Existence and regularity of nonlinear advection problems

A. Aghajani[*] C. Cowan[†] S. H. Lui[‡]

October 9, 2017

Abstract

This paper examines the existence and regularity of classical positive solutions of $-\Delta u = |\nabla u|^p$ on a bounded domain in $\mathbb{R}^N$ with $0 < p < 1$. This appears to be the first paper to discuss dead core solutions for a PDE with a nonlinear advection term. We also give a Liouville-type result for supersolutions on an exterior domain, asymptotics of solutions as $p \nearrow 1$ and $p \searrow 0$, as well as various extensions of the PDE.

1 Introduction

In this article we examine the existence and regularity of positive solutions of the following

$$\begin{cases} -\Delta u &= |\nabla u|^p \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary and $p > 0$. We also consider some variations on (1).

Our motivation for studying this problem comes from the well-studied Lane-Emden equation given by

$$\begin{cases} -\Delta u &= u^p \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{2}$$

where $p > 1$. This equation (and variations of) have played a central role in the development of the calculus of variations, bifurcation theory, the role of the critical Sobolev exponent, Pohazaev identity, see for instance [14, 28, 32, 45, 48]. In the case of p supercritical the existence versus nonexistence of solutions of (2) on non star shaped domains becomes a very nontrivial issue, see for instance [20, 24, 44]. For versions of (2) on exterior domains see [21, 22, 23].

Consider the variation of (2) given by $\Delta u = u^p$ in the case of $0 < p < 1$. In this case there exist dead core solutions; i.e., solutions which (in this case) are zero on a closed set with nonempty interior, see [47].

We now return to (1). The first point is that it is a non variational equation and hence various standard tools are not available anymore. Some relevant monographs for this work include [39, 33, 29]. We now point out a basic feature of of (1). In the case of $p \geq 1$ there is no non-zero classical solution of (1) since we can re-write the equation as $-\Delta u - b(x) \cdot \nabla u(x) = 0$, where $b(x) = |\nabla u|^{p-2} \nabla u$. Hence we can apply the maximum principle, provided $b(x)$ is sufficiently regular, to see that $u = 0$. So with this in mind we restrict our attention to classical solutions of (1) in the case of $0 < p < 1$. One can also examine suitably singular solutions of (1) in the range of $\frac{N}{N-1} < p < 2$ and $p > 2$, see Example 1 and forthcoming paper [9].

We recall some background regarding (1). Many people have studied boundary blow up solutions of the variation of (2) given by $\Delta u = u^p$ (and generalizations of). Similarly people have studied boundary blow

[*]School of Mathematics, Iran University of Science and Technology, Narmak, Tehran, Iran. Email: aghajani@iust.ac.ir.

[†]Department of Mathematics, University of Manitoba, Winnipeg, Manitoba, Canada R3T 2N2. Email: craig.cowan@umanitoba.ca. Research supported in part by NSERC.

[‡]Department of Mathematics, University of Manitoba, Winnipeg, Manitoba, Canada R3T 2N2. Email: shaun.lui@umanitoba.ca. Research supported in part by NSERC.

up versions of (1) where one removes the minus sign in front of the Laplacian; see for instance [40, 49]. See [5, 6, 7, 8, 10, 11, 12, 13, 19, 25, 26, 30, 31, 34, 35, 41, 42, 43, 46] for more results on equations similar to (1). Needless to say, this list is incomplete. We remark that several papers consider a much more general PDE than (1). Those papers tend to discuss existence of solutions which is not an issue in (1) since the trivial solution is always a solution. This paper appears to be the first to discuss dead core solutions to PDEs with a nonlinear advection term.

We now give a brief summary of the results in papers listed above. [5] studied positive weak solutions of $-\Delta u = B|\nabla u|^2/u + f$ for positive B and non-negative f. It was shown that positive solutions exist iff $B < 1$. [6] considers non-negative weak solutions of $-\Delta u + |\nabla u|^2 u^{-\gamma} = f > 0$, where γ is a positive constant. The main result is that such solution exists iff $\gamma < 2$. [7] and [8] consider $-\Delta u = d(x)u + \mu(x)|\nabla u|^2 + h(x)$ and its generalization $-Lu = H(x, u, \nabla u) + h(x)$, where L is a second-order linear elliptic operator and H grows at most quadratically in $|\nabla u|$ and satisfies other appropriate conditions. [10] demonstrates existence of weak solutions of $Au + g(x, u, \nabla u) = h(x)$, where A is a quasi-linear elliptic operator, the nonlinearity g satisfies a growth condition in $|\nabla u|$ and a sign condition.

In the series of papers [11, 12, 13], they consider $-\Delta_q u + |\nabla u|^p = 0$ and its generalizations. Here $-\Delta_q$ is the q-Laplacian and $q - 1 < p < q$. A detailed study of the singularities of non-negative solutions is given. [12] gives a classification isolated singularities of negative solutions of this PDE, which is a generalization of (positive solutions of) (1) for the case $1 < p < 2$. The paper [19] gives existence and a classification of singular non-negative solutions of $\Delta u = u^q|\nabla u|^m$ on $\mathbb{R}^N \setminus 0$, where $q \geq 0$ and $m \in (0, 2)$.

[25, 26] derived existence and a uniform L^∞ estimate of weak solutions of $-\Delta_p = |\nabla u|^p + g - div f$ and its generalization $-div\ a(x, u, \nabla u) = H(x, u, \nabla u) + g - div f$. Here $p > 1$, a satisfies certain coercive and growth conditions and H grows no faster than $|\nabla u|^p$ and satisfies a sign condition. Both f and g must be sufficiently small.

[30, 31] prove existence of weak solutions of $-\Delta u = |\nabla u|^2|u|^{-\theta} + f(x)$ and generalizations. Here $\theta \in (0, 1)$ and $f \in L^m$ may not have a constant sign. [34, 35] prove a priori estimates and existence of a solution for a general class of nonlinear elliptic problems $-div\ a(x, u, \nabla u) = H(x, u, \nabla u)$, where H satisfies a growth condition. In particular, it treats the case $H = |\nabla u|^p$ for $p > 1$.

[41] is a classical paper which considers $-\Delta u + H(x, \nabla u) + \alpha u = 0$, where H satisfies a growth and sign condition and $\alpha \geq 0$. As opposed to other papers in our list, existence theory for both Dirichlet and Neumann boundary conditions are given.

[42, 43] study non-negative solutions of $-\Delta u + H(x, u, \nabla u) = 0$ for some H including the special cases $|\nabla u|^p$, $u^p|\nabla u|^q$ and $u^p + |\nabla u|^q$, where p and q are suitably restricted constants. In particular, when $H = |\nabla u|^p$, assume $p \in (1, 2)$. Finally, [46] contains results for $-\Delta_p u = g(u)|\nabla u|^p + f(x)$, where $p > 1$, $f \in L^m$ and g satisfies a growth condition. It is shown that a weak solution exists, and is bounded or unbounded according to whether $m > N/p$ or $m < N/p$.

We begin by looking at an example.

Example 1. *Let B_1 denote the unit ball centered at the origin in $\mathbb{R}^N$. For $0 < p < 1$ or $p > 2$ define $u(x) := C_p(1 - |x|^q)$ where $q := \frac{2-p}{1-p}$ and $C_p^{p-1} := \frac{N-2+q}{q^{p-1}}$. Then in the case of $0 < p < 1$, $u(x)$ is a C^2 solution of (1) and in the case of $p > 2$ we see that u is a weak $C^{0,q}$ solution of (1). For $\frac{N}{N-1} < p < 2$ we define $u(x) := C_p(|x|^{-q} - 1)$ where $q := \frac{2-p}{p-1} > 0$ and $C_p^{p-1} := \frac{N-2-q}{q^{p-1}}$. Then u is a singular weak solution of (1).*

This example shows there are at least three regimes of the parameter p and what one may expect on general domains. The following definition defines the notion of a dead-core solution.

Definition 1. ***(Dead core solutions)*** *We call a nonnegative nonzero solution u of (1) a* ***dead core*** *solution provided K_u^0 is nonempty; here K_u^0 denotes the interior of $K_u := \{x \in \Omega;\ \ |\nabla u(x)| = 0\}$.*

We now give some explicit examples of dead core solutions on the ball and annulus.

Example 2. *Consider a dead core solution u of (1) in such a way that u is a positive constant for* $r \in [0, r_0]$*, where* $r_0 \in (0,1)$ *and* $0 < p < 1$*. The ODE for* $r \in (r_0, 1)$ *is*

$$-u'' - \frac{N-1}{r}u' = |u'|^p, \qquad u'(r_0) = 0,\ u(1) = 0.$$

Let $v = -u'$ *so that*

$$v' + \frac{N-1}{r}v = v^p, \qquad v(r_0) = 0.$$

Define $z = v^{1-p}$ *so that*

$$z' + \frac{\alpha}{r}z = 1-p, \qquad z(r_0) = 0,$$

where $\alpha = (1-p)(N-1)$*. The solution of this linear ODE for* $r > r_0$ *is*

$$z(r) = \beta(r - r^{-\alpha}r_0^{\alpha+1}), \qquad \beta = \frac{1-p}{\alpha+1}.$$

Consequently,

$$u'(r) = -\beta^{1/(1-p)}(r - r^{-\alpha}r_0^{\alpha+1})^{1/(1-p)}, \qquad r > r_0.$$

Integrate, using the boundary condition $u(1) = 0$*, to get*

$$u(r) = \beta^{1/(1-p)}\int_r^1 (y - y^{-\alpha}r_0^{\alpha+1})^{1/(1-p)}dy, \qquad r > r_0.$$

Note that

$$u''(r) = -\frac{\beta^{1/(1-p)}}{1-p}(r - r^{-\alpha}r_0^{\alpha+1})^{p/(1-p)}(1 + \alpha r^{-\alpha-1}r_0^{\alpha+1}), \qquad r > r_0$$

so that $u \in C^2$*. Indeed,* $u \in C^\infty$ *on* $(0, r_0)$ *and* $(r_0, 1)$*, and for* $\frac{r_0}{2} < r_1 \le r_0 \le r_2 < 1$ *we can write*

$$\begin{aligned} u''(r_2) - u''(r_1) = u''(r_2) &= C_0(r_2^{\alpha+1} - r_0^{\alpha+1})^{p/(1-p)}(r_2^{\alpha+1} + \alpha r_0^{\alpha+1})r_2^{-N} \\ &\le C(r_2 - r_0)^{p/(1-p)} \le C(r_2 - r_1)^{p/(1-p)}, \end{aligned}$$

with C independent of r_1, r_2*. Hence, u belongs to* $C^{2+[\frac{p}{1-p}],\langle\frac{p}{1-p}\rangle}(B_1)$ *if* $\frac{p}{1-p}$ *is not an integer and to* $C^{1+\frac{p}{1-p},\theta}(B_1)$ *for every* $0 < \theta < 1$ *if* $\frac{p}{1-p}$ *is an integer. Here,* $[t]$ *is the integer part of a positive real t and* $\langle t\rangle = \min(t - [t], 1 + [t] - t)$*.*
Note also that

$$h(r_0) := \|u\|_{L^\infty} = \beta^{1/(1-p)}\int_{r_0}^1 (y - y^{-\alpha}r_0^{\alpha+1})^{1/(1-p)}dy.$$

Finally by a direct calculation, if $r_0 = 0$ *and* $q = \frac{2-p}{1-p}$*, then* $u = u_0 = \frac{\beta^{\frac{1}{1-p}}}{q}(1 - r^q)$*, which is not a dead core solution. In fact,* $h(r_0)$*, the height of these positive solutions, is a strictly decreasing function of* r_0 *so that* $h(0)$*, the height of* u_0*, is the highest possible solution, while* $h(1) = 0$ *is the height of the trivial solution.*

We give a brief outline of the paper. In Section 2.1 we discuss the existence of positive classical solutions of (1) in the case of $0 < p < 1$. Our approach is to use a sub/supersolution approach and our main result here is given by Theorem 1, where we also obtain some uniqueness results. In Theorem 2 we obtain some needed estimates which allow us to prove some results regarding dead core solutions as well as the size of a dead core; see Theorem 3. In particular, we show that for any compactly supported subset Ω' of Ω there exists a positive classical solution u of (1) such that $\nabla u \equiv 0$ in Ω'. As we shall see all solutions that we find are indeed dead core solutions, hence we present an open problem about the existence of a non-dead core solution. Also, some Liouville-type results for supersolutions of (1) in unbounded domains such as, the whole space R^N, exterior domains, cone-like domains or, more generally, domains Ω with the property that

$\sup_{x\in\Omega} \text{dist}(x, \partial\Omega) = +\infty$, are given in Theorem 4. In Section 2.2 we consider the question of the regularity of solutions of (1) in the case of $0 < p < 1$. First we show that all positive classical solutions of (1) are in $C^{2,p}(\overline{\Omega})$. We then explore the optimal regularity of solutions of (1) in Theorem 6, where for the proof we use a fundamental result of Caffarelli and Friedman [16] regarding the representation of functions whose Laplacians enjoy a certain inequality. In Section 3 we consider the asymptotics of solutions of (1) in the case of $p \nearrow 1$ and $p \searrow 0$ by means of blow-up arguments and Schauder interior estimates. Finally in Section 4 we consider some generalizations of (1), i.e., $-\Delta u = |\nabla u|^p + f(x, u, \nabla u)$ in Ω with $u = 0$ on $\partial\Omega$, where f satisfies some suitable conditions, and prove the existence of a continuum of positive classical solutions. One example we consider is the convex-concave problem $-\Delta u = |\nabla u|^p + \lambda|\nabla u|^s$ in Ω with $u = 0$ on $\partial\Omega$ where $0 \le \lambda$ is a parameter. We denote this equation by $(Q)_\lambda$ and fix $0 < p < 1 < s < 2$. We obtain solutions indexed by λ which are increasing in λ. These solutions are uniformly bounded in L^∞ and we show as $\lambda \to \infty$ the solutions converge in some sense to a constant in the interior of Ω and they develop some boundary layer behaviour; see Theorem 12 for the existence of solutions and Theorem 13 for the boundary layer result. In Section 4.2 we consider the problem $-\Delta u = |\nabla u|^p + f(x)$ in Ω with zero Dirichlet boundary condition, where $f \in L^q$ (for some $q > N$) is a given function, a non-homogeneous version of (1), and prove the existence of weak solutions u in $W^{2,q}(\Omega) \cap H_0^1(\Omega)$. In Section 4.3 we consider variable exponent versions of (1), i.e., $-\Delta u = |\nabla u|^{p(x)}$ in Ω with $u = 0$ on $\partial\Omega$. Here, $p(x) : \Omega \to [0, 2]$ is a continuous function with $\inf_\Omega p(x) < 1$. We then show that this problem has a continuum of positive classical solutions and there always exists a dead core positive solution.

2 $-\Delta u = |\nabla u|^p$ for $0 < p < 1$

2.1 Existence of solutions

To prove the existence we use sub-super solution method based on the following known result from [38]. We only state a special case of their result. Consider the problem

$$\begin{cases} -\Delta u = G(x, u, \nabla u) \text{ in } \Omega, \\ \quad u = 0 \qquad \text{on } \partial\Omega. \end{cases} \tag{3}$$

Here

$$G(x, s, \xi) = g(x, s) + H(x, s, \xi),$$

where $H(x, s, \xi)$ is continuous in all of its variables for $(x, s, \xi) \in \bar{\Omega} \times \mathbb{R} \times \mathbb{R}^N$ and satisfies

$$|H(x, s, \xi)| \le c(|s|)(1 + |\xi|^2),$$

for some increasing function $c(r) \ge 1$. Moreover, $g(x, s)$ is measurable for $x \in \bar{\Omega}$, continuous for $s \in R$ and for all $r > 0$,

$$\sup_{|s|\le r} |g(x, s)| \in L^q(\Omega), \quad \text{for some} \quad q > N.$$

Now we state Theorem 6.5 of [38].

Theorem A. *[38] Let the above assumptions hold. If $\underline{u}, \bar{u} \in W^{2,q}(\Omega)$ with $q > N$ are, respectively, lower and upper solutions of (3) with $\underline{u} \le \bar{u}$ in Ω, then there is a solution $u \in W^{2,q}(\Omega)$ of (3) such that $\underline{u} \le u \le \bar{u}$. If in addition $G \in C^\infty$ in its variable, then $u \in C^\infty$.*

Theorem 1. *Suppose $0 < p < 1$ and Ω a bounded domain in $\mathbb{R}^N$ with smooth boundary.*
(i) There exists a continuum of positive classical solutions of (1). In particular, for every $x_0 \in \Omega$ there exists a positive solution u with $||u||_{L^\infty(\Omega)} = u(x_0) = \alpha_{N,p}\ r_0^q$, where

$$\alpha_{N,p} := \frac{1}{q}\Big(N + \frac{p}{1-p}\Big)^{\frac{-1}{1-p}}, \quad q := \frac{2-p}{1-p} \quad \text{and} \quad r_0 := dist(x_0, \partial\Omega). \tag{4}$$

(ii) Suppose u and v are two distinct positive classical solutions of (1). Then $K_u \cap K_v \neq \emptyset$.
(iii) Let Ω be a convex bounded domain. Then there exists at most one positive classical solution u with K_u countable. Moreover, if such a solution u exists then all other positive solutions are dead core solutions.

Proof. (i) Let $x_0 \in \Omega$, q and r_0 be as in (4) and set

$$\underline{u}(x) = \alpha_{N,p} \left(r_0^q - |x - x_0|^q \right).$$

Then it is easy to see that $\underline{u} \in W^{2,\infty}(\Omega)$ and $-\Delta \underline{u} = |\nabla \underline{u}|^p$ in Ω with $\underline{u}|_{\partial\Omega} \leq 0$. Hence $\underline{u}$ is a subsolution of (1). Also, $\bar{u} \equiv \alpha_{N,p}\, r_0^q$ is a super-solution. On the other hand we have $\underline{u} \leq \bar{u}$, thus by Theorem A there exists a solution u of (1) (with $u \in W^{2,q}$ for all $q < \infty$) and $\underline{u} \leq u \leq \bar{u}$ in Ω. Note that by the Sobolev imbedding and elliptic regularity we have $u \in C^{2,\alpha}(\overline{\Omega})$ for some $0 < \alpha < 1$. Also, we have $\underline{u}(x_0) \equiv \bar{u}$, implies that $||u||_{L^\infty(\Omega)} = u(x_0) = \alpha_{N,p}\, r_0^q$. This in particular means that we have a continuum of positive classical solutions.

(ii) To the contrary, assume that u and v are two different positive solutions with $K_u \cap K_v = \emptyset$. Take an $y \in \Omega$ such that $u(y) \neq v(y)$. Without loss of generality, assume $u(y) > v(y)$. Then for some $s > 1$ we have $u(y) > sv(y)$. Now set $w = u - sv$. Then we have $w(y) > 0$ and $w|_{\partial\Omega} \leq 0$. Hence, w must take its maximum at a point in Ω, say z. Thus we must have $\nabla w(z) = 0$ and $\Delta w(z) \leq 0$, that give, $\nabla u(z) = s \nabla v(z)$ and

$$0 \geq \Delta w(z) = -|\nabla u(z)|^p + s|\nabla v(z)|^p = -s^p\, |\nabla v(z)|^p + s|\nabla v(z)|^p = (s - s^p)|\nabla v(z)|^p.$$

But, since $s > 1$ and $p < 1$ we have $s - s^p > 0$, hence we must have $\nabla v(z) = 0$ and so $\nabla u(z) = 0$, hence $z \in K_u \cap K_v$, a contradiction.

(iii) Assume that u and v are two positive solutions where K_u and K_v are countable. We show that $u = v$. To do this, for every $\alpha \in (0, 1)$ and $y \in \Omega$ define

$$w_{y,\alpha} : \Omega \to \mathbb{R},$$
$$w_{y,\alpha}(x) = u(\alpha x + (1 - \alpha)y) - v(x).$$

Now, the idea is to show:

$$\text{there exist sequences } y_n \in \Omega \text{ and } \alpha_n \to 1, \text{ such that } w_{y_n,\alpha_n} \text{ takes its maximum at } \partial\Omega. \tag{5}$$

To see how this works, let us for a moment assume (5) holds. Then, using $v|_{\partial\Omega} = 0$,

$$w_{y_n,\alpha_n}(x) = u(\alpha_n x + (1 - \alpha_n)y_n) - v(x) \leq \max_{\partial\Omega} w_{y_n,\alpha_n} = \max_{\xi \in \partial\Omega} u(\alpha_n \xi + (1 - \alpha_n)y_n), \quad x \in \Omega.$$

Now since $\{y_n\}$ is a bounded sequence then letting $n \to \infty$ in the inequality above, we get $u(x) \leq v(x)$, $x \in \Omega$. Changing the rules of u and v we get $v \leq u$, thus $u = v$.
Now we show that (5) holds if the sets K_u and K_v are countable. Take an arbitrary $y \in \Omega \setminus K_u$. Then $|\nabla u(y)| \neq 0$ and by the continuity of $|\nabla u|$, there exists an $\alpha_0 > 0$ sufficiently small such that

$$\text{for } \alpha < \alpha_0, \quad \alpha x + (1 - \alpha)y \notin K_u \quad \text{for every } x \in \Omega. \tag{6}$$

We claim that for $\alpha < \alpha_0$, $w_{y,\alpha}$ takes its maximum on $\partial\Omega$. Otherwise, $w_{y,\alpha}$ takes its maximum at a $z \in \Omega$, hence $\nabla w_{y,\alpha}(z) = 0$, that gives $\alpha(\nabla u)(\alpha z + (1 - \alpha)y) = \nabla v(z)$, and

$$0 \geq \Delta w_{y,\alpha}(z) = (\alpha^p - \alpha)(\nabla u)(\alpha z + (1 - \alpha)y),$$

but $\alpha^p - \alpha > 0$, hence we must have $(\nabla u)(\alpha z + (1 - \alpha)y) = 0$ and $\nabla v(z) = 0$, give $z \in K_v$ and $\alpha z + (1 - \alpha)y \in K_u$ that the later contradicts with (6).
Now for every $y \in \Omega \setminus K_u$ define $\alpha_y := \sup\{\alpha; \;\; w_{y,\alpha} \text{ takes its maximum at } \partial\Omega\}$, and

$$\beta := \sup\{\alpha_y; \;\; y \in \Omega \setminus K_u\}.$$

If $\beta = 1$ then we get a sequence $\alpha_{y_n} \to \beta = 1$ implies (5) holds, hence $u \leq v$. If $\beta < 1$ then by the definition of β, for every $\gamma \in (\beta, 1)$ and every $y \in \Omega \setminus K_u$, $w_{y,\gamma}$ takes its maximum at Ω, hence by the argument above there exits $z_y \in K_v$ such that $\gamma z_y + (1-\gamma)y \in K_u$. This means that $y \in \frac{1}{1-\gamma}K_u - \frac{\gamma}{1-\gamma}K_v$, hence

$$\Omega \setminus K_u \subseteq \frac{1}{1-\gamma}K_u - \frac{\gamma}{1-\gamma}K_v.$$

Since K_u and K_v are countable (and compact) then from the above we get

$$\overline{\Omega} = \overline{\Omega \setminus K_u} \subseteq \frac{1}{1-\gamma}K_u - \frac{\gamma}{1-\gamma}K_v, \quad \text{for every } \gamma \in (\beta, 1). \tag{7}$$

But (7) is impossible because the RHS is a countable set while the LHS is not. This proves the first part of (iii). To prove the moreover part, assume we have a positive solution u with K_u countable and v is a positive solution with $K_v^0 = \emptyset$. We show that $u = v$. First we rewrite (7) as

$$\overline{\Omega} \subseteq \bigcup_{z \in K_u} \Big(\frac{z}{1-\gamma} - \frac{\gamma}{1-\gamma}K_v\Big), \quad \text{for every } \gamma \in (\beta, 1),$$

which is again impossible by Baire's category theorem, since the RHS is a countable union of meager sets, hence $\beta = 1$ and (5) holds implies that $u \leq v$. Now, changing the roles of u and v in the above we get (recall that we have $K_v^0 = \emptyset$)

$$\overline{\Omega} = \overline{\Omega \setminus K_v} \subseteq \bigcup_{z \in K_u} \Big(\frac{1}{1-\gamma}K_v - \frac{\gamma z}{1-\gamma}\Big), \quad \text{for every } \gamma \in (\beta, 1),$$

which is impossible by Baire's theorem, therefore, $v \leq u$. Hence, $u = v$. □

Remark 1. *Note that we know how to get a contradiction from (7) above only when one of K_u or K_v are countable. Generally, if K_u and K_v are two uncountable closed nowhere-dense sets we can not apply the Baire Category Theorem here to get a contradiction. For example take in $\mathbb{R}^2$, two meager sets $E_1 = \mathbb{R} \times \{0\}$ and $E_2 = \{0\} \times \mathbb{R}$, then for every nonzero a, b we have $aE_1 + bE_2 = \mathbb{R}^2$.*

2.1.1 Dead core solutions

Let $\Omega \subset \mathbb{R}^N$ be a bounded smooth domain. For an $x \in \Omega$ we define

$$r_x = dist(x, \partial\Omega), \qquad R_x = \max_{y \in \partial\Omega} |x - y|.$$

Then we have

Theorem 2. *Let u be a positive solution of* (1). *If $x \in \Omega$ is not an interior point of K_u then we have*

$$\alpha_{N,p}\Big(r_x^q - |y-x|^q\Big) \leq u(y) \leq \alpha_{N,p}\Big(R_x^q - |y-x|^q\Big), \quad y \in \Omega, \tag{8}$$

where $\alpha_{N,p}$ and q given in (4). *In particular we have*

$$u(x) \geq \alpha_{N,p}\ r_x^q, \quad x \notin K_u^0. \tag{9}$$

Moreover,

$$||u||_{L^\infty(\Omega)} \leq \alpha_{N,p}\ diam(\Omega)^q. \tag{10}$$

Proof. To prove the second inequality in (8), first fix an $x \in \Omega$ with $\nabla u(x) \neq 0$ and set

$$w(y) := \alpha_{N,p}\Big(R_x^q - |y-x|^q\Big) \quad y \in \Omega.$$

Then we have

$$-\Delta w(y) = |\nabla w(y)|^p, \quad y \in \Omega, \quad w(y)|_{\partial\Omega} \geq 0.$$

We want to show that $u(y) \leq w(y)$ in Ω. If this is not the case then there exist $y_0 \in \Omega$ such that $u(y_0) > w(y_0)$. Now take $1 < s < \frac{u(y_0)}{w(y_0)}$ and $v = u(y) - sw(y)$. Then we have $v(y_0) > 0$ and $v|_{\partial\Omega} \leq 0$. Hence, v must take its maximum at a point in Ω, say z. Thus we must have $\nabla v(z) = 0$ and $\Delta v(z) \leq 0$, that give, $\nabla u(z) = s\nabla w(z)$ and

$$0 \geq \Delta v(z) = -|\nabla u(z)|^p + s|\nabla w(z)|^p = -s^p\ |\nabla w(z)|^p + s|\nabla w(z)|^p = (s - s^p)|\nabla w(z)|^p.$$

But, since $s > 1$ and $p < 1$ we have $s - s^p > 0$, hence we must have $\nabla w(z) = 0$ and so $\nabla u(z) = 0$, that by the definition of w we get $z = x$ implies $\nabla u(x) = 0$, a contradiction. Hence, $u(y) \leq w(y)$ in Ω proves the second inequality in (8) for $x \in \Omega$ with $\nabla u(x) \neq 0$. Similarly we can prove the first inequality for $x \in \Omega$ with $\nabla u(x) \neq 0$. To prove (8) in the case $\nabla u(x) = 0$ but x is not an interior point of $K(u)$, it suffices to take a sequence x_n in Ω such that $\nabla u(x_n) \neq 0$ and $x_n \to x$, then write (8) for x_n and let $n \to \infty$. To prove (9) take $y = x$ in (8). To prove (10), note that $\Omega \setminus K^0(u) \neq \emptyset$ and apply the RHS of (8). □

Theorem 3. (**Dead core solutions**) *Let Ω be an arbitrary bounded domain. Then equation (1) has a continuum of dead core solutions. Indeed, we have*
(i) for every $x_0 \in \Omega$ with $r_{x_0} \neq \sup_{x\in\Omega} r_x := r_\Omega$, there exists a solution u_0 and an open subset Ω_0 in Ω such that

$$u_0 \equiv C := ||u_0||_{L^\infty(\Omega)} \quad in \quad \Omega_0$$

and

$$\{x \in \Omega;\ r_x > r_{x_0}\} \subseteq K^0_{u_0}.$$

(ii) If Ω' is an arbitrary open subset with compact closure in Ω, then there is a solution u of equation (1) such that

$$\nabla u(x) = 0, \quad x \in \Omega'.$$

Proof. (i) Let $x_0 \in \Omega$ with $r_{x_0} \neq r_\Omega$. By Theorem 1 there exists a positive solution u_0 with $||u_0||_{L^\infty(\Omega)} = u_0(x_0) = \alpha_{N,p} r^q_{x_0} := C$. Let $x \in \Omega$ with $r_x > r_{x_0}$ (such an x exists by the assumption that $r_{x_0} \neq r_\Omega$). We claim that x is an interior point of K_{u_0}. Otherwise, from (8) in Theorem 2, we must have $u_0(x) \geq \alpha_{N,p} r^q_x > \alpha_{N,p} r^q_{x_0} = ||u_0||_{L^\infty(\Omega)}$, which is impossible. Hence, we proved that $A_{x_0} := \{x \in \Omega;\ r_x > r_{x_0}\} \subseteq K^0_{u_0}$. Note that, since r_x is a continuous function of x on Ω, then A_{x_0} is an open set and also $x_0 \in \overline{A_{x_0}}$. Let Ω_0 be that connected open component of A_{x_0} for which $x_0 \in \overline{\Omega_0}$. Then we must have $u_0 \equiv C$ in Ω_0. This proves part (i).
(ii) Set for $\epsilon > 0$, $\Omega_\epsilon := \{x \in \Omega,\ dist(x, \partial\Omega) > \epsilon\}$. If Ω' is an arbitrary open subset with compact closure in Ω, then for $\epsilon > 0$ sufficiently small we have $\Omega' \subset \Omega_\varepsilon$. Now take an $x_0 \in \Omega \setminus \Omega_\varepsilon$ and let u_0 be the solution of (1) corresponds to x_0 obtained in part (i). The fact that $x_0 \in \Omega \setminus \Omega_\varepsilon$ gives $\Omega_\varepsilon \subset \{x \in \Omega;\ r_x > r_{x_0}\}$. Hence, by part (i) we must have $\Omega_\varepsilon \subset K^0_{u_0}$, which gives us the desired result.

□

Corollary 1. *Let $\Omega = B_R(x_0)$ be a ball. If u is a positive solution of (1) with $x_0 \notin K^0_u$, then*

$$u(y) = \alpha_{N,p}\Big(R^q - |y - x_0|^q\Big), \quad y \in \Omega.$$

This in particular, means that except for the above radial solution all other solutions are dead core solutions.

Proof. Let $u \not\equiv 0$ be a positive solution of (1). If $x_0 \notin K^0_u$, take $x = x_0$ in (8) then by the fact that $r_{x_0} = R_{x_0} = R$, we get

$$u(y) = \alpha_{N,p}\Big(R^q - |y - x_0|^q\Big).$$

□

As we have seen, in our main theorems on the existence of positive solutions of (1), all solutions we have found are indeed dead core solutions. However, the above Corollary 1 shows that when Ω is a ball then problem (1) has one (and only one) non-dead core solution. Also, in Example 3 below we construct non-dead core solutions of (1) on annular domains in $\mathbb{R}^N$. The following problem would be interesting to consider in connection with the above results.

Open problem. Assume Ω is an arbitrary smooth bounded domain in $\mathbb{R}^N$. Does problem (1) have a non-dead core positive solution?

Example 3. Non-dead core solution on an annular domain in $\mathbb{R}^N$ $(N \geq 2)$.
Consider equation (1) *on the domain* $\Omega = \{x \in \mathbb{R}^N;\ r_1 \leq r \leq 1\}$, $r = |x|$, $(N \geq 2)$. *Set*

$$f_{r_0}(y) := \beta^{1/(1-p)}|y - y^{-\alpha} r_0^{\alpha+1}|^{1/(1-p)},$$

where β, α *given in Example* 2. *Let* $r_0 \in (r_1, 1)$ *(that will be determined later). Taking*

$$u(r) = \int_r^1 f_{r_0}(y), \qquad r > r_0,$$

then as it is shown in Example 2, *u solves equation (1) in* $r_0 < r < 1$, *and* $u(1) = 0$. *Similar as above if we define*

$$u(r) = \int_{r_1}^r f_{r_0}(y), \qquad r_1 < r < r_0,$$

then u solves equation (1) *in* $r_1 < r < r_0$, *and* $u(r_1) = 0$. *Using the computations in Example* 2, *we have* $u'(r_0^-) = u'(r_0^+) = u''(r_0^-) = u'(r_0^+) = 0$. *So, for u being in* C^2 *we need only to find a suitable* r_0 *so that* $u(r_0^-) = u(r_0^+)$, *or*

$$I(r_0) := \int_{r_1}^{r_0} f_{r_0}(y) - \int_{r_0}^1 f_{r_0}(y) = 0.$$

Since $I(r_1^+) < 0,\ I(1^-) > 0$ *and*

$$\begin{aligned} I'(r_0) &= \frac{(\alpha+1)r_0^\alpha}{1-p}\left(\int_{r_1}^{r_0} y^{-\alpha}(r_0^{\alpha+1}y^{-\alpha} - y)^{p/(1-p)}dy + \int_{r_0}^1 y^{-\alpha}(y - y^{-\alpha}r_0^{\alpha+1})^{p/(1-p)}dy\right) \\ &= \frac{(\alpha+1)r_0^\alpha}{1-p}\int_{r_1}^{r_0} y^{-\alpha}\ |r_0^{\alpha+1}y^{-\alpha} - y|^{p/(1-p)}dy \\ &> 0 \end{aligned}$$

there is a unique $r_0^* \in (r_1, 1)$ *so that* $I(r_0^*) = 0$.

2.1.2 Liouville-type results for supersolutions of (1)

In this section we give some Liouville-type results for supersolutions of (1) in the whole space $\mathbb{R}^N$ as well as exterior domains and cone-like domains. Indeed, our result is true in domains Ω containing balls of arbitrary large radius, i.e.,

$$\sup_{x \in \Omega} \operatorname{dist}(x, \partial\Omega) = +\infty. \tag{11}$$

We recall that a domain Ω is called exterior if $\Omega \supset \{|x| > \gamma > 0\}$ for some $\gamma > 0$. By a cone-like domain we mean a domain

$$\mathcal{C}_S = \{(r, \omega) \in \mathbb{R}^N;\ \omega \in S,\ r > 0\},$$

where (r, ω) are the polar coordinates in $\mathbb{R}^N$ and $S \subset S^{N-1}$ is a subdomain (a connected open subset) of the unit sphere S^{N-1} in $\mathbb{R}^N$.

Theorem 4. *Let $0 < p < 1$ and u be a nonnegative smooth solution of the inequality*

$$-\Delta u \geq |\nabla u|^p \quad in \quad \Omega.$$

Then
(i) If $\Omega = \mathbb{R}^N$, u is constant.
(ii) If Ω satisfies condition (11) and u is bounded then there exists an $M > 0$ such that u is constant in $\{x \in \Omega;\ d(x, \partial\Omega) \geq M\}$. Moreover, if u is unbounded then

$$\limsup_{d(x,\partial\Omega)\to\infty} \frac{u(x)}{d(x, \partial\Omega)^q} \geq \alpha_{N,p},$$

where $d(x, \partial\Omega) = dist(x, \partial\Omega)$.

Proof. (i) If we assume that u is not constant then there exists $x_0 \in \mathbb{R}^N$ with $\nabla(x_0) \neq 0$. Let $R > 0$ be arbitrary and set $w_R(x) = \alpha(R^q - |x - x_0|^q)$ where $\alpha := \alpha_{N,p}$. Note we have $-\Delta w_R = |\nabla w_R|^p$ and $w_R \equiv 0$ on $\partial B_R(x_0)$. Take an $s \in (0, 1)$ and set $v_s = u - s w_R$, then we have

$$\Delta v_s \Big|_{\nabla v_s = 0} \leq (s - s^p)|\nabla w|^p,$$

and similar as Theorem 2, since $s - s^p < 0$ and $\nabla w(x) = 0$ only at $x = x_0$, we get $\Delta v_s \Big|_{\nabla v_s = 0} < 0$ implies that v_s takes its minimum at $\partial B_R(x_0)$. But $v_s|_{\partial B_R(x_0)} \geq 0$, hence $v_s \geq 0$ in $B_R(x_0)$. In particular $v_s(x_0) = u(x_0) - s\alpha R^q \geq 0$ which is impossible since R was arbitrary. Hence $\nabla u \equiv 0$ in R^N implies u is constant.
(ii) Let $y \in \Omega$ with $\nabla(y) \neq 0$. Taking $R < d(y, \partial\Omega)$ then similar as part (i) we have $u(y) \geq \alpha R^q$, hence $u(y) \geq \alpha d(y, \partial\Omega)^q$. Now assume u is a bounded solution but the assertion is not true, then there exists a sequence $\{x_n\} \subset \Omega$ such that $d(x_n, \partial\Omega) \to \infty$ with $\nabla u(x_n) \neq 0$, then from the above we get $u(x_n) \geq \alpha d(x_n, \partial\Omega)^q$ implies that u is unbounded, a contradiction. Moreover, if u is an unbounded solution then the above also shows that we must have

$$\limsup_{d(x,\partial\Omega)\to\infty} \frac{u(x)}{d(x, \partial\Omega)^q} \geq \alpha.$$

□

It is worth mentioning here that a rather complete analysis of the positive supersolutions of the more general equation $-\Delta u = f(u)|\nabla u|^p$ in exterior domains has been performed in [15], see also [1, 2, 3, 27].

We now give an example to show the bounded Liouville theorem given in part (ii) of the above theorem is optimal. After the example we give a theorem on general exterior domains which also shows the above Liouville theorem is the best one can hope for.

Example 4. *Let Ω be the region exterior to the unit ball in $\mathbb{R}^N$ and $0 < p < 1$. We look for radial non-negative solutions of (1). The ODE becomes*

$$-u''(r) - \frac{N-1}{r} u'(r) = |u'(r)|^p, \qquad u(1) = 0.$$

We shall show a family of dead core solutions parametrized by $R > 1$, with the solution constant when $r > R$. Since the solution is non-negative, let $v = u' \geq 0$ when $1 \leq r \leq R$. The ODE becomes

$$v' + \frac{N-1}{r} v = -v^p, \qquad v(R) = 0.$$

With $z = v^{1-p}$, the ODE becomes, with $\alpha = (1-p)(N-1) > 0$,

$$z' + \frac{\alpha}{r} z = -(1-p), \qquad z(R) = 0.$$

This is easily solved to get

$$z(r) = \frac{1-p}{\alpha+1}(R^{\alpha+1}r^{-\alpha} - r), \qquad 1 \le r \le R.$$

Then

$$v(r) = \left(\frac{1-p}{\alpha+1}\right)^{1/(1-p)} (R^{\alpha+1}r^{-\alpha} - r)^{1/(1-p)}, \qquad 1 \le r \le R;$$

and so

$$u(r) = \left(\frac{1-p}{\alpha+1}\right)^{1/(1-p)} \int_1^r (R^{\alpha+1}s^{-\alpha} - s)^{1/(1-p)}\, ds, \qquad 1 \le r \le R.$$

By a direct calculation, $u''(R_-) = 0$. Define $M(R) = u(R)$ and

$$u(r) = M(R), \qquad r > R.$$

Thus u is a C^2 solution to (1) on Ω (in fact see Section 2.2 for how regular one expects the solution to be). Note that $M(R) \to \infty$ as $R \to \infty$.

Let us consider the possibility of a solution with a finite (possibly empty) dead core region for $r \in (R, R_2)$ for some $R_2 \ge R$ and a decreasing solution for $r > R_2$. In this instance, let $v = -u'$ when $r > R_2$. A similar calculation as above yields

$$u(r) = M(R) - \left(\frac{1-p}{\alpha+1}\right)^{1/(1-p)} \int_{R_2}^r (s - R_2^{\alpha+1}s^{-\alpha})^{1/(1-p)}\, ds, \qquad r \ge R_2.$$

For all sufficiently large r, it is not difficult to see that $u(r) < 0$ and so this solution can be discarded.

By Hopf's Lemma, there can be no radial solution with a finite dead core region and another dead core at a different value; or a finite dead core region followed by a region of increasing solution. So the family of dead core solutions parmaterized by R found before is the set of all sufficiently regular non-negative radial solutions of (1).

Theorem 5. *Let $0 < p < 1$ and Ω be an exterior domain in $\mathbb{R}^N$, with smooth boundary. Then for every $C > 0$, Problem (1) has a positive bounded solution u with $||u||_{L^\infty(\Omega)} = C$ and u constant in $\{x \in \Omega;\ d(x, \partial\Omega) \ge M\}$ for some $M > 0$.*

Proof. First for the sake of simplicity, consider the exterior domain $\Omega = \{x \in \mathbb{R}^N;\ |x| > 1\}$. For a given $C > 0$ find an r_0 such that $\alpha_{N,p} r_0^q = C$. Now for $R > 1$ set $\Omega_R := \Omega \cap B_R = \{x \in \mathbb{R}^N;\ 1 < |x| < R\}$ and consider the problem

$$\begin{cases} -\Delta u &= |\nabla u|^p \quad \text{in } \Omega_R, \\ \quad u &= 0 \quad\quad\ \ \text{on } \partial\Omega_R, \end{cases}$$

Fix an $x_0 \in \mathbb{R}^N$ with $|x_0| = 1 + r_0$, then for $R > 1 + 3r_0$ we have $x_0 \in \Omega_R$ and $\text{dist}(x_0, \partial\Omega_R) = r_0$. Then, by Theorem 1, the above problem has a positive solution u_R with $||u_R||_{L^\infty(\Omega_R)} = u_R(x_0) = C$. Also, by Proposition 1, $\nabla u_R = 0$ in $\Omega'_R := \{x \in \Omega_R;\ r_x > r_0\} = \{1 + r_0 < |x| < R - r_0\}$. And since $x_0 \in \partial\Omega'_R$ then we get $u(x) = C$ for $x \in \{1 + r_0 < |x| < R - r_0\}$. Now define $u : \Omega \to \mathbb{R}$ by

$$u(x) = \begin{cases} u_R(x), & 1 \le |x| \le 1 + r_0; \\ C, & |x| > 1 + r_0. \end{cases}$$

Then we easily see that u is smooth as much as u_R, also u satisfies Problem (1) in Ω. Moreover, we have $||u||_{L^\infty(\Omega)} = C$ and u is constant ($\equiv C$) in $\{1 + r_0 \le |x| < \infty\}$. A completely similar argument can be applied for a general exterior domain Ω.

□

2.2 Regularity of solutions

In this section we are interested in the regularity of solutions of (1). We begin with an initial regularity result; which we will improve on later.

Lemma 1. *Let $0 < p < 1$ and suppose $u \geq 0$ is a solution obtained via the sub/supersolution approach outlined in the previous section. Then $u \in C^{2,p}(\overline{\Omega})$.*

Proof. First note we have the solution $u \in W^{2,q}$ for all $q > N$; we now fix $N < q < \infty$. Then by Sobolev imbedding we have $|\nabla u|^p \in C^{0,\varepsilon}(\overline{\Omega})$ some $\varepsilon > 0$. By elliptic regularity we have $u \in C^{2,\varepsilon}(\overline{\Omega})$ and hence $|\nabla u| \in C^{0,1}(\overline{\Omega})$ and hence $|\nabla u|^p \in C^{0,p}(\overline{\Omega})$ and we then obtain the desired result after applying elliptic regularity. □

Theorem 6. *Let u be a solution of (1) and $x_0 \in \Omega$ with $|\nabla u(x_0)| = 0$.*
If $\frac{p}{1-p}$ is a non-integer, then we have either
i)

$$|u(x)| \leq C|x - x_0|^{2+\frac{p}{1-p}}, \quad \text{in } B_{r_0}(x_0), \quad r_0 = dist(x_0, \partial\Omega) \tag{12}$$

$$|\nabla u(x)| \leq C|x - x_0|^{1+\frac{p}{1-p}}, \quad \text{in } B_{r_0}(x_0), \tag{13}$$

or
ii)

$$u(x) - u(x_0) = P_m(x) + R_m(x), \tag{14}$$

where $p_m \not\equiv 0$ is a homogeneous harmonic polynomial of precise degree m with $2 \leq m \leq 2 + [\frac{p}{1-p}]$ (here $[x]$ means the integer part of x) and

$$|R_m(x)| = O(|x - x_0|^{m+\delta}),$$
$$|\nabla R_m(x)| = O(|x - x_0|^{m+\delta-1}),$$

for some $\delta > 0$.
If $\frac{p}{1-p}$ is an integer then the same is true if in (12) and (13) we replace $\frac{p}{1-p}$ in the RHS of these estimates by every $\alpha < \frac{p}{1-p}$.

For the proof we need the following fundamental lemma of Caffarelli and Friedman, which is proved in [16] for the case $N = 3$, but as the authors indicated in the introduction, the proof is valid for any number of dimensions, also see [17].

Lemma 2. *([16], Lemma 3.1) Let γ be a positive non-integer, $\gamma \geq \gamma_0 > 0$, and let $v(x)$ be a function satisfying*

$$|\Delta v(x)| \leq C_\gamma |x|^\gamma \quad \text{in } B_1, \quad C_\gamma \geq 2^\gamma. \tag{15}$$

Then

$$v(x) = P(x) + \Gamma(x) \quad \text{in } B_1, \tag{16}$$

where $P(x)$ is a harmonic polynomial of degree (at most) $[\gamma] + 2$ and

$$|\Gamma(x)| \leq CC_\gamma \frac{\gamma^{N-2}}{\langle\gamma\rangle} |x|^{\gamma+2} \quad \text{in } B_1, \tag{17}$$

$$|\nabla\Gamma(x)| \leq CC_\gamma \frac{\gamma^N}{\langle\gamma\rangle} |x|^{\gamma+1} \quad \text{in } B_1, \tag{18}$$

where $\langle\gamma\rangle = \min\{\gamma - [\gamma], 1 + [\gamma] - \gamma\}$ and C is a constant depending only on γ_0, and on upper bounds on $|v(x)|$ and $|\nabla v(x)|$ for $x \in \partial B_1$.

Proof of Theorem 6. For the sake of simplicity take $x_0 = 0$ and $r_0 = 1$.
Assume that (ii) does not hold, then we show u satisfies (12) and (13).
We know that $u \in C^{2,p}$. Thus there exist $C_0 > 0$ such that $|\nabla u(x)| \le C_0|x|$ in B_1. Then from the equation (1) we get

$$|\Delta u(x)| \le C_0^p |x|^p \le C_1 |x|^{\gamma_1}, \quad in\ B_1,$$

where

$$\gamma_1 := p \quad and \quad C_1 := \max\{C_0^p, 2^{\gamma_1}\}.$$

Thus by Lemma 2

$$u(x) = P_1(x) + R_1(x), \quad in\ B_1, \tag{19}$$

where P_1 is a harmonic polynomial of degree $[\gamma_1] + 2 = 2$, and for $x \in B_1$

$$|R_1(x)| \le CC_1 \frac{\gamma_1^{N-2}}{\langle \gamma_1 \rangle} |x|^{\gamma_1+2}, \tag{20}$$

$$|\nabla R_1(x)| \le CC_1 \frac{\gamma_1^N}{\langle \gamma_1 \rangle} |x|^{\gamma_1+1}, \tag{21}$$

where C is a constant depending only on γ_1, and on upper bounds on $|u(x)|$ and $|\nabla u(x)|$ for $x \in \partial B_1$. Since P_1 is of degree 2, by our assumption and (19), we must have $P_1 \equiv u(0)$ or is of degree 1, i.e, $P_1(x) = u(0) + A.x$ for some $0 \ne A \in R^N$. But in later case from equation (1) we get

$$-\Delta R_1 = |A + \nabla R_1|^p, \quad in\ B_1,$$

that gives $\Delta R_1(0) \ne 0$, which is a contradiction because from (21) and the fact that $R_1 = u - P_1$ is a C^2 function we get $\Delta R_1(0) = 0$. So $P_1 \equiv u(0)$. It then follows that $u(x) - u(0) = R_1(x)$ in B_1, hence

$$|\Delta u(x)| = |\nabla R_1(x)|^p \le (CC_1 \frac{\gamma_1^N}{\langle \gamma_1 \rangle})^p |x|^{p(\gamma_1+1)} \le C_2 |x|^{\gamma_2}, \quad in\ B_1,$$

where

$$\gamma_2 := p(\gamma_1 + 1) > \gamma_1, \quad C_2 := \max\{(CC_1 \frac{\gamma_1^N}{\langle \gamma_1 \rangle})^p, 2^{\gamma_2}\}.$$

Replacing γ_2 (in the case it is an integer) with $\gamma_2 - \varepsilon_1$ to get a non-integer, where $0 \le \varepsilon_1 < 1$ is sufficiently small so that $\gamma_2 - \varepsilon_1 > \gamma_1$, and applying Lemma 2 once again, we conclude that,

$$u(x) = P_2(x) + R_2(x), \quad in\ B_1,$$

where P_2 is a harmonic polynomial of degree at most $2 + [\gamma_2 - \varepsilon_1]$ and

$$|R_2(x)| \le CC_2 \frac{(\gamma_2 - \varepsilon_1)^{N-2}}{\langle \gamma_2 - \varepsilon_1 \rangle} |x|^{2+\gamma_2-\varepsilon_1},$$

$$|\nabla R_2(x)| \le CC_2 \frac{(\gamma_2 - \varepsilon_1)^{N}}{\langle \gamma_2 - \varepsilon_1 \rangle} |x|^{1+\gamma_2-\varepsilon_1}.$$

Note that again we have

$$2 \le 2 + [\gamma_2 - \varepsilon_1] \le 2 + [p(p+1)] \le 2 + [\frac{p}{1-p}].$$

Hence by our assumption (that (ii) does not hold) and similar as the above we must have $P_2 \equiv u(0)$ and $u(x) - u(0) = R_2(x)$ in B_1. Repeating the above argument, we are able to find sequences C_j,β_j and a decreasing sequence ε_j such that

$$|u(x) - u(0)| = |R_j(x)| \le CC_j \frac{(\gamma_j - \varepsilon_{j-1})^{N-2}}{\langle \gamma_j - \varepsilon_{j-1} \rangle} |x|^{2+\gamma_j-\varepsilon_{j-1}}, \quad in\ B_1, \tag{22}$$

and

$$|\nabla u(x)| = |\nabla R_j(x)| \leq CC_j \frac{(\gamma_j - \varepsilon_{j-1})^N}{\langle \gamma_j - \varepsilon_{j-1} \rangle} |x|^{1+\gamma_j - \varepsilon_{j-1}}, \quad in\ B_1, \tag{23}$$

where $\gamma_1 = p, \varepsilon_0 = 0$ and for $j \geq 1$

$$\gamma_{j+1} = p(1 + \gamma_j - \varepsilon_{j-1}), \quad 0 \leq \varepsilon_j < \frac{1}{j}, \tag{24}$$

$$C_{j+1} = \max\{\Big(CC_j \frac{(\gamma_j - \varepsilon_{j-1})^N}{\langle \gamma_j - \varepsilon_{j-1} \rangle}\Big)^p, 2^{\gamma_j}\}. \tag{25}$$

From (24) it is not hard to see that γ_j is a bounded increasing sequence. Indeed, we have

$$\gamma_j < \frac{p}{1-p}, \quad for\ every\ j \geq 1. \tag{26}$$

To see this, by induction, we have $\gamma_1 = p < \frac{p}{1-p}$ and assuming $\gamma_j < \frac{p}{1-p}$ then from (24) we get

$$\gamma_{j+1} < p(1 + \frac{p}{1-p}) = \frac{p}{1-p},$$

that proves (26) . Now from (24) and the facts that ε_j is decreasing and $\gamma_1 < \gamma_2$, then by induction it is easy to see that γ_j is also an increasing sequence. Thus γ_j is convergent, say to γ. Taking limit from (24) as $j \to \infty$ we get

$$\gamma_j \to \gamma = \frac{p}{1-p} \quad as \quad j \to \infty. \tag{27}$$

Now suppose $\frac{p}{1-p}$ is a non-integer then from (27) for a $j_0 \in \mathcal{N}$, $\langle \gamma_j \rangle \geq \frac{1}{2} \langle \frac{p}{1-p} \rangle$ for every $j \geq j_0$. Thus from (25) we get

$$C_{j+1} \leq aC_j^p + 2^\gamma, \quad j \geq j_0,$$

for a constants a independent of j. Then $p < 1$ implies that C_j is a bounded sequence. Now, letting $j \to \infty$ in (22) and (23) we arrive at

$$|u(x) - u(0)| \leq C|x|^{2+\frac{p}{1-p}}, \quad in\ B_1,$$

and

$$|\nabla u(x)| \leq C|x|^{1+\frac{p}{1-p}}, \quad in\ B_1,$$

where $C = C(p, u)$ is a constant independent of x_0.
Now consider the case when $\frac{p}{1-p}$ is an integer and take $\alpha < \frac{p}{1-p}$. Find a j such that $\alpha < \gamma_i < \frac{p}{1-p} = \gamma$ then (22) and (23) give us the desired result. □

Remark 2. *Note that in the above we used the fact that if P_m is a harmonic polynomial of degree m then we can write*

$$P_m(x) = \sum_{k=0}^{m} Q_k(x - x_0), \quad Q_k \ \textit{is a homogeneous polynomials of degree} \ \ k.$$

Also note that if u is a solution of (1) the u is C^∞ at $x \in \Omega$ with $\nabla u(x) \neq 0$ or $x \in K_u^0$. Thus we need only to check the regularity for $x \in K_u \setminus K_u^0$. In this regard we have

Corollary 2. *If x_0 is an extremum point of u or $x_0 \in \partial K_u^0$, then u satisfies the estimates in part (i) of Theorem 6 at x_0.*

Proof. If $x_0 \in \Omega$ is an extremum point of u then $u(x) - u(x_0)$ does not change sign in a neighborhood $B_r(x_0)$ of x_0 for $r < r_0$ sufficiently small. Now if u satisfies part (ii) of Theorem 6, then from (14) we have $P_m(x_0) = 0$, $P_m(x)$ does not change sign in $B_r(x_0)$ and $\Delta P_m = 0$ in $B_r(x_0)$ thus by the maximum principle $P_m(x) \equiv 0$, a contradiction.
Now assume $x_0 \in \partial K_u^0$ and u satisfies part (ii) of Theorem 6 at x_0. Then we have $u(x) - u(x_0) = P_m(x) + R_m(x)$ in a small neighborhood $B_r(x_0)$ of x_0, where $P_m \not\equiv 0$ is a homogeneous harmonic polynomial of precise degree m with $2 \leq m \leq 2 + [\frac{p}{1-p}]$ and $|R_m(x)| = O(|x - x_0|^{m+\delta})$. Since $x_0 \in \partial K_u^0$ then we have $u(x) - u(x_0) = \nabla u(x) = 0$ for $x \in D := B_{r'}(x) \cap K_u^0$ for $r' < r$ small. But then we also have $P_m(x) = \nabla P_m(x) = 0$ for $x \in D$ means that the dimension of the singular set of P_m is N, contradicts with the fact that the singular set of a harmonic homogeneous polynomial is of dimension less than or equal to $N - 2$ when $N \geq 3$ ([37], page 5) and is isolated when $N = 2$ (see [36], Lemma 2.4.1). □

3 Asymptotics of solutions as $p \nearrow 1$ and $p \searrow 0$

Theorem 7. *Let $p_m \nearrow 1$ and suppose $u_m > 0$ is a classical solution of (1). Then there is some $0 < \alpha < 1$ such that $u_m \to 0$ in $C^{2,\alpha}(\overline{\Omega})$. Moreover, the following bound on the rate of convergence to zero holds*

$$\left(\sup_{\Omega} |\nabla u_m|\right)^{1-p_m} \to 0.$$

Proof. First note that by (10) Theorem 1 we have $\|u_m\|_{L^\infty} \leq \alpha_{N,p_m} diam(\Omega)^{q_m}$ which gives

$$\|u_m\|_{L^\infty} \leq \frac{1-p_m}{2-p_m}\left(N + \frac{p_m}{1-p_m}\right)^{\frac{-1}{1-p_m}} diam(\Omega)^{\frac{2-p_m}{1-p_m}} \to 0,$$

and hence $u_m \to 0$ uniformly in Ω. We now show the gradients are bounded. Let $T_m := \sup_\Omega |\nabla u_m| = |\nabla u_m(x_m)|$ and set

$$v_m(x) = \frac{u_m(x_m + x)}{T_m}, \qquad x \in \Omega_m := \{x \in \mathbb{R}^N : x + x_m \in \Omega\}.$$

Then $|\nabla v_m| \leq 1$ in Ω_m and $|\nabla v_m(0)| = 1$. Now suppose $T_m \to \infty$. Then we have

$$-\Delta v_m(x) = \frac{1}{T_m^{1-p_m}} |\nabla v_m(x)|^{p_m} \quad \Omega_m, \qquad v_m = 0 \quad \partial\Omega_m.$$

We need to consider two cases:
Case 1. $T_m^{1-p_m} \to \infty$,
In this case we see that $-\Delta v_m \to 0$ uniformly in Ω_m and hence $v_m \to 0$ in $C^{1,\alpha}$ and hence $|\nabla v_m(0)| \to 0$ (a contradiction).

Case 2. Here we assume that $\gamma_m := T_m^{1-p_m} \to \gamma \in [1, \infty)$. Then in this case there is some $v \in C^{1,\alpha}$ such that $v_m \to v$ in $C^{1,\alpha}$ and v solves $-\Delta v = \frac{1}{\gamma}|\nabla v|^1$ in Ω_∞ with $v = 0$ on $\partial\Omega_\infty$. But the only solution of this is $v = 0$ and hence we'd have $v_m \to 0$ in $C^{1,\alpha}$ which again would contradict the fact that $|\nabla v_m(0)| = 1$. So from this we see there is some $C > 0$ such that $\sup_\Omega |\nabla u_m| \leq C$. We can then apply elliptic regularity to see that u_m is bounded in $C^{1,\alpha}$. By passing to a subsequence we can assume there is some $u \in C^{1,\alpha}$ such that $u_m \to u$ in $C^{1,\alpha}$. But note u would satisfy $-\Delta u = |\nabla u|^1$ in Ω with $u = 0$ on $\partial\Omega$; but $u = 0$ is the only classical solution of this. So we have convergence of u_m to 0 in $C^{1,\alpha}$. Since $u_m \to 0$ in $C^{1,\alpha}$ we have $|\nabla u_m| \to 0$ in $C^{0,\alpha}$ and since $p_m \to 1$ we have $|\nabla u_m|^{p_m} \to 0$ in some slightly larger $C^{0,\beta}$ space. From this we get the convergence to 0 in $C^{2,\beta}$.

To prove the moreover part, as before let $T_m := \sup_\Omega |\nabla u_m| = |\nabla u_m(x_m)|$ and we have $T_m \to 0$. Set $v_m(x) := \frac{u_m(x_m+x)}{T_m}$ for $x \in \Omega_m := \{x \in \mathbb{R}^N : x_m + x \in \Omega\}$. Then we have (as above) $|\nabla v_m| \leq 1$ in Ω_m and $|\nabla v_m(0)| = 1$. Also we have

$$-\Delta v_m(x) = \frac{1}{\gamma_m} |\nabla v_m(x)|^{p_m} \quad \Omega_m, \qquad v_m = 0 \quad \partial\Omega_m,$$

where $\gamma_m := T_m^{1-p_m}$. Note that since $0 < T_m < 1$ we have $\gamma_m \in (0, 1]$.
Now suppose we have $\gamma_m \to \gamma \in (0, 1]$. In this case we can find some v and Ω_∞ such that $-\Delta v = \frac{|\nabla v|^1}{\gamma}$ in Ω_∞ with $v = 0$ on the boundary. (As above we would then have $v = 0$; and since we have convergence of the gradient we'd have $|\nabla v_m(0)| \to 0$ a contradiction). Hence we must have $\gamma_m \to 0$. □

We are now ready to determine the asymptotics of the solutions as $p \searrow 0$. Two lemmas are needed.

Lemma 3. *Let $\frac{1}{2} > p_m \searrow 0$. Then for all $0 < \alpha < 1$ there is some $C_\alpha > 0$ such that for any positive classical solution u_m of (1), with $p = p_m$, we have $\|u_m\|_{C^{1,\alpha}} \leq C_\alpha$.*

Proof. First note that u_m is bounded in L^∞ from the above results. We now obtain L^∞ bounds on the gradients. Let $T_m := \sup_\Omega |\nabla u_m| = |\nabla u_m(x_m)|$ and suppose $T_m \to \infty$. Set

$$v_m(x) := \frac{u_m(x_m + x)}{T_m}, \qquad x \in \Omega_m := \{x \in \mathbb{R}^N : x_m + x \in \Omega\}.$$

Then note that $\sup_\Omega |v_m| \to 0$. Also note that $|\nabla v_m| \leq 1$ and $|\nabla v_m(0)| = 1$. Also note that

$$-\Delta v_m(x) = \frac{1}{T_m^{1-p_m}} |\nabla v_m(x)|^{p_m} \quad \Omega_m,$$

with $v_m = 0$ on $\partial\Omega_m$. By elliptic regularity we have $v_m \to 0$ in some $C^{1,\alpha}(\Omega_m)$ From this we'd have $|\nabla v_m(0)| \to 0$; a contradiction. Set $f_m(x) := |\nabla u_m(x)|^{p_m}$ and note that f_m is bounded in $L^\infty(\Omega)$. Hence from this we have u_m is bounded in $C^{1,\alpha}$ for all $0 < \alpha < 1$ after considering L^p elliptic regularity and the Sobolev imbedding theorem. □

To go any further on the asymptotics as $p \searrow 0$ we need some nondegeneracy results. We already have shown that for positive solutions of equation (1) in every point $x_0 \in \Omega$ we have the $q := \frac{2-p}{1-p}$ growth estimate. The next result excludes the possibility that $u(x_0) - u(x)$ may decay faster than q at every non-interior point of singular set.

Lemma 4. *(Nondegeneracy) Let $u \not\equiv 0$ be a positive solution of (1). Then for every $x_0 \notin K_u^0$ we have*

$$u(x_0) \geq m_r + \alpha_{N,p}\, r^q, \quad 0 < r < dist(x_0, \partial\Omega), \tag{28}$$

where

$$m_r = \min_{\partial B_r(x_0)} u(x).$$

Also, for every $x \in \Omega$ we have

$$u(x) \leq M_r + \alpha_{N,p}\, r^q, \quad 0 < r < dist(x, \partial\Omega), \tag{29}$$

where

$$M_r = \max_{\partial B_r(x_0)} u(x).$$

In particular from (28) we get

$$\sup_{B_r(x_0)} |u(x) - u(x_0)| \geq \alpha_{N,p}\, r^q, \quad \textit{for any } x_0 \in \Omega,$$

and

$$\sup_{B_r(x_0)} |\nabla u(x)| \geq \alpha_{N,p}\, r^{\frac{1}{1-p}}, \quad \textit{for any } x_0 \in \Omega.$$

Proof. To prove (28) it is sufficient to consider the case $\nabla u(x_0) \neq 0$, then by the continuity of u we get it for all $x_0 \notin K_u^0$. For simplicity set $\alpha := \alpha_{N,p}$ and take

$$v(x) = \alpha(r^q - |x - x_0|^q), \quad x \in B_r(x_0).$$

Take an arbitrary $s \in (0, 1)$ and set $w = u - sv$. We show that w takes its minimum at $\partial B_r(x_0)$. If this is not the case and w takes its minimum at a point $z \in B_r(x_0)$ then we must have $\nabla w(z) = 0$ and $\Delta w(z) \geq 0$, that give, $\nabla u(z) = s\nabla v(z)$ and

$$0 \leq \Delta w(z) = -|\nabla u(z)|^p + s|\nabla v(z)|^p = -s^p \; |\nabla v(z)|^p + s|\nabla v(z)|^p = (s - s^p)|\nabla v(z)|^p.$$

But, since $s < 1$ and $p < 1$ we have $s - s^p < 0$, hence we must have $\nabla v(z) = 0$ and so $\nabla u(z) = 0$, that by the definition of v we get $z = x_0$ implies $\nabla u(x_0) = 0$, that contradicts by our assumption. So, using $v|_{\partial B_r(x_0)} = 0$ we get

$$w(x) \geq \min_{B_r(x_0)} w = \min_{\partial B_r(x_0)} w = m_r, \quad x \in B_r(x_0),$$

or equivalently

$$u(x) \geq m_r + s(\alpha r^q - \alpha|x - x_0|^q), \quad x \in B_r(x_0).$$

And since $s \in (0, 1)$ was arbitrary we get

$$u(x) \geq m_r + \alpha r^q - \alpha|x - x_0|^q, \quad x \in B_r(x_0).$$

Taking $x = x_0$ in the inequality above we get (28). To prove (29), it suffices to note that the functions $\underline{u}(y) := u(y) - M_r$ and $\bar{u} \equiv \max_\Omega u - M_r$ are sub and supersolutions of the equation $-\Delta v = |\nabla v|^p$ on $B_r(x)$ with Dirichlete BC., respectively. Hence there exists a solution v of this equation so that $u(y) - M_r \leq v(y)$ for all $y \in B_r(x)$. But we know that the function $\alpha_{N,p}(r^q - |y - x|^q)$ is the largest pointwise solution to this equation. Thus we get $u(y) \leq M_r + v(y) \leq M_r + \alpha_{N,p}(r^q - |y - x|^q)$ for $y \in B_r(x)$. Taking $y = x$ in the later inequality gives the desired result.
Now take an $x_r \in \partial B_r(x_0)$ such that $u(x_r) = m_r$ then from (28) we have

$$\sup_{B_r(x_0)} |u(x) - u(x_0)| \geq u(x_0) - u(x_r) \geq \alpha_{N,p} \; r^q, \quad \text{for any } x_0 \in \Omega,$$

and

$$\alpha_{N,p} \; r^q \leq u(x_0) - u(x_r) \leq |x_0 - x_r| \sup_{B_r(x_0)} |\nabla u(x)| = r \sup_{B_r(x_0)} |\nabla u(x)|,$$

that gives

$$\sup_{B_r(x_0)} |\nabla u(x)| \geq \alpha_{N,p} \; r^{\frac{1}{1-p}}.$$

□

For the next result we need the following $W^{2,q}$ estimate from Gilbarg-Trudinger [33].

Theorem 8. *Suppose $1 < q < \infty$, Ω is a bounded domain in $\mathbb{R}^N$ and $K \subset\subset \Omega$. Suppose $u \in L^q(\Omega)$ satisfies $\Delta u = f$ in Ω in the sense of distributions, where $f \in L^q(\Omega)$. Then $u \in W^{2,q}_{loc}(\Omega)$ and there is some $C = C(q, N, K, \Omega)$ such that*

$$||u||_{W^{2,q}(K)} \leq C\Big(||u||_{L^q(\Omega)} + ||f||_{L^q(\Omega)}\Big)$$

.

Theorem 9. *Let $p_m \to 0$ and $u_m := u_{p_m} > 0$ denote the positive solution such that $u_m \to u_0$ in $C^{1,\alpha}$ for some $0 < \alpha < 1$. If $B_r(x_0) \subset \Omega$ then we have*

$$|\nabla u_0| = 0 \;\; on \; B_r(x_0) \quad \Rightarrow \quad |\nabla u_m| = 0 \;\; on \; B_{\frac{r}{2}}(x_0), \quad m \geq m_0, \tag{30}$$

for sufficiently large m_0. This in particular gives

$$K^0_{u_0} \subseteq \limsup_{m\to\infty} K^0_{u_m} := \bigcap_{m\geq 1} \bigcup_{j\geq m} K^0_{u_j}.$$

Moreover, u_0 solves the problem

$$\begin{cases} -\Delta u_0 &= \chi_{(\Omega\setminus K_{u_0})} \quad a.e. \quad in\ \Omega, \\ u_0 &= 0 \qquad\qquad\qquad on\ \partial\Omega, \end{cases} \tag{31}$$

.

Proof. Suppose (30) fails, then there exists $m_j \to \infty$ such that $|\nabla u_{m_i}(x_j)| > 0$ for some $x_j \in B_{\frac{r}{2}}(x_0)$. Applying the nondegeneracy of the gradient at x_j and using the fact that $B_{\frac{r}{4}}(x_j) \subset B_{\frac{3r}{4}}(x_0)$ we obtain

$$\sup_{B_{\frac{3r}{4}}(x_0)} |\nabla u_{m_j}| \geq \alpha_{N,p}\ (\frac{3}{4})^{\frac{1}{1-p_{m_j}}}\ r^{\frac{1}{1-p_{m_j}}}.$$

Then passing to the limit, we get

$$\sup_{B_{\frac{3r}{4}}(x_0)} |\nabla u_0| \geq \frac{3r}{8N},$$

contradicting our assumption. To show that u_0 solves (31), note that by Lemma 3, $|\nabla u_m|$ is uniformly bounded from above, then by Theorem 8, we may assume that $\{u_m\}$ is uniformly bounded in $W^{2,q}(\Omega')$ ($q > N$) for any $\Omega' \subset\subset \Omega$ and therefore $u_0 \in W^{2,q}_{loc}(\Omega)$. Suppose $x \in K^0_{u_0}$ then for some $r > 0$ we have $|\nabla u_0| = 0$ on $B_r(x_0)$, thus from the first part $|\nabla u_m| = 0$ on $B_{\frac{r}{2}}(x_0)$, $m \geq m_0$ for sufficiently large m_0. Consequently,

$$-\Delta u_m = 0 \quad in \quad B_{\frac{r}{2}}(x_0),\ m \geq m_0,$$

which implies $\Delta u_0 = 0$ in $B_{\frac{r}{2}}(x_0)$. Hence,

$$\Delta u_0 = 0 \quad in \quad K^0_{u_0}.$$

If $x_0 \notin K_{u_0}$ then for some $\delta > 0$ we have $|\nabla u_0| > c_0 > 0$ in $B_\delta(x_0)$ for some constant c_0. In particular, since $u_m \to u_0$ in $C^{1,\alpha}$, we must have $|\nabla u_m| > \frac{c_0}{2}$ in $B_\delta(x_0)$ for sufficiently large m. On the other hand, by Lemma 3, $|\nabla u_m|$ is uniformly bounded from above, hence $|\nabla u_m|^{p_m} \to 1$ uniformly in $B_\delta(x_0)$. Therefore, $-\Delta u_0 = 1$ in $B_\delta(x_0)$, implies

$$-\Delta u_0 = 1, \quad in \quad \Omega \setminus K_{u_0}.$$

Now the fact that ∂K_{u_0} has a Lebesgue measure zero completes the proof. □

4 Some variations of (1)

4.1 Extension 1.

Consider positive solutions of the PDE

$$\begin{cases} -\Delta u &= |\nabla u|^p + f(x,u,\nabla u) \text{ in } \Omega, \\ u &= 0 \qquad\qquad\qquad\quad \text{on } \partial\Omega, \end{cases} \tag{32}$$

where $0 < p < 1$, Ω is a bounded smooth domain in $\mathbb{R}^N$. Also we assume $f(x,s,\xi)$ continuous in all of its variables for $(x,s,\xi) \in \bar{\Omega} \times \mathbb{R} \times \mathbb{R}^N$ satisfies

$$0 \leq f(x,s,\xi) \leq c(|s|)(1 + |\xi|^2), \tag{33}$$

for some increasing function $c(r) \geq 1$. Note a special case of (32) is given by the convex-concave problem

$$(Q_\lambda) \qquad \begin{cases} -\Delta u &= |\nabla u|^p + \lambda |\nabla u|^s \text{ in } \Omega, \\ u &= 0 \qquad\qquad\quad \text{ on } \partial\Omega, \end{cases}$$

where $0 < p < 1 < s < 2$. This is a variation of a similar problem $-\Delta u = u^p + \lambda u^s$ first studied by [4]. They used mostly calculus of variations to prove their results. Unfortunately, this tool is not available for our problem.

Theorem 10. *Suppose p, f satisfy the above assumptions, $f(x, s, \xi) \geq 0$ and $f(x, s, 0) = 0$. Then (32) has a continuum of positive classical solutions. In particular, for every $x_0 \in \Omega$ there exists a positive solution u with $||u||_{L^\infty(\Omega)} = u(x_0) = \alpha_{N,p}\ r_0^q$, where $\alpha_{N,p}, r_0, q$ are defined in (4).*

Proof. The proof is completely similar to the proof of Theorem 2. Let $x_0 \in \Omega$ and set

$$\underline{u}(x) = \alpha_{N,p}\left(r_0^q - |x - x_0|^q\right).$$

Then we have $\Delta\underline{u} + |\nabla\underline{u}|^p + f(x, \underline{u}, \nabla\underline{u}) = f(x, \underline{u}, \nabla\underline{u}) \geq 0$ in Ω and $\underline{u}|_{\partial\Omega} \leq 0$. Hence, $\underline{u}$, that belongs to $W^{2,\infty}(\Omega)$, is a sub-solution of (32). Also by the assumption that $f(x, s, 0) = 0$ we easily see that $\bar{u} \equiv \alpha_{N,p}\ r_0^{\frac{2-p}{1-p}}$ is a supersolution. On the other hand we have $\underline{u} \leq \bar{u}$, thus by Theorem A there exists a solution u of (1) in $W^{2,s}(\Omega)$ for every $s > N$, with $\underline{u} \leq u \leq \bar{u}$. Obviously u is not the trivial solution. Indeed, u is a solution of (1) and $u|_{\partial\Omega} = 0$, we get $u \geq 0$ in Ω. Also, we have $\underline{u}(x_0) \equiv \bar{u}$, implies that $||u||_{L^\infty(\Omega)} = u(x_0) = \alpha_{N,p}\ r_0^{\frac{2-p}{1-p}}$. This implies that we have a continuum of positive classical solutions. Note that the growth assumption (33) is needed in Theorem A. □

Theorem 11. *Suppose $0 < p < 1 < s < 2$ and Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary. Define $M := \sup\{\|u\|_{L^\infty} : u$ is a classical solution of (1) $\} \in (0, \infty)$. For all $\lambda \geq 0$ and $t \in (0, M]$ (Q_λ) has a nonnegative classical u with $\|u\|_{L^\infty} = t$.*

Proof. Let $\underline{u}$ be a solution of $-\Delta u = |\nabla u|^p$ with $\|\underline{u}\|_\infty = t$ for some $t \in (0, M)$ (such a solution exists noticing the fact that if w is a solution and $0 < \alpha < 1$ then the functions αw and $\alpha\|w\|_{L^\infty}$ are sub and supersolutions, respectively). Then $\underline{u}$ is a subsoln of (Q_λ) with supersoln $\overline{u} := t$. Hence there is a soln of (Q_λ) whose supremum norm is t. □

We now obtain uniform estimates on solutions of (Q_λ) for small λ. We first recall

$$d_\Omega = diam\Omega, \ \ R_x = \max_{y \in \partial\Omega} |x - y| \ \ and \ \ \alpha := \frac{1-p}{2-p}\Big(N + \frac{p}{1-p}\Big)^{\frac{-1}{1-p}}, \ \ q := \frac{2-p}{1-p}.$$

Lemma 5. *Assume that*

$$\lambda\Big(\alpha q\ d_\Omega^{q-1}\Big)^{s-p} < \gamma := \max_{0<t<1} t^{s-1} - t^{s-p}, \tag{34}$$

Let $u_\lambda \not\equiv 0$ be a positive solution of (Q_λ). If $x \in \Omega$ is not an interior point of K_{u_λ} then we have

$$u_\lambda(y) \leq \beta\ \alpha\Big(R_x^q - |y - x|^q\Big), \ \ y \in \Omega, \ \ where \ \ \beta := (\frac{s-p}{s-1})^{\frac{1}{1-p}}. \tag{35}$$

In particular we have

$$||u_\lambda||_{L^\infty(\Omega)} \leq \beta\alpha\ d_\Omega^q. \tag{36}$$

Note that the RHS of (36) is independent of u_λ. Also, γ is attained at the point $t = (\frac{s-1}{s-p})^{\frac{1}{1-p}}$.

Proof. To prove (35) first fix an $x \in \Omega$ with $\nabla u_\lambda(x) \neq 0$. Set

$$w(y) := \alpha\Big(R_x^q - |y-x|^q\Big) \quad y \in \Omega.$$

Then we have

$$-\Delta w(y) = |\nabla w(y)|^p, \quad y \in \Omega, \quad w(y)|_{\partial\Omega} \geq 0.$$

We want to show that $u(y) \leq \beta w(y)$ in Ω. If this is not the case then there exists $y_0 \in \Omega$ such that $u(y_0) > \beta w(y_0)$. Now set $v = u(y) - \beta w(y)$. Then we have $v(y_0) > 0$ and $v|_{\partial\Omega} \leq 0$. Hence, v must take its maximum at a point in Ω, say z. Thus we must have $\nabla v(z) = 0$ and $\Delta v(z) \leq 0$, that give, $\nabla u(z) = \beta \nabla w(z)$ and

$$\begin{aligned} 0 \quad \geq \quad \Delta v(z) \;&=\; -|\nabla u(z)|^p - \lambda |\nabla u|^s + \beta |\nabla w(z)|^p \\ &= \; -\beta^p\, |\nabla w(z)|^p - \lambda \beta^s |\nabla w(z)|^s + \beta |\nabla w(z)|^p \\ &= \; (\beta - \beta^p)|\nabla w(z)|^p - \lambda\beta^s |\nabla w(z)|^s. \end{aligned}$$

Suppose $\nabla w(z) \neq 0$. From the above we get

$$\lambda |\nabla w(z)|^{s-p} \geq (\frac{1}{\beta})^{s-1} - (\frac{1}{\beta})^{s-p} = \gamma.$$

But by the definition of w we have $|\nabla w(z)| = q\alpha_{N,p}|z-x|^{q-1} \leq q\alpha_{N,p} d_\Omega^{q-1}$, hence from the above we must have

$$\lambda\Big(\alpha q\; d_\Omega^{q-1}\Big)^{s-p} \geq \gamma,$$

which contradicts with the assumption (34). Therefore, we must have $\nabla w(z) = 0$ and so $\nabla u_\lambda(z) = 0$, that by the definition of w we get $z = x$ implies $\nabla u_\lambda(x) = 0$, a contradiction. Hence, $u_\lambda(y) \leq \beta w(y)$ in Ω that proves the inequality (35). To prove (35) in the case $\nabla u_\lambda(x) = 0$ but x is not an interior point of K_{u_λ}, it suffices to take a sequence x_n in Ω such that $\nabla u_\lambda(x_n) \neq 0$ and $x_n \to x$, then write (36) for x_n and let $n \to \infty$. To prove (36), note that $\Omega \setminus K_{u_\lambda}^0 \neq \emptyset$ and apply (35). □

We now consider a more detailed study of $(Q)_\lambda$. We begin with a lemma.

Lemma 6. *Suppose $0 < p < 1 < s < 2$. Let $\lambda_m \uparrow \lambda_0$ and $u_m \in H_0^1(\Omega)$ be an increasing bounded (in L^∞) sequence of positive solutions of $(Q)_{\lambda_m}$. Then there exists a $u_0 \in H_0^1(\Omega)$ such that $u_m \to u$ in $H_0^1(\Omega)$ and u_0 is a weak solution of $(Q)_{\lambda_0}$. If $s < \frac{N+2}{N}$, then u_0 is a classical solution.*

Proof. Let, for every m, $||u_m||_\infty \leq M$. Multiply (Q_λ) by u_m and integrate we get (using $p < s$)

$$\int_\Omega |\nabla u_m|^2 = \int_\Omega u_m(|\nabla u_m|^p + \lambda_m |\nabla u_m|^s) \leq M \int_\Omega |\nabla u_m|^p + M\lambda_0 \int_\Omega |\nabla u_m|^s \leq C_1 + C_2 \int_\Omega |\nabla u_m|^s,$$

where C_1, C_2 are constants independent of u_m. Hence, using $s < 2$ we get

$$\int_\Omega |\nabla u_m|^2 \leq C,$$

where C is independent of m. So $\{u_m\}$ is bounded in $H_0^1(\Omega)$. By the reflexivity of $H_0^1(\Omega)$ there exists a $u_0 \in H_0^1(\Omega)$ such that (up to a subsequence)

$$\begin{aligned} u_m \;&\rightharpoonup\; u_0 \;\; \textit{weakly in} \;\; H_0^1(\Omega), \\ u_m \;&\to\; u_0 \;\; \textit{in} \;\; L^t(\Omega), \;\; t \in [1, \frac{2N}{N-2}), \\ u_m \;&\to\; u_0 \;\; \textit{a.e.}\; x \in \Omega. \end{aligned}$$

Indeed, using the assumption that u_m is increasing (hence $u_m \le u_0$), we can show that $u_m \to u_0$ strongly in $H_0^1(\Omega)$. To see this, we write

$$\int_\Omega |\nabla(u_m - u_0)|^2 = \int_\Omega \nabla u_0 \cdot \nabla(u_m - u_0) - \int_\Omega \nabla u_0 \cdot \nabla u_m + \int_\Omega |\nabla u_m|^2.$$

Now, from the weak convergence of u_m to u_0, we have

$$\int_\Omega \nabla u_0 \cdot \nabla(u_m - u_0) \to 0$$

and also we have

$$\int_\Omega |\nabla u_m|^2 = \int_\Omega u_m(-\Delta u_m) \le \int_\Omega u_0(-\Delta u_m) = \int_\Omega \nabla u_0 \cdot \nabla u_m,$$

hence

$$\int_\Omega |\nabla(u_m - u_0)|^2 \to 0.$$

Thus we proved $u_m \to u_0$ strongly in $H_0^1(\Omega)$. This also implies, (passing to a subsequence),

$$u_m(x) \to u_0(x) \;\; and \;\; \nabla u_m(x) \to \nabla u_0(x), \quad a.e.\ x \in \Omega.$$

Note also that, since $\nabla u_m \to \nabla u_0$ in $L^2(\Omega)$ we can also assume there exists a function $h \in L^2(\Omega)$ such that

$$|\nabla u_m| \le h(x), \quad a.e.\ x \in \Omega. \tag{37}$$

Now we show that u_0 is a weak solution of the equation

$$-\Delta u = |\nabla u|^p + \lambda_0 |\nabla u|^s, \qquad u = 0 \text{ on } \partial\Omega,$$

or equivalently

$$\int_\Omega \nabla u_0 \cdot \nabla \phi = \int_\Omega |\nabla u_0|^p \phi + \lambda_0 \int_\Omega |\nabla u_0|^s \phi, \quad \phi \in C_0^\infty(\Omega).$$

To show the above, take a $\phi \in C_0^\infty(\Omega)$ and note that since u_m is a solution of (Q_{λ_m}) then we have

$$\int_\Omega \nabla u_m \cdot \nabla \phi = \int_\Omega |\nabla u_m|^p \phi + \lambda_m \int_\Omega |\nabla u_m|^s \phi, \quad \phi \in C_0^\infty(\Omega). \tag{38}$$

From the weak convergence of u_m to u_0 we get

$$\int_\Omega \nabla u_m \cdot \nabla \phi \to \int_\Omega \nabla u_0 \cdot \nabla \phi.$$

Also, using (37), $p, s < 2$ and the Lebesgue dominated convergence theorem we have

$$\int_\Omega |\nabla u_m|^p \phi \to \int_\Omega |\nabla u_0|^p \phi \;\; and \;\; \int_\Omega |\nabla u_m|^s \phi \to \int_\Omega |\nabla u_0|^s \phi.$$

Hence, letting $m \to \infty$ in (38) we get

$$\int_\Omega \nabla u_0 \cdot \nabla \phi = \int_\Omega |\nabla u_0|^p \phi + \lambda_0 \int_\Omega |\nabla u_0|^s \phi, \quad \phi \in C_0^\infty(\Omega),$$

which means that $u_0 \in H_0^1(\Omega)$ is a weak solution of $-\Delta u = |\nabla u|^p + \lambda_0 |\nabla u|^s$.
Now we show that if $s < \frac{N+2}{N}$, then u_0 is a classical solution. Take $f(x) := |\nabla u|^p + \lambda_0 |\nabla u|^s$ and $p_1 = 2$. Since $\nabla u_0 \in L^{p_1}(\Omega)$ and Ω is bounded then $f \in L^{\frac{p_1}{s}}$, hence from elliptic regularity we get $u_0 \in W^{2,\frac{p_1}{s}}$ that gives $\nabla u_0 \in W^{1,\frac{p_1}{s}}$. Then by the Rellich-Kondrachov theorem, $\nabla u_0 \in L^q$ for every $q < \frac{N\frac{p_1}{s}}{N - \frac{p_1}{s}} = \frac{Np_1}{sN - p_1} := p_2$.

Note that by the assumption that $s < \frac{N+2}{N}$ we have $p_1 < p_2$. Now if $p_2 < sN$ then repeating the argument above we get $\nabla u_0 \in L^q$ for every $q < \frac{Np_2}{sN-p_2} := p_3 > p_2$. Continuing the procedure, if in each step we have $p_k < sN$, then we obtain an increasing sequence p_k such that $\nabla u_0 \in L^q$ for every $q < \frac{Np_k}{sN-p_k} = p_{k+1}$. Hence, p_k is convergent, say to γ. From the recursive formula we get $\gamma = (s-1)N$, but we must have $\gamma > p_1 = 2$ implies that $s > \frac{N+2}{N}$ that contradicts with our assumption. This implies that there exists a k_0 so that $p_{k_0} \geq sN$. Then, by the above we get $\nabla u_0 \in L^q$ for every $q < sN$. Hence $f(x) \in L^t$ for every $t < N$ so from elliptic regularity we get $u_0 \in W^{2,t}$ for $t < N$ that gives $\nabla u_0 \in W^{1,t} \subset L^{\frac{tN}{N-t}}$ for $t < N$. Taking t sufficiently close to N we see that $f(x) \in L^q$ for $q > N$, therefore, $u \in C^{2,\alpha}$. □

Remark 3. *Note that by the above lemma, if $u \in H_0^1(\Omega)$ is a solution of*

$$-\Delta u = |\nabla u|^p + \lambda |\nabla u|^s, \qquad u = 0 \text{ on } \partial\Omega,$$

in the weak sense, then it is a classical solution when $p < 1 < s < \frac{N+2}{N}$ (it suffices to take $\lambda_m = \lambda$ for every m). Also note that, if a weak solution $u \in W^{1,q}$ $(q > s)$ then completely similar as above we can prove that u is a classical solution when $1 < s < \frac{N+q}{N}$. Hence if we want to remove the constraint $1 < s < \frac{N+2}{N}$ in the statement of the above Lemma we need to show that $u_0 \in W^{1,q}$ for some $q > N$.
Also note the term $|\nabla u|^p$ used only for the existence of solution while the regularity depends only to the larger exponent s. This also can be done for the problem when the term $|\nabla u|^p$ does not appear, i.e., for the problem

$$-\Delta u = |\nabla u|^s, \qquad u = 0 \text{ on } \partial\Omega.$$

We know that the problem above does not have a nontrivial classical solution when $s > 1$, so by the above lemma and remark, if $1 < s < \frac{N+q}{N}$ $(q > s)$ then this problem also does not have a singular solution $u \in W^{1,q}(\Omega)$ (otherwise it must be $C^{2,\alpha}$ a contradiction). Recall that the function

$$u(x) = C(1 - |x|^s), \quad t = \frac{s-2}{s-1}, \quad C = \frac{N-2+t}{t},$$

for $\frac{N}{N-1} < s < 2$ is a singular solution of

$$-\Delta u = |\nabla u|^s \text{ in } B, \qquad u = 0 \text{ on } \partial B,$$

but note that $u \notin W^{1,q}(B(0,1))$ for $(q > s)$. Indeed, $u \in W_0^{1,s}(B)$. The following simple observation shows that if the latter equation has a singular solution $u \in W_0^{1,s}(\Omega)$ in a domain Ω then we must have $\frac{N}{N-1} < s < 2$. Thus for $1 < s < \frac{N}{N-1}$, there is no singular solution $u \in W_0^{1,s}(\Omega)$.

Lemma 7. *Suppose Ω a bounded domain in $\mathbb{R}^N$ with smooth boundary and $1 < s < \frac{N}{N-1}$. Then the problem*

$$-\Delta u = \lambda |\nabla u|^s \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega, \tag{39}$$

does not have any singular solution $u \in W_0^{1,s}(\Omega)$. The same proof shows that there is no distributional singular solution which is sufficiently regular near $\partial\Omega$.

Proof. Assume $1 < s < \frac{N}{N-1}$ and the problem has a singular solution $u \in W_0^{1,s}(\Omega)$. Then $\nabla u \in L^s$ or $|\nabla u|^s \in L^1$ and from elliptic regularity we get $u \in W^{2,1}$ that gives $\nabla u_0 \in W^{1,1}$. Then by the Rellich-Kondrachov theorem, $\nabla u \in L^q$ for every $q < \frac{N}{N-1} := p_2 > p_1 := s$. Repeating the argument above we get $\nabla u \in L^q$ for every $q < \frac{Np_2}{sN-p_2} := p_3 > p_2$. Continuing the procedure we obtain an increasing sequence p_k with $p_1 = s$ and $p_{k+1} = \frac{Np_k}{sN-p_k}$ so that $\nabla u \in L^q$ for every $q < p_k$. If p_k is bounded and tends to a $\gamma > 0$ then we must have $\gamma = (s-1)N$, but we need $\gamma > p_1 = s$ gives $s > \frac{N}{N-1}$, a contradiction. Hence $p_k \to \infty$ and thus $\nabla u \in L^q$ for every $q < \infty$. But this implies $u \in W^{2,t}$ for $t < \infty$ gives $u \in C^{2,\alpha}$, a contradiction. To see the claim regarding distributional solutions note that since u is a distributional solution one sees that $|\nabla u| \in L^s_{loc}(\Omega)$ and we then carry on as above. □

Theorem 12. *Let $0 < p < 1 < s < 2$ and $t \in [0, M]$ where M is defined as in Theorem 11. Then for each $0 \le \lambda$ there exists a $u_\lambda \in H_0^1(\Omega)$ weak solution of $(Q)_\lambda$ with $\|u_\lambda\|_{L^\infty} = t$. Moreover we have u_λ increasing in λ and $\|\nabla u_\lambda\|_{L^\infty} \to \infty$. Additional if $s < \frac{N+2}{N}$ then u_λ is a classical solution.*

Proof. First we construct an increasing sequence of solutions for all nonnegative integers λ. Let u_0 be a classical solution of problem $(Q)_0$ with $||u_0||_\infty = t$. Since $\underline{u} = u_0$ and $\overline{u} = t$ are sub and super-solutions of $(Q)_1$, respectively, with $\underline{u} \le \overline{u}$ and $||\underline{u}||_\infty = ||\overline{u}||_\infty = t$, then we have a classical solution u_1 of $(Q)_1$ with $u_0 \le u_1$ and $||u_1||_\infty = t$. Continuing this procedure we find an increasing sequence of classical solutions u_n $(n = 0, 1, 2, ...)$ of $(Q)_n$ with $||u_n||_\infty = t$. Now consider the sequence of parameters $\{\lambda_n = \frac{n}{2},\ \ n \text{ is odd}\}$. Since every solution of $(Q)_\lambda$ is a sub-solution of $(Q)'_\lambda$ for $\lambda > \lambda'$, then similar as above we can construct an increasing sequence of classical solutions $u_{\frac{n}{2}}$, $n = 1, 3, 5, ...$ of $(Q)_{\frac{n}{2}}$ with the property that $u_{\frac{n}{2}}$ $(n \in \mathcal{N} \cup \{0\})$ remains increasing. We can do this because the number of non-integers $\frac{m}{2}$ between two integers is finite. Continuing as above we can construct an increasing sequence of classical solutions $u_{\frac{n}{m}}$ of $(Q)_{\frac{n}{m}}$. Now assume λ_0 is an irrational. Take a sequence of rational λ tends to λ_0. By Lemma 6, there exists a $u_{\lambda_0} \in H_0^1(\Omega)$ such that $u_\lambda \nearrow u_{\lambda_0}$ in $H_0^1(\Omega)$, $u_\lambda \nearrow u_{\lambda_0}$, a.e. and u_{λ_0} is a weak solution of Q_{λ_0}. This also gives $\|u_{\lambda_0}\|_{L^\infty} = t$. The remaining parts of the theorem follow from Lemma 6 and the next theorem. □

Theorem 13. *(Boundary layer) Suppose $0 < p < 1 < s < 2$ and Ω a bounded domain in $\mathbb{R}^N$ with smooth boundary. Let $\lambda_m \nearrow \infty$ and u_m be an increasing bounded (in L^∞) sequence of positive H_0^1 solutions of $(Q)_{\lambda_m}$. Then $u_m(x) \to T := \sup ||u_m||_{L^\infty(\Omega)}$ for a.e. $x \in \Omega$. We also have $||\nabla u_m||_{L^2(\Omega')} \to 0$ for all $\Omega' \subset\subset \Omega$. Moreover, $||\nabla u_m||_{L^\infty(\Omega)} \to \infty$.*

Proof. Let u_m be a positive solution of $(Q)_{\lambda_m}$ and $0 \le \phi \in C_c^\infty(\Omega)$. Multiply the equation by ϕ^q (where $\frac{1}{s} + \frac{1}{q} = 1$) we get

$$\int_\Omega \nabla u_m \cdot \nabla(\phi^q) = \int_\Omega |\nabla u_m|^p \phi^q + \lambda_m \int_\Omega |\nabla u_m|^s \phi^q.$$

From above and $\nabla(\phi^q) = q\phi^{q-1}\nabla\phi$ we get

$$\lambda_m \int_\Omega |\nabla u_m|^s \phi^q \le \int_\Omega \nabla u_m \cdot \nabla(\phi^q) = q \int_\Omega (\nabla u_m \cdot \nabla\phi)\phi^{q-1} \le q\Big(\int_\Omega |\nabla\phi|^q\Big)^{\frac{1}{q}} \Big(\int_\Omega |\nabla u_m|^s \phi^q\Big)^{\frac{1}{s}},$$

which implies that,

$$\int_\Omega |\nabla u_m|^s \phi^q \le \frac{q^q}{\lambda_m^q} \int_\Omega |\nabla\phi|^q. \tag{40}$$

From above, by choosing a cut-off function ϕ so that $\phi \equiv 1$ in Ω' we easily get

$$\|\nabla u_m\|_{L^s(\Omega')} \to 0.$$

Now take

$$T'_m := \frac{1}{|\Omega'|} \int_{\Omega'} u_m dx$$

and note that T_m is an increasing bounded sequence of real numbers, hence $T'_m \nearrow T' \le T$. By imbedding theorems we have

$$\|u_m - T'_m\|_{L^{\frac{Ns}{N-s}}(\Omega')} \le C\|\nabla u_m\|_{L^s(\Omega')} \to 0.$$

So from this we have

$$\|u_m - T'\|_{L^{\frac{Ns}{N-s}}(\Omega')} \to 0.$$

Note by passing to a subsequence we can assume that $u_{m_k} \to T'$ a.e. in Ω' and since u_m is increasing and Ω' was arbitrary, then $u_m \to T_0$ a.e. in Ω for some $T_0 \le T$. We show that $T_0 = T$. Assume $T_0 < T$ and take $T_0 < T_1 < T$, $m_0 \in \mathcal{N}$ and $x_0 \in \Omega$ s.t $T_0 < T_1 < u_{m_0}(x_0) < T$ with $u_m(x_0) \to T_0$. But u_m is increasing

hence $u_m(x_0) \geq u_{m_0}(x_0)$ for $m > m_0$, gives $T_0 \geq T_1$, a contradiction. So we have $u_m \to T$ a.e. in Ω.
In the following, using the results above and the PDE, we show that

$$||\nabla u_m\|_{L^2(\Omega')} \to 0.$$

To show this, take a $\phi \in C_0^\infty(\Omega)$, then first we prove

$$\lim_{m\to\infty} \int_{\Omega'} |\nabla u_m|^2 \phi \to 0. \tag{41}$$

Multiply the PDE of u_m by $u_m\phi$, then we get

$$\int_\Omega u_m \phi(-\Delta u_m) = \int_\Omega u_m \phi |\nabla u_m|^p + \int_\Omega \lambda_m u_m \phi |\nabla u_m|^s. \tag{42}$$

Take $\Omega_0 :=$ supp ϕ, then we have

$$\int_\Omega u_m \phi(-\Delta u_m) = \int_{\Omega_0} \phi |\nabla u_m|^2 + \int_{\Omega_0} u_m \nabla\phi \cdot \nabla u_m. \tag{43}$$

Using (43) in (42) we get

$$\int_{\Omega_0} \phi |\nabla u_m|^2 = \int_{\Omega_0} u_m \phi |\nabla u_m|^p + \int_{\Omega_0} \lambda_m u_m \phi |\nabla u_m|^s - \int_{\Omega_0} u_m \nabla\phi \cdot \nabla u_m. \tag{44}$$

Note that by choosing a suitable cut-off function in Ω_0 in (40) we easily get

$$\int_{\Omega_0} \lambda_m |\nabla u_m|^s \leq \frac{C_0}{\lambda_m^{q-1}} \to 0.$$

Now by using this, $u_m \leq T$, ϕ and $\nabla\phi$ are bounded and the dominated convergence theorem we see that the RHS of (44) tends to zero as $m \to \infty$. Hence,

$$\int_{\Omega_0} \phi |\nabla u_m|^2 \to 0 \quad as \quad m \to \infty.$$

Now taking ϕ so that $\Omega' \subset\subset \Omega_0 \subset\subset \Omega$ and $\phi = 1$ in Ω', $0 \leq \phi \leq 1$ then we get

$$||\nabla u_m\|_{L^2(\Omega')} \to 0.$$

To prove the last part, assume there is some $A > 0$ such that $\sup_\Omega |\nabla u_m| \leq A$ (where A is independent of m). For $\varepsilon > 0$ small set $\Omega_\varepsilon := \{x \in \Omega : dist(x, \partial\Omega) > \varepsilon\}$. Let $\varepsilon > 0$ and assume $0 \leq \phi \in C_c^\infty(\Omega)$ is such that $\phi = 1$ in Ω_ε and $|\nabla\phi| \leq \frac{C}{\varepsilon}$ (C independent of ε). Then we have, using (40),

$$\begin{aligned} \int_\Omega |\nabla u_m|^s dx &= \int_{\Omega_\varepsilon} |\nabla u_m|^s \phi^q + \int_{\Omega_\varepsilon^c} |\nabla u_m|^s \\ &\leq \int_{\Omega_\varepsilon} |\nabla u_m|^s \phi^q + A^s |\Omega_\varepsilon^c| \\ &\leq \frac{q^q}{\lambda_m^q} \int_\Omega |\nabla\phi|^q dx + A^s |\Omega_\varepsilon^c|, \end{aligned}$$

hence we have

$$\limsup_{m\to\infty} \int_\Omega |\nabla u_m|^s dx \leq A^s |\Omega_\varepsilon^c|$$

and then we can send $\varepsilon \searrow 0$ to see that

$$\limsup_{m\to\infty} \int_\Omega |\nabla u_m|^s dx = 0$$

and hence using the Sobolev imbedding we have

$$\int_\Omega u_m^{\frac{Ns}{N-s}} dx \to 0$$

as $m \to \infty$. But this is a contradiction since u_m is increasing in m. □

We now consider $(Q)_\lambda$ in the case of $0 < p < s < 1$. Fix $0 < \lambda < \infty$ and let $u_m > 0$ denote a classical solution of $(Q)_\lambda$ with $t_m := \|u_m\|_{L^\infty} \to \sup\{\|u\|_{L^\infty} : u$ a classical solution of $(Q)_\lambda\}$. Since $0 < p < s < 1$ one can argue directly the existence of a maximal solution u_λ; by maximal here we mean a solution with maximum L^∞ norm.

Theorem 14. *Let $0 < p < s < 1$. Let u_λ denote the a maximal solution of $(Q)_\lambda$ (in the L^∞ sense). Then $\|u_\lambda\|_\infty \nearrow \infty$ as $\lambda \nearrow \infty$.*

Proof. Define $\underline{u}$ as the maximal solution of $-\Delta u = \lambda |\nabla u|^s$ in Ω with $u = 0$ on $\partial\Omega$. Note that by scaling, this can be converted to our standard $-\Delta v = |\nabla v|^s$ and that $\|\underline{u}\|_{L^\infty} \nearrow \infty$ as $\lambda \nearrow \infty$. It is easy to check that $\underline{u}$ is a subsolution of $-\Delta u = |\nabla u|^p + \lambda |\nabla u|^s$, while a supersolution is $M_\lambda = \sup_\Omega \underline{u}$. Hence there must be a solution in the sector $\langle \underline{u}, M_\lambda \rangle$. Since $\underline{u}$ blows up as $\lambda \nearrow \infty$, so does u_λ. □

4.2 Extension 2

We now consider equation

$$\begin{cases} -\Delta u &= |\nabla u|^p + f(x) \text{ in } \Omega, \\ u &= 0 \quad\quad\quad\quad \text{on } \partial\Omega, \end{cases} \tag{45}$$

where Ω a bounded domain in $\mathbb{R}^N$ with smooth boundary, $0 < p < 1$ and where $f(x)$ is given.

Theorem 15. *Suppose p, Ω are as above and suppose $f \in L^q(\Omega)$ for some $N < q < \infty$. Then there exists a weak solution $u \in W^{2,q}(\Omega) \cap H_0^1(\Omega)$ of (45).*

Proof. Let v be the unique solution of the problem

$$\begin{cases} -\Delta v &= f(x) \quad \text{in } \Omega, \\ v &= 0 \quad\quad \text{on } \partial\Omega, \end{cases} \tag{46}$$

and ψ the unique positive solution of

$$\begin{cases} -\Delta \psi &= 1 \quad\quad \text{in } \Omega, \\ \psi &= 0 \quad\quad \text{on } \partial\Omega. \end{cases} \tag{47}$$

Since $f(x) \in L^q(\Omega)$ with $q > N$, by the elliptic regularity we have $v \in W^{2,q}(\Omega)$ that also gives $v \in C^1(\bar{\Omega})$. For $\gamma > 0$ large define $\overline{u} = v + \gamma\psi$ and $\underline{u} = v - \gamma\psi$. Then note both are zero on $\partial\Omega$ and $\overline{u} \geq \underline{u}$ in Ω and we also note both are in $W^{2,q}(\Omega)$. For $\overline{u}$ to be a supersolution and $\underline{u}$ to be a subsolution we require that

$$\gamma \geq |\nabla v + \gamma \nabla \psi|^p \quad \text{and} \quad -\gamma \leq |\nabla v - \gamma \nabla \psi|^p \quad \text{in } \Omega.$$

But since $\gamma, v \in C^1(\overline{\Omega})$ and since $0 < p < 1$ we see this holds for large enough γ. □

4.3 Extension 3

In this section we are interested in variable exponent versions of the above, namely

$$\begin{cases} -\Delta u(x) &= |\nabla u(x)|^{p(x)} \text{ in } \Omega, \\ u &= 0 \qquad\qquad \text{on } \partial\Omega. \end{cases} \tag{48}$$

Proposition 1. *Let $p(x) : \Omega \to [0, 2]$ be a continuous function with $\inf_\Omega p(x) < 1$. Then (48) has a positive classical solution. Also, there always exists a dead core positive solution.*

Proof. By the assumption there exists $x_0 \in \Omega$ such that $p(x_0) < 1$, and just for the sake of simplicity we let $\mathrm{dist}(x_0, \partial\Omega) = 1$. Take a $p \in (p(x_0), 1)$, then by the continuity of $p(x)$ find a $\delta_0 > 0$ such that $p(x) < p$ in $B_{\delta_0}(x_0)$. Now let $r_0 \in [0, \delta_0)$, then from example 2 we know that the radial function

$$u(r) = \beta^{1/(1-p)} \int_r^1 (y - y^{-\alpha} r_0^{\alpha+1})^{1/(1-p)} dy, \qquad r_0 < r < \infty,$$

$$u(r) \equiv \beta^{1/(1-p)} \int_{r_0}^1 (y - y^{-\alpha} r_0^{\alpha+1})^{1/(1-p)} dy, \qquad 0 \le r \le r_0,$$

where $r = |x - x_0|$, is a solution of $-\Delta u = |\nabla u|^p$ in R^N with $u(r) \le 0$ for $r \ge 1$ (with equality at $r = 1$). Define $\underline{u} = \frac{\gamma}{\theta}(e^{\theta u} - 1)$. We find suitable $\gamma > 0$ and $\theta > 0$ for which $\underline{u}$ is a sub-solution of (48). Note that we have $\underline{u}|_{\partial\Omega} \le 0$, hence we need

$$\Delta\underline{u} + |\nabla\underline{u}|^{p(x)} = \gamma\theta e^{\theta u}|\nabla u|^2 - \gamma e^{\theta u}|\nabla u|^p + \gamma^{p(x)} e^{\theta p(x) u}|\nabla u|^{p(x)} \ge 0, \quad x \in \Omega,$$

or equivalently,

$$\theta|\nabla u|^2 + \gamma^{p(x)-1} e^{(p(x)-1)\theta u}|\nabla u|^{p(x)} \ge |\nabla u|^p, \quad x \in \Omega. \tag{49}$$

To find suitable γ and θ for which (49) holds for all $x \in \Omega$ we divide Ω to two parts $B_{\delta_0}(x_0)$ and $\Omega \setminus B_{\delta_0}(x_0)$. First consider the part $\Omega \setminus B_{\delta_0}(x_0)$. From the definition of u we have

$$|\nabla u|^{2-p} = \beta^q (r - r^{-\alpha} r_0^{\alpha+1})^q \ge \beta^q (\delta_0 - \delta_0^{-\alpha} r_0^{\alpha+1})^q > 0, \quad for\ x \in \Omega \setminus B_{\delta_0}(x_0).$$

Hence, taking $\theta = \Big(\beta^q (\delta_0 - \delta_0^{-\alpha} r_0^{\alpha+1})^q\Big)^{-1}$, which is independent of γ, we get

$$\theta|\nabla u|^2 > |\nabla u|^p, \quad when \ \ x \in \Omega \setminus B_{\delta_0}(x_0).$$

Hence, (49) holds for $|x - x_0| \ge \delta_0$ with this θ and for every $\gamma > 0$. Now fix θ as above. For $|x - x_0| < \delta_0$ we find a suitable γ for which the second term in the LHS of (49) to be larger than the RHS. For this we need, from (49) (noticing that we have $\nabla u = 0$ in $B_{r_0}(x_0)$, hence (49) obviously holds in $B_{r_0}(x_0) \subset B_{\delta_0}(x_0)$)

$$\frac{1}{\gamma} \ge e^{\theta u}|\nabla u|^{\frac{p-p(x)}{1-p(x)}} := g(x), \quad r_0 < |x - x_0| < \delta_0. \tag{50}$$

Notice that to get (50) we used that $p(x) < p < 1$ for $|x - x_0| < \delta_0$. Now since for $|x - x_0| < \delta_0$ we have $0 < \frac{p-p(x)}{1-p(x)} < \frac{p}{1-p}$, then $g(x)$ is bounded on $B_{\delta_0}(x_0)$. Hence it suffices to take $\frac{1}{\gamma} = \sup_{B_{\delta_0}(x_0)} g(x)$. So we showed that, with the above choice of β and γ, the function $\underline{u}$ is a subsolution. As a supersolution take $\bar{u} = ||\underline{u}||_{L^\infty(\Omega)}$. Then by Theorem A there exists a solution v of (48) that belongs to $W^{2,q}(\Omega)$ (for every $q > N$), with

$$\underline{u} \le v \le ||\underline{u}||_{L^\infty(\Omega)}. \tag{51}$$

Now note that if in the above we consider $r_0 > 0$ then the fact that $u(r) \equiv ||u||_{L^\infty(\Omega)}$ in $[0, r_0]$ implies that $\underline{u} = ||\underline{u}||_{L^\infty(\Omega)}$ in $[0, r_0]$. Then from (51) we see that $v(x) \equiv ||\underline{u}||_{L^\infty(\Omega)}$ in $B_{r_0}(x_0)$. Hence, v is a dead core positive solution. □

Remark 4. *Note that in the above we constructed a solution v corresponds to every x_0 with $p(x_0) < 1$, hence we should have a continuum of positive classical solutions. Also note that the assumption $\inf_\Omega p(x) < 1$ is necessary for the existence of nontrivial classical solutions, otherwise we have $p(x) \ge 1$ and then similarly to the case of constant p we can use the maximum principle to arrive at a contradiction.*

Acknowledgement

We thank the referee for a meticulous reading of the manuscript and for the suggestions which have greatly improved the presentation.

References

[1] S. Alarcn, J. Garca-Melin and A. Quaas, Nonexistence of positive supersolutions to some nonlinear elliptic problems, J. Math. Pures Appl. 99 (2013) 618-634.

[2] S. Alarcn, J. Garca-Melin and A. Quaas, Existence and non-existence of solutions to elliptic equations with a general convection term, Proceedings of the Royal Society of Edinburgh, 144A (2014) 225-239.

[3] S. Alarcn, J. Garca-Melin and A. Quaas, Liouville Type Theorems for Elliptic Equations with Gradient Terms, Milan J. Math. 81 (2013) 171-185.

[4] Ambrosetti, A, Brezis, H., Cerami, G., Combined effects of concave and convex nonlinearities in some elliptic problems. J. Funct. Anal., 122 (1994) 519-543.

[5] D. Arcoya, L. Boccardo, T. Leonori and A. Porretta, Some elliptic problems with singular natural growth lower order terms, J. Diff. Eqns., 249 (2010) 2771-2795.

[6] D. Arcoya, J. Carmona, T. Leonori, P. J. Martinez-Aparicio, L. Orsina and F. Petitta, Existence and non-existence of solutions for singular quadratic quasilinear equations, J. Diff. Eqns., 246 (2009) 4006-4042.

[7] D. Arcoya, C. De Coster, L. Jeanjean and K. Tanaka, Remarks on the uniqueness for quasilinear elliptic equations with quadratic growth conditions, J. Math. Anal. Appl., 420 (2014) 772-780.

[8] D. Arcoya, C. De Coster, L. Jeanjean and K. Tanaka, Continuum of solutions for an elliptic problem with critical growth in the gradient, J. Funct. Anal., 268 (2015) 2298-2335.

[9] A. Aghajani, C. Cowan and S. H. Lui, Singular solutions of elliptic equations involving nonlinear gradient terms on perturbations of the ball, preprint (2017).

[10] A. Bensoussan, L. Boccardo and F. Murat, On a nonlinear partial differential equation having natural growth terms and unbounded solution, Ann. Inst. Henri Poincare, 5 (1988) 347-364.

[11] M.-F. Bidaut-Veron, M. Garcia-Huidobro and L. Veron, Remarks on some quasilinear equations with gradient terms and measure data. In Contemp. Math, 595 (2013) 31-53.

[12] M.-F. Bidaut-Veron, M. Garcia-Huidobro and L. Veron, Local and global properties of solutions of quasilinear Hamilton-Jacobi equations, J. Funct. Anal., 267 (2014) 3294-3331.

[13] M.-F. Bidaut-Veron, M. Garcia-Huidobro and L. Veron, Boundary singularities of positive solutions of quasilinear Hamilton-Jacobi equations, Calc. Var., 54 (2015) 3471-3515.

[14] H. Brezis and L. Nirenberg, Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents, CPAM, 36 (1983) 437-477.

[15] M.A. Burgos-Prez, J. Garca-Melin and A. Quaas, Classification of supersolutions and Liouville theorems for some nonlinear elliptic problems, Discrete and Continuous Dynamical Systems (2016), DOI: 10.3934/dcds.2016004.

[16] Caffarelli, L., Friedman, A., The free boundary in the Thomas-Fermi atomic model. J. Diff. Eqns., 32 (1979) 335-356.

[17] L. Caffarelli, A. Friedman, Partial Regularity of the Zero Set of Linear and Superlinear Elliptic Equations, J. Diff. Eqns., 60 (1985) 420-439.

[18] L. Caffarelli, B. Gidas and J. Spruck. *Asymptotic symmetry and local behaviour of semilinear elliptic equations with critical Sobolev growth.* Commun. Pure Appl. Math. 42 (1989), 271297.

[19] J. Ching and F. C. Cirstea, Existence and classification of singular solutions to nonlinear elliptic equations with a gradient term, Anal. PDE, 8 (2015) 1931-1962.

[20] J.M. Coron, *Topologie et cas limite des injections de Sobolev.* C.R. Acad. Sc. Paris, 299, Series I, 209212.(1984).

[21] J. Dávila, M. del Pino and M. Musso, *The Supercritical LaneEmdenFowler Equation in Exterior Domains*, Communications in Partial Differential Equations, 32:8, 1225-1243, (2007).

[22] J. Dávila, M. del Pino, M. Musso and J. Wei, *Fast and slow decay solutions for supercritical elliptic problems in exterior domains*, Calculus of Variations and Partial Differential Equations August 2008, Volume 32, Issue 4, pp 453-480.

[23] J. Dávila, Manuel del Pino, M. Musso and J. Wei, *Standing waves for supercritical nonlinear Schrödinger equations*, Journal of Differential Equations 236(2007),no.1, 164-198.

[24] M. del Pino, P. Felmer and M. Musso, *Two-bubble solutions in the super-critical Bahri-Corons problem.* Calc. Var. Partial Differential Equations 16(2):113145.(2003).

[25] V. Ferone and F. Murat, Nonlinear problems having natural growth in the gradient: an existence result when the source terms are small, Nonlinear Anal., 42 (2000) 13309-1326.

[26] V. Ferone, M. R. Posteraro and J. M. Rakotoson, L^∞-estimates for nonlinear elliptic problems with p-growth in the gradient, J. Inequal. Appl., 3 (1999) 109-125.

[27] R. Filippucci, Nonexistence of positive weak solutions of elliptic inequalities, Nonlinear Anal. 70 (2009) 2903-2916.

[28] M. Ghergu and V. Radulescu, Singular Elliptic Problems, Oxford University Press, New York, 2008.

[29] M. Ghergu and V. Radulescu, Nonlinear PDEs, Springer-Verlag, Berlin Heidelberg, 2012.

[30] D. Giachetti, F. Petitta and S. Segura de Leon, Elliptic equations having a singualar quadratic gradient term and a changing sign datum, 11 (2012) 1875-1895.

[31] D. Giachetti, F. Petitta and S. Segura de Leon, A priori estimates for elliptic problems with a strongly singular gradient term and a general datum, Diff. Integral Eqns., 226 (2013) 913-948.

[32] B. Gidas and J. Spruck, *Global and local behavior of positive solutions of nonlinear elliptic equations.* Comm. Pure Appl. Math., 34(4):525-598, 1981.

[33] D. Gilbarg and N. Trudinger, Elliptic partial differential equations of second order, Clasiics in Mathematics, Springer-Verlag, Berlin, 2001.

[34] N. Grenon, F. Murat and A. Porretta, Existence and a priori estimate for elliptic problems with subquadratic gradient dependent terms, C. R. Acad. Sci. Paris, Ser. I, 342 (2006) 23-28.

[35] N. Grenon and C. Trombetti, Existence results for a class of nonlinear elliptic problems with p-growth in the gradient, Nonlinear Anal., 52, (2003) 931-942.

[36] Q. Han, F.-H. Lin, Nodal sets of solutions of elliptic differential equations. Available at http://nd.edu/qhan/nodal.pdf.

[37] R. Hardt, M. Hoffmann-Ostenhof, T. Hoffmann-Ostenhof and N. Nadirashvili, Critical sets of solutions to elliptic equations. J. Differential Geom., 51 (2) (1999), 359-373.

[38] J.L. Kazdan, R.J. Kramer, Invariant criteria for existence of solutions to second-order quasilinear elliptic equations, Comm. Pure Appl. Math., 31 (1978) 619-645.

[39] O. L. Ladyzhenskaya and N. N. Ural'ceva, Linear and Quasilinear Elliptic Equations, Academic Press, New York, 1968.

[40] J. M. Lasry and P. L. Lions, Nonlinear elliptic equations with singular boundary conditions and stochastic control with state constraints, Math. Ann., 283 (1989) 583-630.

[41] P. L. Lions, Quelques remarques sur les problemes elliptiques quasilineaires du second ordre, J. Anal. Math., 45 (1985) 234-254.

[42] M. Marcus and P. T. Nguyen, Elliptic equations with nonlinear absorption depending on the solution and its gradient, Proc. London Math. Soc., 111 (2015) 205-239.

[43] P. T. Nguyen and L. Veron, Boundary singularities of solutions to elliptic viscous Hamilton-Jacobi equations, J. Funct. Anal., 263 (2012) 1487-1538.

[44] D.Passaseo, *Nonexistence results for elliptic problems with supercritical nonlinearity in nontrivial domains.* J. Funct. Anal. 114(1):97105.(1993).

[45] S. Pohozaev, On the eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$, Dokl. Akad. Nauk SSSR, 165 (1965) 36-39.

[46] A. Porretta and S. Segura de Leon, Nonlinear elliptic equations having a gradient term with natural growth, J. Math. Pures Appl., 85 (2006) 465-492.

[47] P. Pucci and J. Serrin. The maximum principle. Progress in Nonlinear Differential Equations and their Applications, 73. Birkhäuser Verlag, Basel, 2007.

[48] M. Struwe, Variational Methods, Springer-Verlag, Berlin Heidelberg, 4th edition, 2008.

[49] Z. Zhang, Boundary blow-up elliptic problems with nonlinear gradient terms, J. Diff. Eqns., 228 (2006) 661-684.