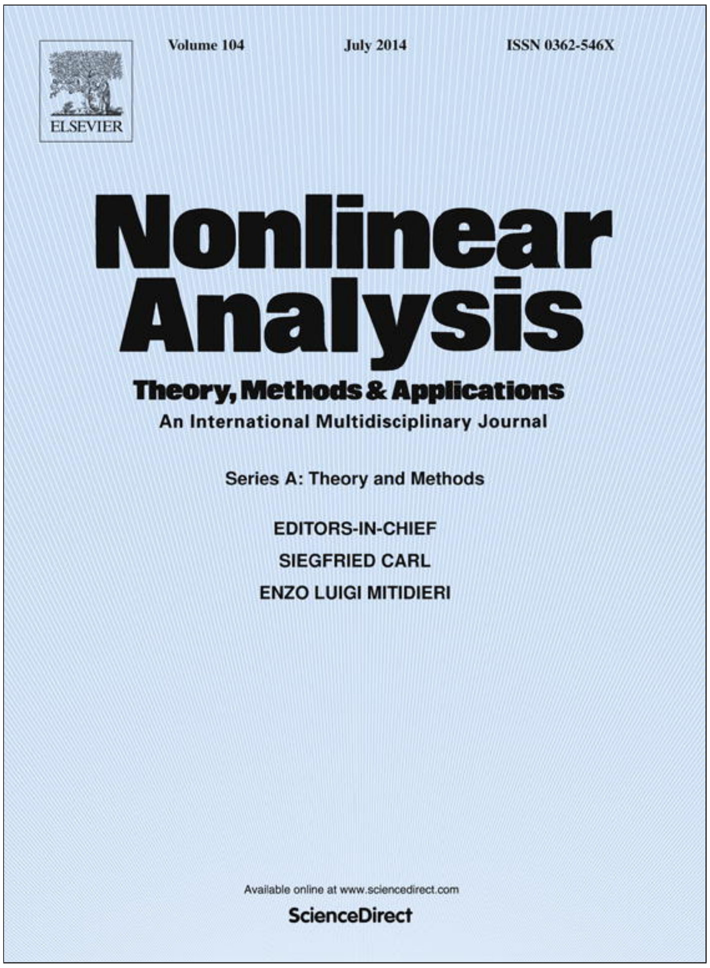
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Stability of entire solutions to supercritical elliptic problems involving advection

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ABSTRACT

We examine the equation given by

$$-\Delta u + a(x) \cdot \nabla u = u^p \quad \text{in } \mathbb{R}^N, \tag{1}$$

where $p > 1$ and $a(x)$ is a smooth vector field satisfying some decay conditions. We show that for $p < p_c$, the Joseph–Lundgren exponent, there is no positive stable solution of (1) provided one imposes a smallness condition on a along with a divergence free condition. In the other direction we show that for $N \geq 4$ and $p > \frac{N-1}{N-3}$ there exists a positive solution of (1) provided a satisfies a smallness condition. For $p > p_c$ we show the existence of a positive stable solution of (1) provided a is divergence free and satisfies a smallness condition.

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1. Introduction and results

In this article we are interested in the existence versus nonexistence of positive stable solutions of

$$-\Delta u + a(x) \cdot \nabla u = u^p \quad \text{in } \mathbb{R}^N, \tag{2}$$

where $p > 1$ and $a(x)$ is a smooth vector field satisfying some decay conditions. We now define the notion of stability and for this we prefer to work on a general domain.

Definition 1. Let u denote a nonnegative smooth solution of (2) in an open set $\Omega \subset \mathbb{R}^N$. We say u is a stable solution of (2) in Ω provided there is some smooth positive function E such that

$$-\Delta E + a(x) \cdot \nabla E \geq p u^{p-1} E \quad \text{in } \Omega. \tag{3}$$

We begin by recalling some facts in the case where $a(x) = 0$. There has been much work done on the existence and nonexistence of positive classical solutions of

$$-\Delta u = u^p, \quad \text{in } \mathbb{R}^N. \tag{4}$$

For $N \geq 3$ there exists a critical value of p, given by $p_S = \frac{N+2}{N-2}$, such that for $1 < p < p_S$ there is no positive classical solution of (4) and for $p > p_S$ there exist positive classical solutions, see [1–4]. By definition we call a nonnegative solution u of (4) stable if

$$\int p u^{p-1} \phi^2 \leq \int |\nabla \phi|^2 \quad \forall \phi \in C_c^\infty(\mathbb{R}^N), \tag{5}$$

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which is nothing more than the stability of u using (3), after using a variational principle. The additional requirement that the solution be stable drastically alters the existence versus nonexistence results. It is known that there is a new critical exponent, the so called Joseph–Lundgren exponent p_c, such that for all $1 < p < p_c$ there is no positive stable solution of (4) and for $p > p_c$ there exist positive stable solutions of (4). The value of the p_c is given by

$$p_c = \begin{cases} \dfrac{(N-2)^2 - 4N + 8\sqrt{N-1}}{(N-2)(N-10)} & N \geq 11 \\ \infty & 3 \leq N \leq 10. \end{cases}$$

The first implicit appearance of p_c was in the work [5] where they examined $-\Delta u = \lambda(u+1)^p$ on the unit ball in $\mathbb{R}^N$ with zero Dirichlet boundary conditions. The exponent p_c first explicitly appeared in the works [6,7] where they examined the stability of radial solutions to a parabolic version of (4). Their results easily imply the existence of a positive radial stable solution of (4) when $p > p_c$ and the nonexistence of positive radial stable solutions in the case of $p < p_c$. More recently there has been interest in finite Morse index solutions of either (4) and the generalized version given by

$$-\Delta u = |u|^{p-1}u, \quad \text{in } \mathbb{R}^N. \tag{6}$$

In [8] they completely classified the finite Morse index solutions of (6) and again the critical exponent p_c was involved. For results regarding singular nonlinearities, general nonlinearities, or quasilinear equation see [9–14].

In the work [15] the nonexistence of nontrivial solutions of

$$-div(\omega_1 \nabla u) = \omega_2 u^p \quad \text{in } \mathbb{R}^N,$$

was examined where ω_i are some nonnegative functions. In the special case where $\omega_1 = \omega_2$ this equation reduces to

$$-\Delta u + \nabla\gamma(x)\cdot\nabla u = u^p \quad \text{in } \mathbb{R}^N, \tag{7}$$

where γ is a scalar function. Even though (7) and (2) are similar a major difference is that (7) is variational in nature; critical points of

$$E(u) = \frac{1}{2}\int e^{-\gamma}|\nabla u|^2 - \frac{1}{p+1}\int e^{-\gamma}|u|^{p+1},$$

are solutions of (7). This variational structure of (7) allows one to prove various nonexistence results for (7) by slightly modifying the nonexistence proofs used in proving similar results for $-\Delta u = u^p$ in $\mathbb{R}^N$. This approach will generally not work for (2) since in general there will not be a variational structure.

In [16] the regularity of the extremal solution, u^*, associated with problems of the form

$$\begin{cases} -\Delta u + a(x)\cdot\nabla u = \lambda f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

was examined for various nonlinearities f. Here $a(x)$ was an arbitrary smooth advection and the main difficulty was to utilize the stability of u^* in a meaningful way. As mentioned earlier, this is not a problem when $a(x)$ is the gradient of a scalar function. The main tool used was the generalized Hardy inequality from [17]. This same approach was extended to more general nonlinearities in [18].

We now list our results.

Theorem 1. *Suppose $3 \leq N \leq 10$ or $N \geq 11$ and $1 < p < p_c$. Suppose $a(x)$ is a smooth divergence free vector field satisfying $|a(x)| \leq \frac{C}{|x|+1}$ with $0 < C$ sufficiently small. Then there is no positive stable solution of (2).*

The next result gives a decay estimate in the case of $p < p_c$. We are including this result since it may allow one to use a Lane–Emden type of change of variables to obtain a nonexistence result without a smallness condition on the advection.

Theorem 2. *Suppose $\frac{N+2}{N-2} < p < p_c$, $a(x)$ is a smooth divergence free vector field with $|a(x)| \leq \frac{C}{|x|+1}$ and $|a| \in L^N(\mathbb{R}^N)$. Then any positive stable solution u of (2) satisfies*

$$\lim_{|x|\to\infty} |x|^{\frac{2}{p-1}} u(x) = 0. \tag{8}$$

The approach to solve Theorem 1 will be to combine the methods used in [8] with the techniques from [16] which relied on generalized Hardy inequalities from [17]. The same approach will be used in the proof of Theorem 2 with an added scaling argument.

Our final result gives an existence result.

Theorem 3. 1. *Suppose $N \geq 4$, $p > \frac{N+1}{N-3}$ and $a(x)$ is some smooth vector field with $|a(x)| \leq \frac{C}{|x|+1}$. If $0 < C$ is sufficiently small there exists a positive solution of (2).*

2. *Suppose $N \geq 11$, $p > p_c$ and let $a(x)$ denote some smooth divergence free vector field with $|a(x)| \leq \frac{C}{|x|+1}$. For $0 < C$ sufficiently small (2) has a positive stable solution.*

The idea of the proof will be to look for a solution u as a perturbation of the positive radial solution w of $-\Delta w = w^p$ in $\mathbb{R}^N$ with $w(0) = 1$. See the beginning of Section 3 for details on w. The framework we will use to prove the existence of a positive solution will be the approach developed in [19]. Their interest was in the existence of positive solutions of $-\Delta u = u^p$ in $\Omega \subset \mathbb{R}^N$ an exterior domain with zero Dirichlet boundary conditions.

Open problem. It would be interesting to see if these smallness conditions on $a(x)$ can be removed, possibly at the expense of adding some additional decay requirements.

2. Nonexistence proofs

Remark 1. A computation shows that $p < p_c$ is equivalent to the condition

$$\frac{N}{2} < 1 + \frac{2p}{p-1} + \frac{2}{p-1}\sqrt{p^2 - p}. \tag{9}$$

For our nonexistence results it will be easier to deal with (9).

Theorems 1 and 2 will depend on the following energy estimate, which we state for a general domain.

Proposition 1. *Suppose u is a smooth positive stable solution of (2) and $a(x)$ is a smooth divergence free vector field. Then for all $1 \le T$, $0 < \beta < 1$, $0 < \varepsilon$, $0 < \delta$, $\frac{1}{2} < t$ and $0 \le \psi \in C_c^\infty(\Omega)$ we have*

$$\begin{aligned}&\left(\beta p - \frac{Tt^2}{2t-1}\right)\int u^{2t+p-1}\psi^2 + \beta(1-\beta-\varepsilon)\int \frac{|\nabla E|^2}{E^2}u^{2t}\psi^2 + (T-1)\int |\nabla(u^t\psi)|^2 \\ &\quad \le \left(\frac{\beta}{4\varepsilon} + \frac{Tt\delta}{2t-1}\right)\int |a|^2 u^{2t}\psi^2 + \left(T + \frac{Tt}{4\delta(2t-1)}\right)\int u^{2t}|\nabla\psi|^2 + \frac{T|t-1|}{2(2t-1)}\int u^{2t}|\Delta\psi^2|.\end{aligned} \tag{10}$$

Define the following parameters

$$t_-(p) = p - \sqrt{p^2 - p} \quad \text{and} \quad t_+(p) = p + \sqrt{p^2 - p}.$$

A computation shows that for $t_-(p) < t < t_+(p)$ we have $p - \frac{t^2}{2t-1} > 0$. This restriction on t will be related to the restrictions on t we must impose if one wants to obtain an estimate from Proposition 1.

Proof of Proposition 1. Suppose u is a smooth positive stable solution of (2) in Ω and let $E > 0$ satisfy (3). From [17] we have the following generalized Hardy inequality

$$\beta \int \frac{-\Delta E}{E}\phi^2 + (\beta - \beta^2)\int \frac{|\nabla E|^2}{E^2}\phi^2 \le \int |\nabla\phi|^2, \quad \forall \phi \in C_c^\infty(\Omega), \tag{11}$$

for all $\beta \in \mathbb{R}$. Adding $T\int |\nabla\phi|^2$ to both sides of the inequality, using the fact that E satisfies (3), and taking $\phi = u^t\psi$, where $\psi \in C_c^\infty(\Omega)$, gives

$$\beta p \int u^{p-1}u^{2t+p-1}\psi^2 - \beta\int \frac{a\cdot\nabla E}{E}u^{2t}\psi^2 + (\beta - \beta^2)\int \frac{|\nabla E|^2}{E^2}u^{2t}\psi^2 + (T-1)\int |\nabla(u^t\psi)|^2 \le T\int |\nabla(u^t\psi)|^2.$$

Note that the right side expands as

$$Tt^2\int u^{2t-2}|\nabla u|^2\psi^2 + 2tT\int u^{2t-1}\psi\nabla u\cdot\nabla\psi + T\int u^{2t}|\nabla\psi|^2.$$

We now wish to eliminate the term $\int u^{2t-2}|\nabla u|^2\psi^2$ from the inequality. To do this we multiply (2) by $u^{2t-1}\psi^2$ and integrate over Ω to arrive at

$$(2t-1)\int u^{2t-2}|\nabla u|^2\psi^2 = \int u^{p+2t-1}\psi^2 - \int a\cdot\nabla u u^{2t-1}\psi^2 - 2\int \nabla u\cdot\nabla\psi u^{2t-1}\psi.$$

Using this equality we replace the desired term in the inequality to arrive at an inequality of the form

$$\left(\beta p - \frac{Tt^2}{2t-1}\right)\int u^{2t+p-1}\psi^2 + \beta(1-\beta)\int \frac{|\nabla E|^2}{E^2}u^{2t}\psi^2 + (T-1)\int |\nabla(u^t\psi)|^2 \le T\int u^{2t}|\nabla\psi|^2 + \sum_{k=1}^{3} I_k \tag{12}$$

where

$$I_1 = \left(2Tt - \frac{2Tt^2}{2t-1}\right) \int u^{2t-1} \psi \nabla u \cdot \nabla \psi,$$

$$I_2 = -\frac{Tt^2}{2t-1} \int a(x) \cdot \nabla u u^{2t-1} \psi^2,$$

$$I_3 = \beta \int \frac{a(x) \cdot \nabla E}{E} u^{2t} \psi^2.$$

An integration by parts shows that

$$I_1 = \frac{T(1-t)}{2(2t-1)} \int u^{2t} \Delta(\psi^2).$$

An integration by parts also shows

$$|I_2| \le \frac{Tt}{2t-1} \int |a| \psi |\nabla \psi| u^{2t},$$

and an application of Young's inequality shows this is less than or equal to

$$\frac{Tt\delta}{2t-1} \int |a|^2 \psi^2 u^{2t} + \frac{Tt}{(2t-1)4\delta} |\nabla \psi|^2 u^{2t},$$

for any $\delta > 0$. An application of Young's inequality shows that

$$|I_3| \le \beta \varepsilon \int \frac{|\nabla E|^2}{E^2} u^{2t} \psi^2 + \frac{\beta}{4\varepsilon} \int |a|^2 u^{2t} \psi^2,$$

for any $\varepsilon > 0$. Using these upper bounds in (12) and re-grouping gives the desired result. □

Proof of Theorem 1. We assume that u is a positive stable solution of (2). Firstly note that

$$\int |a|^2 u^{2t} \psi^2 \le C^2 \int \frac{u^{2t} \psi^2}{|x|^2},$$

after considering the conditions on a. Also note by Hardy's inequality we have

$$\int |\nabla (u^t \psi)|^2 \ge C_N \int \frac{u^{2t} \psi^2}{|x|^2},$$

where $C_N = \frac{(N-2)^2}{4}$. Putting these into (10) gives

$$\left(\beta p - \frac{Tt^2}{2t-1}\right) \int u^{2t+p-1} \psi^2 + \beta(1-\beta-\varepsilon) \int \frac{|\nabla E|^2}{E^2} u^{2t} \psi^2 + C_1 \int \frac{u^{2t} \psi^2}{|x|^2} \le C_2 \int u^{2t} \left(|\nabla \psi|^2 + |\Delta(\psi^2)|\right) \quad (13)$$

where

$$C_1 = (T-1) C_N - C^2 \left(\frac{\beta}{4\varepsilon} + \frac{Tt\delta}{2t-1}\right),$$

and $C_2 = C_2(T, t, \delta)$. Note that for each $t_-(p) < t < t_+(p)$ we have $\beta p - \frac{Tt^2}{2t-1} > 0$ provided $\beta < 1$ and $T > 1$ are chosen sufficiently close to 1. We now pick $\varepsilon > 0$ small enough such that $1 - \beta - \varepsilon > 0$. We now assume $C > 0$ is sufficiently small such that $C_1 \ge 0$. We then arrive at an estimate of the form

$$\left(\beta p - \frac{Tt^2}{2t-1}\right) \int u^{2t+p-1} \psi^2 \le C_2 \int u^{2t} \left(|\nabla \psi|^2 + |\Delta(\psi^2)|\right), \quad (14)$$

for all $\psi \in C_c^\infty(\mathbb{R}^N)$. We now assume that ϕ is a smooth cut-off function with, $0 \le \phi \le 1$, $\phi = 1$ in B_R (the open ball of radius R centered at the origin) and compactly supported in B_{2R} such that $|\nabla \phi| \le \frac{C}{R}$ and $|\Delta \phi| \le \frac{C}{R^2}$ where C is independent of R. Putting $\psi = \phi^m$ where m is a large integer into (14) gives

$$\left(\beta p - \frac{Tt^2}{2t-1}\right) \int u^{2t+p-1} \phi^{2m} \le C_2 C_m \int u^{2t} \phi^{2m-2} \left(|\nabla \phi|^2 + |\Delta \phi|\right),$$

where C_m depends only on m. We now apply Hölder's inequality to see the right hand side of this inequality is bounded above by

$$C_2C_m\left(\int u^{2t+p-1}\phi^{\frac{(m-1)(2t+p-1)}{t}}dx\right)^{\frac{2t}{2t+p-1}}\left(\int(|\nabla\phi|^2+|\Delta\phi|)^{\frac{2t+p-1}{p-1}}dx\right)^{\frac{p-1}{2t+p-1}}.$$

Now note that for sufficiently large m we have that $\frac{(m-1)(2t+p-1)}{t}>2m$ and hence we can replace the first term on the right hand side of the inequality with

$$\left(\int u^{2t+p-1}\phi^{2m}dx\right)^{\frac{2t}{2t+p-1}},$$

which allows one to cancel terms to arrive at

$$\left(\beta p-\frac{Tt^2}{2t-1}\right)^{\frac{2t+p-1}{p-1}}\int u^{2t+p-1}\phi^{2m}\le\tilde{C}_m\int\left(|\nabla\phi|^2+|\Delta\phi|\right)^{\frac{2t+p-1}{p-1}}.$$

We now take into account the support of ϕ and how ϕ scales to arrive at

$$\int_{B_R}u^{2t+p-1}\le C_0R^{N-2-\frac{2(2t+p-1)}{p-1}},$$

where C_0 depends on the various parameters but is independent of R. Now provided $N-2-\frac{2(2t+p-1)}{p-1}<0$ we can send $R\to\infty$ to arrive at a contradiction. Now note we can pick a $t\in(t_-(p),t_+(p))$ such that this exponent is negative provided

$$\frac{N(p-1)}{2}<2\left(p+\sqrt{p^2-p}\right)+p-1,$$

which is precisely (9). □

Proof of Theorem 2. Suppose $0<u$ is a smooth stable solution of (2) and $E>0$ solves (3). Let $|x_k|\to\infty$ and set $r_k:=\frac{|x_k|}{4}$. By passing to a subsequence we can assume that $\{B(x_k,r_k):k\ge 1\}$ is a disjoint family of balls. We now define the re-scaled functions

$$u_k(x)=r_k^{\frac{2}{p-1}}u(x_k+r_kx),\qquad a_k(x)=r_ka(x_k+r_kx),\qquad E_k(x)=E(x_k+r_kx),$$

and we restrict $|x|<2$. Then Eqs. (2) and (3) are satisfied on B_2 with u_k,a_k,E_k replacing u,a,E. Note that $a_k(x)$ is a sequence of smooth divergence free vector fields which satisfy the bound $|a_k(x)|\le C$ for all $|x|<2$. From this we see the term involving a_k in (10) will be a lower order term as far as powers of u are concerned and hence will cause no issues. With the conditions on N and p there is some $t_-(p)<t<t_+(p)$ such that $2t+p-1>\frac{N}{2}(p-1)>0$ and by taking $T=1$ (we can take $T=1$ since the advection term is lower order) and $\beta<1$ sufficiently close to 1 we can assume $\beta p-\frac{t^2}{2t-1}>0$. Let $0\le\phi\le 1$ be compactly supported in B_2 with $\phi=1$ on B_1 and put $\psi=\phi^m$, where m is a large integer, into (10) where now u,a,E are given by u_k,a_k,E_k. Arguing as in the proof of Theorem 1 one can obtain a bound of the form

$$\int_{B_1}u_k^{2t+p-1}\le C_0,$$

where C_0 depends on the various parameters but is independent of k. Now note that $u_k>0$ is a sequence of smooth positive solutions of

$$-\Delta u_k+a_k(x)\cdot\nabla u_k=C_k(x)u_k\quad\text{in }B_2,$$

where $C_k(x)=u_k^{p-1}$. The above integral estimate shows that C_k is bounded in $L^q(B_1)$ for some $q>\frac{N}{2}$. We can now apply a Harnack inequality from [20] to see that

$$\sup_{B_{\frac{1}{2}}}u_k\le C\inf_{B_{\frac{1}{2}}}u_k.\tag{15}$$

If we can show that $\inf_{B_{\frac{1}{2}}}u_k\to 0$ then one has $\sup_{B_{\frac{1}{2}}}\to 0$ and in particular this gives

$$|x_k|^{\frac{2}{p-1}}u(x_k)\le 4^{\frac{2}{p-1}}\sup_{B_{\frac{1}{2}}}u_k\to 0$$

which gives us the desired decay estimate. To show $\inf_{B_{\frac{1}{2}}} u_k \to 0$ we will show

$$\int_{B_1} u_k^{\frac{(p-1)N}{2}} \to 0.$$

Using a change of variables shows that

$$\int_{B_1} u_k^{\frac{(p-1)N}{2}} = \int_{B(x_k,r_k)} u^{\frac{(p-1)N}{2}},$$

and if we show that $u \in L^{\frac{(p-1)N}{2}}(\mathbb{R}^N)$ then we had have the desired result since

$$\int_{\mathbb{R}^N} u^{\frac{(p-1)N}{2}} \ge \sum_{k=1}^{\infty} \int_{B(x_k,r_k)} u^{\frac{(p-1)N}{2}}.$$

Towards this we now set $t = \frac{(p-1)(N-2)}{4}$ and note that the condition on N and p implies that $t_-(p) < t < t_+(p)$. We now pick $\beta < 1$ but sufficiently close such that $\beta p - \frac{t^2}{2t-1} > 0$ and pick $\varepsilon > 0$ sufficiently small such that $1 - \beta - \varepsilon > 0$. Let ϕ be the smooth cut-off function from the proof of Theorem 1, which is equal to 1 in B_R and compactly supported in B_{2R}. We now put $\psi = \phi^m$, where m is a large integer, into (10) taking $T = 1$, to arrive at inequality of the form

$$\int u^{2t+p-1}\phi^{2m} \le C_0 \int |a|^2 u^{2t}\phi^{2m} + C_0 \int u^{2t}\phi^{2m-2}\left(|\nabla\phi|^2 + |\Delta\phi|\right). \tag{16}$$

We now let τ be such that $2t\tau = 2t + p - 1$ and let τ' denote the conjugate index of τ. Applying Hölder's inequality to the right hand side of (16) and arguing as in the proof of Theorem 1 we arrive at an inequality, for sufficiently large m, of the form

$$\int u^{2t+p-1}\phi^{2m} \le C_0 \int_{B_{2R}} |a|^{2\tau'} + C_0 \int \left(|\nabla\phi|^2 + |\Delta\phi|\right)^{\tau'},$$

where C_0 is a constant which depends on the various parameters but is independent of R. A computation shows that $\tau' = \frac{N}{2}$ and $2t + p - 1 = \frac{N}{2}(p-1)$. Using these explicit values and the scaling of ϕ we arrive at

$$\int_{B_R} u^{\frac{N(p-1)}{2}} \le C_0 \int_{B_{2R}} |a|^N + C_0,$$

and from this we obtain the desired bound on u after recalling that $|a| \in L^N(\mathbb{R}^N)$. $\square$

3. Existence proofs

The positive radial solution.

For $p > \frac{N+2}{N-2}$ let $w = w(r)$ denote the positive radial decreasing solution of $-\Delta w = w^p$ in $\mathbb{R}^N$ with $w(0) = 1$. Asymptotics of w as $r \to \infty$ are given by

$$w(r) = \beta^{\frac{1}{p-1}} r^{\frac{-2}{p-1}}(1 + o(1)),$$

where

$$\beta = \beta(p, N) = \frac{2}{p-1}\left(N - 2 - \frac{2}{p-1}\right).$$

In the case where $p > p_c$ the refined asymptotics are given by

$$w(r) = \beta^{\frac{1}{p-1}} r^{\frac{-2}{p-1}} + \frac{a_1}{r^{\mu_0^-}} + o\left(\frac{1}{r^{\mu_0^-}}\right),$$

where $a_1 < 0$ and $\mu_0^- > \frac{2}{p-1}$; see [7].

We begin by analyzing the radial solution w as defined above. Let $v(r) = \beta^{\frac{1}{p-1}} r^{\frac{-2}{p-1}}$ where β is defined as above.

Lemma 1. *Suppose $p > p_c$, $v(r) = \beta^{\frac{1}{p-1}} r^{\frac{-2}{p-1}}$ and β is defined as in the definition of w.*

1. *Then $v \ge w$ in $\mathbb{R}^N$.*
2. *There is some $\varepsilon > 0$ such that*

$$\int (p+\varepsilon) w^{p-1}\phi^2 \le \int |\nabla\phi|^2 \quad \forall\phi \in C_c^\infty(\mathbb{R}^N). \tag{17}$$

Proof. (1) Note that $v(r) > w(r)$ for large r and small r. Towards a contradiction we assume that there is $0 < r_0 < r_1$ such that $w(r) > v(r)$ for all $r_0 < r < r_1$ with $w = v$ at $r = r_0, r_1$. A computation shows that for $p > p_c$ there is some $\varepsilon > 0$ such that $(p+\varepsilon)\beta \le \frac{(N-2)^2}{4}$ and then from Hardy's inequality we obtain

$$\int (p+\varepsilon)v^{p-1}\phi^2 \le \int |\nabla\phi|^2 \quad \forall \phi \in C_c^\infty(\mathbb{R}^N). \tag{18}$$

From this we see that v is a stable singular solution of $-\Delta v = v^p$ in $\mathbb{R}^N$ and in particular it is a stable solution of

$$-\Delta v = v^p \quad \text{in } r_0 < r < r_1 \text{ with } v = w \text{ on } r = r_0, r_1.$$

It is possible to use the stability of v to show that v is the minimal solution of this equation with the given prescribed boundary conditions; minimal solution means smallest in the pointwise sense. This fact relies on the strict convexity of the nonlinearity, see Lemma 2.4 [21] for details. Noting that w satisfies the same equation with the prescribed boundary conditions one must have $v \le w$ on $r_0 < r < r_1$ since v is a minimal solution. This gives us the desired contradiction.

(2) The result is immediate after combining the pointwise comparison between w and v and using (18). □

For the remainder w always refers to the above radial solution and L to the linear operator $L(\phi) = -\Delta\phi - pw^{p-1}\phi$. We now define the various function spaces. For $\sigma > 0$ but small, define

$$\|\phi\|_{\tilde{X}_\sigma} := \sup_{|x|\le 1} |x|^\sigma |\phi(x)| + \sup_{|x|\ge 1} |x|^{\frac{2}{p-1}} |\phi(x)|,$$

and

$$\|f\|_{Y_\sigma} := \sup_{|x|\le 1} |x|^{\sigma+2} |f(x)| + \sup_{|x|\ge 1} |x|^{\frac{2}{p-1}+2} |f(x)|.$$

Let $\tilde{X}_\sigma$ and Y_σ denote the completions of $C_c^\infty(\mathbb{R}^N\setminus\{0\})$ under the appropriate norms.

The following linear estimate is from [19] and is a key starting point for their work. They also obtain results in the case of $\frac{N+2}{N-2} < p < \frac{N+1}{N-3}$ in [19] and also in another of their works [22]. This case is harder to deal with but luckily our main interest is in the case of $p > p_c$ which allows us to avoid the harder case.

Theorem A ([19]). *Suppose $N \ge 4$ and $p > \frac{N+1}{N-3}$. There exists some small $\sigma > 0$ such that for any $f \in Y_\sigma$ there exists some $\phi \in \tilde{X}_\sigma$ such that $L(\phi) = f$ in $\mathbb{R}^N$. Moreover the linear map $T : Y_\sigma \to \tilde{X}_\sigma$, given by $T(f) = \phi$, is continuous.*

For our approach we will not work directly with $\tilde{X}_\sigma$ but instead work with a slight variant that allows us to handle the advection term. So towards this define the norm

$$\|\phi\|_{X_\sigma} := \sup_{|x|\le 1} \left(|x|^\sigma |\phi(x)| + |x|^{\sigma+1}|\nabla\phi(x)|\right) + \sup_{|x|\ge 1} \left(|x|^{\frac{2}{p-1}} |\phi(x)| + |x|^{\frac{2}{p-1}+1}|\nabla\phi(x)|\right)$$

and let X_σ denote the completion of $C_c^\infty(\mathbb{R}^N\setminus\{0\})$ with respect to this norm.

Lemma 2. *Suppose $N \ge 4$ and $p > \frac{N+1}{N-3}$. For sufficiently small $\sigma > 0$ and for all $f \in Y_\sigma$ there exists some $\phi \in X_\sigma$ such that $L(\phi) = f$ in $\mathbb{R}^N$. Moreover the linear map $T : Y_\sigma \to X_\sigma$ defined by $T(f) = \phi$ is continuous.*

Proof of Lemma 2. Suppose $f \in Y_\sigma$ and let $\phi \in \tilde{X}_\sigma$ be such that $L(\phi) = f$ in $\mathbb{R}^N$. Then there exists some $C > 0$, independent of f and ϕ, such that $\|\phi\|_{\tilde{X}_\sigma} \le C\|f\|_{Y_\sigma}$. Our goal is to now show there is some $C_1 > 0$, independent of f and ϕ, such that $\|\phi\|_{X_\sigma} \le C_1\|f\|_{Y_\sigma}$ and this will complete the proof. Define the re-scaled functions $\phi_m(x) = \phi(x_m + r_m x)$ where $|x_m| > 0$, $r_m = \frac{|x_m|}{4}$ for $|x| < 1$. Note that

$$-\Delta\phi_m(x) = pr_m^2 w(x_m + r_m x)^{p-1}\phi(x_m + r_m x) + r_m^2 f(_m + r_m x) =: g_m(x),$$

for all $x \in B_1$. We now obtain some estimates on ϕ_m using the following result, which is just an elliptic regularity result coupled with the Sobolev embedding theorem: for $t > N$ there is some C_t such that

$$\sup_{B_{\frac{1}{2}}} |\nabla\phi_m(x)| \le C_t \left(\int_{|x|<1} |\Delta\phi_m(x)|^t dx\right)^{\frac{1}{t}} + C_t \int_{|x|<1} |\phi_m(x)| dx. \tag{19}$$

We now assume we are in the case of $|x_m| \ge 1$. Using the fact that $f \in Y_\sigma$ and $\phi \in \tilde{X}_\sigma$ one sees that $|x_m|^{\frac{2}{p-1}}|g_m(x)| \le C$ for all $|x| < 1$ and m. Putting these estimates into (19) gives $\sup_{B_{\frac{1}{2}}} |\nabla\phi_m(x)| \le C|x_m|^{\frac{-2}{p-1}}$ and from this we see that

$$|x_m|^{\frac{2}{p-1}+1}|\nabla\phi(x_m)| \le C_1.$$

The case of $|x_m| \le 1$ is handled as above. Combining these results gives us the desired bound. □

Proof of Theorem 3. (1) To solve (2) we first consider solving the related problem given by

$$-\Delta u + a(x)\cdot\nabla u = |u|^p \quad \text{in } \mathbb{R}^N. \tag{20}$$

To do this we perturb off the radial solution w of the advection free problem. So we look for a solution of the form $u=\phi+w$. So we need to find a solution ϕ of

$$L(\phi) = -a\cdot\nabla w - a\cdot\nabla\phi + |w+\phi|^p - pw^{p-1}\phi - w^p \quad \text{in } \mathbb{R}^N, \tag{21}$$

where $L(\phi)=-\Delta\phi - pw^{p-1}\phi$. Letting T be defined as in Lemma 2 we are looking for a $\phi\in X_\sigma$ such that

$$\phi = -T(a\cdot\nabla w) - T(a\cdot\nabla\phi) + T(|w+\phi|^p - pw^{p-1}\phi - w^p). \tag{22}$$

To find such a ϕ we define $J(\phi)$ to be the mapping on the right hand side of (22) and we will now show that for a suitable R that J is a contraction mapping on the closed ball B_R, centered at the origin, in X_σ. We will then argue that $u=w+\phi$ is positive. We begin by showing J is into B_R. In what follows C can depend on p, a, w but not on x, R, ϕ and σ provided σ is small. Let $R>0$ and let $\phi\in B_R$. Then note that there is some $C>0$ such that

$$\|J(\phi)\|_{X_\sigma} \le C\|a\cdot\nabla w\|_{Y_\sigma} + C\|a\cdot\nabla\phi\|_{Y_\sigma} + C\||w+\phi|^p - pw^{p-1}\phi - w^p\|_{Y_\sigma}. \tag{23}$$

We now estimate the terms on the right hand side.

$$\begin{aligned}\|a\cdot\nabla w\|_{Y_\sigma} &\le \sup_{|x|\le 1}|a(x)||x|\sup_{|x|\le 1}|x|^{1+\sigma}|\nabla w(x)| + \sup_{|x|\ge 1}|x||a(x)|\sup_{|x|\ge 1}|x|^{\frac{2}{p-1}+1}|\nabla w(x)|\\ &\le \left(\sup_x |a(x)||x|\right)\|w\|_{X_\sigma}.\end{aligned}$$

The same argument shows that

$$\|a\cdot\nabla\phi\|_{Y_\sigma} \le \left(\sup_x |x||a(x)|\right)\|\phi\|_{X_\sigma}.$$

We now approximate the last term in (23). For this we need the following real analysis result. There exists some $C=C_p$ such that for all numbers $w>0$ and $\phi\in\mathbb{R}$ we have

$$\left||w+\phi|^p - pw^{p-1}\phi - w^p\right| \le C\left(w^{p-2}\phi^2 + |\phi|^p\right).$$

Set $\Gamma = |w+\phi|^p - pw^{p-1}\phi - w^p$. Then one sees

$$\begin{aligned}\|\Gamma\|_{Y_\sigma} &\le C\sup_{|x|\le 1}|x|^{\sigma+2}\left(w^{p-2}\phi^2+|\phi|^p\right) + C\sup_{|x|\ge 1}|x|^{\frac{2}{p-1}+2}\left(w^{p-2}\phi^2+|\phi|^p\right)\\ &:= CI_1 + CI_2.\end{aligned}$$

Then note that

$$\begin{aligned}I_1 &= \sup_{|x|\le 1}\left(|x|^{2-\sigma}w^{p-2}(|x|^\sigma\phi(x))^2 + |x|^{\sigma+2-\sigma p}(|\phi(x)||x|^\sigma)^p\right)\\ &\le \sup_{|x|\le 1}\left(|x|^{2-\sigma}w^{p-2}\|\phi\|_{X_\sigma}^2 + |x|^{\sigma+2-\sigma p}\|\phi\|_{X_\sigma}^p\right)\\ &\le C\|\phi\|_{X_\sigma}^2 + C\|\phi\|_{X_\sigma}^p\end{aligned}$$

for sufficiently small $\sigma>0$. One can similarly show that

$$\begin{aligned}I_2 &\le \sup_{|x|\ge 1}\left(|x|^{\frac{2}{p-1}}w\right)^{p-2}\|\phi\|_{X_\sigma}^2 + \|\phi\|_{X_\sigma}^p\\ &\le C\|\phi\|_{X_\sigma}^2 + \|\phi\|_{X_\sigma}^p.\end{aligned}$$

So combining these results we arrive at

$$\|J(\phi)\|_{X_\sigma} \le C\sup_x |x||a(x)| + C\sup_x |x||a(x)|\|\phi\|_{X_\sigma} + C\|\phi\|_{X_\sigma}^2 + C\|\phi\|_{X_\sigma}^p. \tag{24}$$

Before choosing R we examine the condition on J to be a contraction on B_R. First note there is some $C=C_p$ such that for all numbers $w>0$ and $\hat\phi,\phi\in\mathbb{R}$ one has

$$\left||\hat\phi+w|^p - |\phi+w|^p - pw^{p-1}(\hat\phi-\phi)\right| \le CM|\hat\phi-\phi| \tag{25}$$

where

$$M = w^{p-2}\left(|\hat{\phi}| + |\phi|\right) + |\hat{\phi}|^{p-1}| + |\phi|^{p-1}.$$

Let $\hat{\phi}, \phi \in B_R$. Then

$$J(\hat{\phi}) - J(\phi) = -T(a \cdot \nabla(\hat{\phi} - \phi)) + T(|w + \hat{\phi}|^p - |w + \phi|^p - pw^{p-2}(\hat{\phi} - \phi)),$$

and so

$$\begin{aligned}\|J(\hat{\phi}) - J(\phi)\|_{X_\sigma} &\le C\|a \cdot \nabla(\hat{\phi} - \phi)\|_{Y_\sigma} + C\||w + \hat{\phi}|^p - |w + \phi|^p - pw^{p-1}(\hat{\phi} - \phi)\|_{Y_\sigma} \\ &=: CJ_1 + CJ_2.\end{aligned}$$

Arguing as above one easily sees that $J_1 \le \sup_x(|x||a(x)|)\|\hat{\phi} - \phi\|_{X_\sigma}$. Using (25) we see that

$$J_2 \le C \sup_{|x|\le 1} |x|^{2+\sigma} M|\hat{\phi} - \phi| + C \sup_{|x|\ge 1} |x|^{\frac{2}{p-1}+2} M|\hat{\phi} - \phi| =: CJ_3 + CJ_4.$$

We now compute the various terms in J_3 and J_4.

$$\sup_{|x|\le 1} |x|^{2+\sigma} w^{p-1}|\hat{\phi}||\hat{\phi} - \phi| \le \sup_{|x|\le 1}(|x|^{2-\sigma} w^{p-2})\|\hat{\phi}\|_{X_\sigma}\|\hat{\phi} - \phi\|_{X_\sigma}.$$

Also we have

$$\begin{aligned}\sup_{|x|\le 1} |x|^{2+\sigma}|\hat{\phi}|^{p-1}|\hat{\phi} - \phi| &\le \sup_{|x|\le 1} |x|^{2-\sigma-\sigma(p-1)}\|\hat{\phi}\|^{p-1}_{X_\sigma}\|\hat{\phi} - \phi\|_{X_\sigma}, \\ &\le \|\hat{\phi}\|^{p-1}_{X_\sigma}\|\hat{\phi} - \phi\|_{X_\sigma},\end{aligned}$$

for small enough $\sigma > 0$. Combining these results we obtain

$$\begin{aligned}J_3 &\le \left(\sup_{|x|\le 1} |x|^{2-\sigma} w^{p-2} 2R + 2R^{p-1}\right)\|\hat{\phi} - \phi\|_{X_\sigma} \\ &\le \left(CR + 2R^{p-1}\right)\|\hat{\phi} - \phi\|_{X_\sigma}.\end{aligned}$$

One can argue in a similar fashion to show

$$\begin{aligned}J_4 &\le \sup_{|x|\ge 1}\left(|x|^{\frac{2}{p-1}} w\right)^{p-2}(\|\hat{\phi}\|_{X_\sigma} + \|\phi\|_{X_\sigma})\|\hat{\phi} - \phi\|_{X_\sigma} + \left(\|\hat{\phi}\|^{p-1}_{X_\sigma} + \|\phi\|^{p-1}_{X_\sigma}\right)\|\hat{\phi} - \phi\|_{X_\sigma} \\ &\le \left(CR + 2R^{p-1}\right)\|\hat{\phi} - \phi\|_{X_\sigma}.\end{aligned}$$

Combining the results we obtain an inequality of the form

$$\|J(\hat{\phi}) - J(\phi)\|_{X_\sigma} \le C\left(\sup_x |x||a(x)| + R + R^{p-1}\right)\|\hat{\phi} - \phi\|_{X_\sigma}. \tag{26}$$

We now pick R and put conditions on a. Fix R sufficiently small such that $CR^2 + CR^p \le \frac{R}{10}$ and such that $CR + CR^{p-1} < \frac{1}{2}$. Now impose a smallness condition on a such that $C\sup_x |x||a(x)| + C\sup_x |x||a(x)|R \le \frac{R}{10}$ and $C\sup_x |x||a(x)| < \frac{1}{4}$. These conditions are sufficient to show that J is a contraction mapping on B_R in X_σ and hence by the Contraction Mapping Principle there is some $\phi \in B_R$ such that $J(\phi) = \phi$, which was the desired result. So we have $\phi \in B_R$ such that

$$-\Delta(w + \phi) + a \cdot \nabla(w + \phi) = |w + \phi|^p \quad \text{in } \mathbb{R}^N. \tag{27}$$

By taking $R > 0$ smaller, which imposes a further smallness condition on a, we can assume that

$$\sup_{|x|\ge 1} |x|^{\frac{2}{p-1}}|\phi(x)| \le \frac{1}{10}\inf_{|y|\ge 1} |y|^{\frac{2}{p-1}} w(y). \tag{28}$$

Using this one sees that $\phi + w > 0$ on $|x| \ge 1$. Note there are some possible regularity issues for ϕ near the origin. But taking $\sigma > 0$ small enough and applying elliptic regularity theory, along with a bootstrap, one sees that ϕ is at least $C^{2,\alpha}$ in a ball around the origin for some $\alpha > 0$. One can now apply the maximum principle to see that $u = w + \phi$ is a positive solution of (2). □

Proof of Theorem 3. (2) First note that a computation shows that $p_c > \frac{N+1}{N-3}$. For $R > 0$ sufficiently small there is some $u_R > 0$ which satisfies (2) and as R gets small one imposes smallness conditions on a. For $m \ge 2$ an integer let $E = E_{m,R} > 0$ denote the first eigenfunction of $L(E) := -\Delta E + a \cdot \nabla E - pu_R^{p-1}E$ on the ball B_m with $E = 0$ on ∂B_m and let $\mu_{m,R}$ denote the first eigenvalue. We now multiply the equation for E by E and integrate over B_m. Using the fact that a is divergence free (this is only spot we utilize this fact) one sees, after a suitable L^2 normalization of E, that

$$\int |\nabla E|^2 = \int p u_R^{p-1} E^2 + \mu_{m,R}.$$

We now extend E outside B_m by setting it to be zero and we use the fact that w satisfies (17) to arrive at

$$(p+\varepsilon)\int w^{p-1}E^2 \le \int p u_R^{p-1} E^2 + \mu_{m,R},$$

for some fixed $\varepsilon > 0$. Note that $u_R \to w$ in X_σ as $R \to 0$ and so we can argue as in (28), that for any $\delta > 0$ we can pick R small enough such that $u_R(x) \le (\delta+1)w(x)$ for all $|x| \ge 1$. Using elliptic regularity and Sobolev embedding one sees that the restriction of u_R to the unit ball converges to the restriction of w uniformly. And so we can assume that $u_R^{p-1} \le w^{p-1} + \delta$ for $|x| \le 1$ for small enough R. Using these estimates and breaking the integrals into the regions $|x| \ge 1$ and $|x| \le 1$ one arrives at

$$\left((p+\varepsilon) - p(1+\delta)^{p-1}\right)\int_{|x|\ge 1} w^{p-1}E^2 + (\varepsilon - p\delta)\int_{|x|\le 1} w^{p-1}E^2 \le \mu_{m,R},$$

for sufficiently small R. Now by taking $\delta > 0$ small enough one sees that for fixed R small enough we have $\mu_{m,R} \ge 0$ for all $m \ge 2$. We now fix this R and let $u = u_R$, $E_m = E_{m,R}$ and $\mu_m = \mu_{m,R}$. So we have that $E_m > 0$ satisfies

$$\begin{cases} -\Delta E_m + a(x)\cdot\nabla E_m = p u^{p-1} E_m + \mu_m E_m & \text{in } B_m \\ E_m = 0 & \text{on } \partial B_m. \end{cases}$$

Let us assume that $\mu_m \to 0$. By suitably scaling E_m we can assume that $E_m(0) = 1$. Now fix $k \ge 1$ an integer and let $m \ge k+2$. Now note that E_m satisfies the same equation on B_{k+1} and hence by the Harnack inequality there is some $C_k > 0$ such that

$$\sup_{B_k} E_m \le C_k \inf_{B_k} E_m \le C_k,$$

for all $m \ge k+2$. Using elliptic regularity one can show that E_m is bounded in $C^{1,\alpha}(B_k)$ and by a diagonal argument there is some subsequence of E_m, which we still denote by E_m, which converges to some $E \ge 0$ locally in $C^{1,\beta}$ for some $\beta > 0$ and $E(0) = 1$. One can then argue that E satisfies

$$-\Delta E + a(x)\cdot\nabla E = p u^{p-1} E \quad \text{in } \mathbb{R}^N,$$

and then we can apply the strong maximum principle to see that $E > 0$. This shows that u is a stable solution of (2) which was the desired result. We now show $\mu_m \to 0$. We begin by putting E_m, which we L^2 normalize, into (11) with $\beta = \frac{1}{2}$ to arrive at

$$\mu_m \int \phi^2 + \frac{1}{2}\int \frac{|\nabla E_m|^2}{E_m^2}\phi^2 \le 2\int |\nabla\phi|^2 + \int \frac{a\cdot\nabla E_m}{E_m}\phi^2,$$

for all $\phi \in C_c^\infty(B_m)$. We now use Young's inequality to arrive at

$$\mu_m \int \phi^2 + \frac{1}{2}\int \frac{|\nabla E_m|^2}{E_m^2}\phi^2 \le 2\int |\nabla\phi^2| + \varepsilon\int \frac{|\nabla E_m|^2}{E_m^2}\phi^2 + \frac{1}{4\varepsilon}\int |a|^2\phi^2.$$

By taking $\varepsilon > 0$ small enough and re-grouping terms and by using the fact that $|a(x)| \le \frac{c^2}{|x|^2}$ along with Hardy's inequality, one can obtain

$$\mu_m \int \phi^2 \le C\int |\nabla\phi|^2 \quad \forall \phi \in C_c^\infty(B_m),$$

where C is independent of m. From this we can conclude that $\limsup_m \mu_m \le 0$ but we already have $\mu_m \ge 0$ and hence we have the desired result. □

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