the vertex angle of a 2005-gon. The area of a 2005-gon of edge-length s may be determined by dividing it into 4010 right triangles of sides r (the distance from the center to the midpoint of an edge) and $\frac{s}{2}$ (half an edge), opposite the central angle of $\frac{\pi}{2005}$; accordingly, $r = \frac{s}{2}\cot\frac{\pi}{2005}$; the area of the polygon is thus $G(s) = 2005\frac{s^2}{4}\cot\frac{\pi}{2005}$. Removing an inscribed 2005-gon of side c from one of side $a+b$ leaves 2005 copies of the original triangle; thus its area is

$$\begin{aligned} A &= \frac{1}{2005}(G(a+b) - G(c)) \\ &= \frac{2005((a+b)^2 - c^2)}{\cot}\frac{\pi}{2005\cdot 4}\cot\pi 2005 \\ &= \frac{(a+b+c)(a+b-c)}{4}\cot\frac{\pi}{2005}. \end{aligned}$$

Thus, $K = \frac{1}{4}$.

2. By induction we show that, for all n, $n! > \left(\frac{n}{3}\right)^n$. This is true for $n = 1$. Suppose it is true for n. Then $(n+1)! > (n+1)n! > (n+1)\left(\frac{n}{3}\right)^n$. Now, $\left(\frac{n+1}{3}\right)^{n+1}\left(\frac{3}{n}\right)^n = \frac{n+1}{3}\left(1+\frac{1}{n}\right)^n < \frac{n+1}{3}e < n+1$ because $(1+\frac{1}{n})^n \uparrow e$. Thus, $(n+1)\left(\frac{n}{3}\right)^n$, and so we have $(n+1)! > \left(\frac{n+1}{3}\right)^{n+1}$, which justifies the claim by induction.

   Therefore, $2005! > 2004! > 668^{2004} > (512)^{2004} = 2^{9\cdot 2004} = 2^{18036} > 2^{18000}$. Therefore 2005! is the greater of the two.

3. Without loss of generality $a, b, c > 0$. Write $s = (a, b)$, $a = sp$ and $b = sq$. Then

$$\frac{1}{a^2} + \frac{1}{b^2} = \frac{p^2+q^2}{s^2(p^2q^2)} = \frac{1}{c^2}.$$

   Since $(p,q) = 1$, $(p^2q^2, p^2+q^2) = 1$, so that $s^2 = p^2+q^2$ and $c = pq$. Since (p,q,s) form a Pythagorean triple, there exist m, n, $m > n$, such that (again, without loss of generality) $p = m^2 - n^2$, $q = 2mn$, $s = m^2 + n^2$; accordingly,

$$(a, b, c) = (m^4 - n^4, 2mn(m^2+n^2), 2mn(m^2-n^2))$$

   and, conversely, such a triple gives

$$\frac{1}{a^2} + \frac{1}{b^2} = \frac{(m^2-n^2)^2 + 4m^2n^2}{4m^2n^2(m^2-n^2)^2(m^2+n^2)^2} = \frac{1}{(m^2-n^2)^2} = \frac{1}{c^2}.$$

4. Let $p(n) = (n+1)(n+2)\cdots(n+2005)$. If $-2005 \le n < 0$ then $p(n) = 0$ and the claim is true. If $n < -2005$ then $p(n) = p(-n-2005)$, so without loss of generality, $n \ge 0$. The number of ways of selecting 2005 objects from a set of $2005+n$ is $\binom{n+2005}{2005} = p(n)/2005!$. The result follows.

5. Note: the problem is stated incorrectly; it should have excluded constant polynomials.

   Solution: Suppose that there is such a polynomial $p(x)$, and that its degree is $n \ge 1$. It is therefore determined by its values at n distinct points; taking these to be $x =$

$1, 2, \ldots, n+1$ there results a linear equation with integer coefficients, whose solution determines the coefficients of $p(x)$, which are therefore rational (e.g., by Cramer's rule). Hence, by multiplying $p(x)$ by the LCM of the coefficients (call it L), we conclude that there is a polynomial $q(x)$ of degree n with integer coefficients such that $q(n) = r_n L$, $n \geq 1$, where r_n's are all prime and $L \in \mathbb{N}$.

Denote $q(1) = r_1 L =: M$ and consider $q(1+kM)$, $k \in \mathbb{N}$. Since $kM | (q(1+kM) - q(1))$, we conclude that $M | q(1+kM)$ and hence $r_1 | r_{1+kM}$, for all $k \in \mathbb{N}$. This can only happen if $r_{1+kM} = \pm r_1$ for all $k \in N$ and, in particular, for $1 \leq k \leq 2n+1$. By the PHP, there are $n+1$ values of k such that r_{1+kM} are the same (and are equal to r_1 or $-r_1$). Therefore, the polynomial $q(x)$ of degree n assumes the same value at $n+1$ points, which can only happen if $q(x)$ is constant, and so $n = 0$. Contradiction.

6. We consider the problem modulo 2 and then modulo 5.

modulo 2: The recurrence proceeds 110001001101011$\boxed{1100\ldots}$, and so is periodic with period 15.

modulo 5: We consider the same recurrence with initial values 1,0,0,0 and nth term a_n; this proceeds

$$\begin{array}{l}1000100110121133241010111122234402342 02\\ 1223340124131044143002302032230403443 23\boxed{2000\ldots},\end{array}$$

and $a_n = a_{n+78}$. Thus $a_{n+4\cdot 78} = 2^4 a_n = a_n$. The sequence is thus periodic with period $4 \cdot 78 = 2^3 \cdot 39$. Since this sequence contains subsequences 0100, 0010 and 0001, the recurrences starting with each of these 4-tuples has the same period. Since these form a basis for the solution space to this recurrence, the period of all solutions is a divisor of $2^3 \cdot 39$.

If $\{a_n\}$ is any nonzero solution to the recurrence modulo 5 we have $a_{k+\cdot 2\cdot 78} = 4a_k \neq a_k$, for $a_k \neq 0$, so the recurrence with given initial conditions, modulo 5, cannot have period that is a divisor of $2^2 \cdot 39$, and it cannot be 8, since the sequence begins 341420012.... Therefore its period is $2^3 \cdot 39$.

By the Chinese Remainder Theorem, given sequences of residues modulo 2 and modulo 5 there corresponds a unique sequence modulo 10 whose entries, taken modulo 2 and 5, are these sequences. The period of this sequence modulo 10 will thus be the least common multiple of the periods modulo 5 and modulo 2, which is $2^3 \cdot 3 \cdot 5 \cdot 13 = 1560$.

7. Let $n_1 < n_2 < \cdots < n_{2005}$. Suppose $p(x) = a(x)b(x)$ is a factorization of $p(x)$ over $\mathbb{Z}[x]$. Then $a(n_i)b(n_i) = p(n_i) = \pm 1$, and so$a(n_i) = \pm 1$ for all i. Now $(n_1 - n_j | (a(n_1) - a(n_j))$ If $a(n_1) \neq a_{n_j}$, then $a(n_1) - a(n_j) = \pm 2$, from which it follows that $n_j < n_1 + 3$, which can happen only for $j = 1, 2$. Thus, $a(x) = a_{n_1}$ for $x = n_1, n_3, n_4, \ldots, n_{2005}$ — 2004 values. The same is true for $b(x)$. One of a, b must have degree less than 2003; wlog this is a. A polynomial of degree n taking on the same value c in $n+1$ positions is equal to the constant polynomial $f(x) = c$; it follows that $p(x)$ cannot be so factored.

8. If f is a constant function, the result follows. Therefore let us assume that f is not constant, in which case (being a polynomial) $|f|$ grows without bound. Therefore, $f(x)$ attains the values of infinitely many integers. Suppose the degree of f is n; let $f(x_i) = a_i \in \mathbb{Z}$, $i = 0, 1, \ldots, n$, with the a_i's distinct. Since each a_i is rational so is x_i.

The Lagrange interpolation formula gives

$$f(x) = \sum_{i=0}^{n} \frac{\prod_{j\neq i}(x - x_j)}{\prod_{j\neq i}(x_i - x_j)} a_i,$$

a polynomial with rational coefficients. Therefore it is of the form $p(x)/m$, where $p(x) \in \mathbb{Z}[x]$ and $m \in \mathbb{Z}$.

Without loss of generality (since the hypothesis about f is unaffected by multiplication by rational numbers), $m = 1$, so $f(x) \in \mathbb{Z}[x]$. Let $f(x)$ have leading coefficient $a \in \mathbb{Z}$. Since $|f(x)|$ grows without bound, we may assume without loss that $\lim_{x\to\infty} f(x) = \infty$.

Now suppose f is not a linear polynomial. Then $\lim_{x\to\infty} f'(x) = \infty$, so that there exists $M \in \mathbb{R}$ such that $f'(x) > a$ for all $x \geq M$. Let p be any integer greater than $f(M)$. Then $f(x) = p$ has a real root $r > M$, which must be rational by hypothesis. By the rational roots theorem r can be expressed in the form $\frac{s}{t}$, where t is a divisor of a; without loss of generality, $t = a$. Similarly there must be a rational root of $f(x) = p + 1$, $r' = \frac{s'}{a} > M$.

By the mean value theorem, there is a value x between $\frac{s}{a}$ and $\frac{s'}{a}$ such that $f'(x) = \frac{f(\frac{s'}{a}) - f(\frac{s}{a})}{\frac{s'}{a} - \frac{s}{a}} = \frac{a}{s'-s}$. Since $x > M$, $f'(x) > a$, so $|s' - s| < 1$ and s and s' cannot both be integers, a contradiction.

It follows, therefore, that f must be a linear polynomial.

9. [Credit to B. Wolk] Observe that $(1, 1, \ldots, 1)$ is a solution to the problem without the size condition. What if we alter just one number? $(1, \ldots, 1, x)$ gives $30 + x^2 = 31x$ giving a monic integer polynomial $x^2 - 31x + 30 = 0$ for which we have an a priori solution, $x = 1$; polynomial division gives the other root, $x = 30$. But then, $(1, \ldots, 1, y, 30)$ must also give a monic integer quadratic in y with a root $y = 1$, which forces another integer solution: $y^2 - 31 \cdot 30y + 929 = (y - 1)(y - 929)$. We can iterate this process as many times as needed:

$$(1, \ldots, 929, z) \quad \to \quad z^2 - 31 \cdot 929z + 929^2 + 29 = (z - 30)(z - *)$$

Polynomial long division (or synthetic subsitution, if you know how) gives the other root, $z = 28769$; thus one solution is $(1, \ldots, 929, 28769)$.

10. Let $|XB| = r$. Construct circles C_a, C_b, C_c of radius r at Z, Y, X respectively. Then A is on C_a, B is on C_B and C is on C_c. Given point A, points B and C are uniquely determined by the intersection of lines AX and AZ with the appropriate circles.

Now observe that A, B, C cannot lie on the sector of these circles between the points where the extended sides of triangle XYZ meet them. This leaves two arcs on each of these circles, symmetrically arranged, in which vertices A, B, C may lie. We may assume that A lies on the arc closest to X; the arcs on which B and C lie are determined by this.

Let us call the angles at A, B, C α_1, α_2 and α_3 respectively. Call the angle at X in triangle AXZ β_1 and the corresponding angles in triangles BXY and CYZ β_2 and β_3.

Now a configuration of this type can be constructed, in which $\alpha_1 = \alpha_2 = \alpha_3 = 60$ degrees. We show that there are no others.

Observe that, as A moves along circle C_1 away from the point X, α_1 decreases. At the same time B and C, as located from A, move away from Y and Z, respectively. Thus α_i decreases, $i = 1, 2, 3$. Now suppose two such configurations exist. By the above observation they must have different values of α_i, differing by some quantity $\Delta\alpha_i$, and similarly the angles β_i differ between the two configurations by $\Delta\beta_i$, $i = 1, 2, 3$.

By the outside angle theorem, $\beta_i + 60 = \alpha_{i+1} + \beta_{i+1}$, with indices reduced mod 3. Accordingly, we have $\Delta\beta_i = \Delta\alpha_{i+1} + \Delta\beta_{i+1}$. Adding these three equations yields $\Delta\alpha_1 + \Delta\alpha_2 + Delta\alpha_3 = 0$. But, as observed above, the α_i's all decrease (and conversely increase) simultaneously, so the $\Delta\alpha_i$'s must all have the same sign. It follows that $\Delta\alpha_i = 0$, $i = 1, 2, 3$. It follows that any two such configurations must have $\alpha_i = 60$ degrees; the result follows.