<u>Remark</u>: We close this introduction to external/internal weak direct products by remarking on the distinction between the two:

Suppose that G is the internal weak direct product of N_i. By definition, it follows that $N_i \lhd G$ for all $i \in I$ and $G \cong \prod_{i \in I}^{w} N_i$.

However, technically speaking <u>G is not equal</u> to $\prod_{i \in I}^{w} N_i$ because $\prod_{i \in I}^{w} N_i$ does not contain N_i $(i \in I)$, it contains a group isomorphic to N_i (namely $\iota_i(N_i)$) (the image under the canonical injection).

Thus the distinction between internal/external direct products really only comes into play when we are required to keep careful track of elements, homomorphisms, subgroups and images of homomorphisms, etc.

Hungerford §1.9

<u>Free groups</u>, free products, generators and relations.

There are many, many topics one could address relating to presentations of groups. We only aim to give definitions and a few basic examples.

As a first step towards discussing group presentations, we construct what is called a *free group*.

Construction:

Let X be a set, here is how to construct a certain group $F(X)$ called "the free group on X".

If $X = \emptyset$, then set $F(X) = \{e\}$ (trivial group).

If $X \neq \emptyset$, then create a new set denoted by X^{-1} that contains exactly one element x^{-1} for each $x \in X$. I.e. the map $X \longrightarrow X^{-1}$

given by $x \longmapsto x^{-1}$

is a bijection.

At this point the element x^{-1} is not an inverse of x, it's simply some element of a newly constructed set. The newly constructed set X^{-1} contains one such new element for each $x \in X$. Last, take some one-element set disjoint from $X \cup X^{-1}$, call the element of this set id.

Define a *word on X* to be a sequence $(a_1, a_2, a_3, \dots)$ with $a_i \in X \cup X^{-1} \cup \{1\}$ such that $\exists\, n \in \mathbb{N}$ satisfying $a_k = id$ for all $k \geq n$. The constant sequence $(id, id, id, \dots)$ is called the *empty word* and will be denoted by id.

A word is <u>reduced</u> if:

(i) for all $x \in X$ there is no $i > 0$ such that $a_i = x$ and $a_{i+1} = x^{-1}$ or $a_i = x^{-1}$ and $a_{i+1} = x$. (ie x and x^{-1} are not adjacent).

(ii) $a_k = 1$ implies $a_i = 1$ for all $i \geq k$.

In particular, note that (id, id, id, ...) is reduced. Now since every nonempty reduced word is of the form

$$(x_1^{\varepsilon_1}, x_2^{\varepsilon_2}, \dots, x_n^{\varepsilon_n}, id, id, id, \dots)$$

where $x_i \in X$ and $\varepsilon_i = \pm 1$ (where x^1 means x), so from here on we write this as

$$x_1^{\varepsilon_1} x_2^{\varepsilon_2} \dots x_n^{\varepsilon_n} \text{ (just concatenate)}.$$

Also note that, by definition of equality of sequences,

$$x_1^{\varepsilon_1} x_2^{\varepsilon_2} \dots x_n^{\varepsilon_n} = y_1^{\delta_1} y_2^{\delta_2} \dots y_m^{\delta_m} \quad (x_i, y_i \in X,\ \varepsilon_i, \delta_i = \pm 1)$$

means that $x_i = y_i$ and $\varepsilon_i = \delta_i$ for all i, (and $m = n$) because you compare them entry-by-entry.

Now let $F(X)$ denote the set of all reduced words. Note that $X \subset F(X)$, since we identify each $x \in X$ with the reduced word

$$(x, id, id, id, \dots)$$

which we simply write as

$$x.$$

Now we want to define a binary operation that makes $F(X)$ a group. We want to say: just concatenate words, ie

$$(x_1^{\varepsilon_1} x_2^{\varepsilon_2} \cdots x_n^{\varepsilon_n})(y_1^{\delta_1} y_2^{\delta_2} \cdots y_m^{\delta_m}) = x_1^{\varepsilon_1} \cdots x_n^{\varepsilon_n} y_1^{\delta_1} \cdots y_m^{\delta_m}$$

But this is not well defined, since the right hand side is not reduced. E.g.

$$(x_1 x_2 x_3^{-1} x_2)(x_2^{-1} x_3 x_3) = \underbrace{x_1 x_2 x_3^{-1} x_2 x_2^{-1} x_3 x_3}_{\text{not reduced}}$$

It is clear what we should do, however: cancel $x_2 x_2^{-1}$, and define

$$(x_1 x_2 x_3^{-1} x_2)(x_2^{-1} x_3 x_3) = x_1 x_2 x_3^{-1} x_3 x_3 = \underbrace{x_1 x_2 x_3}_{\text{reduced}}.$$

So this is our definition of the product on $F(X)$:

Suppose $x_1^{\varepsilon_1} \cdots x_n^{\varepsilon_n}$ and $y_1^{\delta_1} \cdots y_m^{\delta_m}$ are reduced words; say $n \le m$. Let k be the largest integer such that

$$x_{n-j}^{\varepsilon_{n-j}} = y_{j+1}^{-\delta_{j+1}} \quad \text{for } j = 1, \ldots, k-1. \text{ Then set:}$$

$$(x_1^{\varepsilon_1} \cdots x_n^{\varepsilon_n})(y_1^{\delta_1} \cdots y_m^{\varepsilon_m}) = \begin{cases} x_1^{\varepsilon_1} \cdots x_{n-k}^{\varepsilon_{n-k}} y_{k+1}^{\delta_{k+1}} \cdots y_m^{\delta_m} & \text{if } k < n \\ y_{n+1}^{\delta_{n+1}} \cdots y_m^{\delta_m} & \text{if } k = m < n \\ \mathrm{id} & \text{if } k = m = n \end{cases}$$

if $n \le m$, and if $m < n$ then make an analagous definition.

Also set $w(\mathrm{id}) = (\mathrm{id})w = w$ for all $w \in F(X)$)

<u>Theorem</u>: The set $F(X)$ with the binary operation above is a group, called "<u>the free group on</u> X".

<u>Proof</u>: It's clear that $id = (id, id, id, \dots)$ serves as the identity element, and that $(x_1^{\varepsilon_1} \cdots x_n^{\varepsilon_n})^{-1} = x_n^{-\varepsilon_n} x_{n-1}^{-\varepsilon_{n-1}} \cdots x_1^{-\varepsilon_1}$.

We need only verify that the operation is associative. There are two ways:

① A tedious induction on the length of reduced words and many case arguments (try this to get a feeling for the difficulty).

② A clever, difficult argument. This is what we'll do.

For each $x \in X$ and $\varepsilon = \pm 1$, let $\varphi_{x^\varepsilon} : F(X) \longrightarrow F(X)$ be the function:

$$\varphi_{x^\varepsilon}(id) = x^\varepsilon, \text{ and}$$

$$\varphi_{x^\varepsilon}(x_1^{\delta_1} \cdots x_n^{\delta_n}) = \begin{cases} x^\varepsilon x_1^{\delta_1} \cdots x_n^{\delta_n} & \text{if } x_1^{-\delta_1} \neq x^\varepsilon \\ x_2^{\delta_2} \cdots x_n^{\delta_n} & \text{if } x_1^{-\delta_1} = x^\varepsilon . \end{cases}$$

Then note that $(\varphi_x)^{-1} = \varphi_{x^{-1}}$ and $(\varphi_{x^{-1}})^{-1} = \varphi_x$, so these maps are actually bijections $F(X) \longrightarrow F(X)$.

Let $S(F(X))$ be the group of all permutations of $F(X)$, and let $F_0 \subset S(F(X))$ be the subgroup generated by the set $\{\varphi_x \mid x \in X\}$. Then the map $\varphi: F(X) \longrightarrow F_0$ given by ~~$\varphi F x$~~

$$\varphi(id) = \mathbb{1}_{F(X)} : F(X) \longrightarrow F(X) \text{ (identity permutation)}$$

and

$$\varphi(x_1^{\delta_1} \cdots x_n^{\delta_n}) = \varphi_{x_1^{\delta_1}} \circ \varphi_{x_2^{\delta_2}} \circ \cdots \circ \varphi_{x_n^{\delta_n}}$$

is an onto function that satisfies $\varphi(w_1 w_2) = \varphi(w_1)\varphi(w_2)$ for all reduced words $w_1, w_2 \in F(X)$ (this last claim requires a check, but it's easy to do!) ~~Thus~~ In fact φ is one-to-one as well. This is easy to see because the permutation $\varphi(x_1^{\delta_1} \cdots x_n^{\delta_n})$ sends the element $id \in F(X)$ to the reduced word $x_1^{\delta_1} \cdots x_n^{\delta}$ (again, check this!). So now since $\varphi: F(X) \longrightarrow F_0$ is onto, one-to-one and satisfies $\varphi(w_1 w_2) = \varphi(w_1)\varphi(w_2)$ it follows that the operation on $F(X)$ is associative (since F_0 is a group, so its operation is associative).

Next: What universal property makes free groups interesting/meaningful?