

Algebra 3

Lecture 1 Hungerford Section 1.8.

Recall the following:

Definition: Let G, H be groups. Then the set $G \times H$ becomes a group called the direct product when we equip it with the operation

$$(g_1, h_1)(g_2, h_2) = (g_1 g_2, h_1 h_2)$$

for all $g_1, g_2 \in G$ and $h_1, h_2 \in H$. The identity is (e_G, e_H) where e_G, e_H are the identities of G and H respectively, the inverse is: $(g, h)^{-1} = (g^{-1}, h^{-1})$.

We would like to extend this to arbitrary families of groups $\{G_i | i \in I\}$, not just products of two groups.

Recall that we can think of elements in the Cartesian product $\prod_{i \in I} G_i$ as functions $f: I \rightarrow \bigcup_{i \in I} G_i$ satisfying $f(i) \in G_i$ for all i . For example, in the case of a finite product:

$$(2, (1, 3), \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}) \in \mathbb{Z} \times S_3 \times SL_2(\mathbb{R})$$

↑
integers

↑
symmetric
group

↑
2x2 matrices
with det 1

can be identified with

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$$f: \{1, 2, 3\} \longrightarrow \mathbb{Z} \cup S_3 \cup SL_2(\mathbb{R})$$

given by $f(1) = 2$, $f(2) = (1, 3)$ and $f(3) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

Or

$$(1, 2, 3, 4, \dots) \in \mathbb{Z} \times \mathbb{Z} \times \dots$$

can be identified with $f: \mathbb{N} \longrightarrow \mathbb{Z} \cup \mathbb{Z} \cup \dots$ where $f(i) = i$, where i lies in the i^{th} copy of \mathbb{Z} .

Definition: Let $\{G_i \mid i \in I\}$ be a family of groups.

Define an operation on $\prod_{i \in I} G_i$ as follows: Given

$f, g \in \prod_{i \in I} G_i$ (so $f, g: I \longrightarrow \prod_{i \in I} G_i$ with $f(i)$ and $g(i)$ in G_i for all i) define $fg: I \longrightarrow \prod_{i \in I} G_i$ to be the map $fg(i) = f(i)g(i)$ for all $i \in I$, where multiplication on the right hand side is in G_i . This is called componentwise multiplication, and in the

finite case boils down to

$$(g_1, \dots, g_n)(h_1, \dots, h_n) = (g_1 h_1, \dots, g_n h_n)$$

for elements (g_1, \dots, g_n) and (h_1, \dots, h_n) of $G_1 \times \dots \times G_n$.

Remark: In place of $\prod_{i \in I} G_i$, when I is finite (say $|I| = n$) it is customary to write

$$G_1 \times \dots \times G_n$$

or $G_1 \oplus \dots \oplus G_n$ if the G_i 's are abelian.

Theorem: If $\{G_i \mid i \in I\}$ are groups, then

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(i) $\prod_{i \in I} G_i$ is a group, and

(ii) For each $k \in I$, the map $\pi_k: \prod_{i \in I} G_i \rightarrow G_k$ given by $\pi_k(f) = f(k)$ (or given by

$$\pi_k(g_1, \dots, g_n) = g_k \text{ if } I \text{ is finite and } |I|=n)$$

is an onto homomorphism, called a canonical projection.

Proof: A straightforward check.

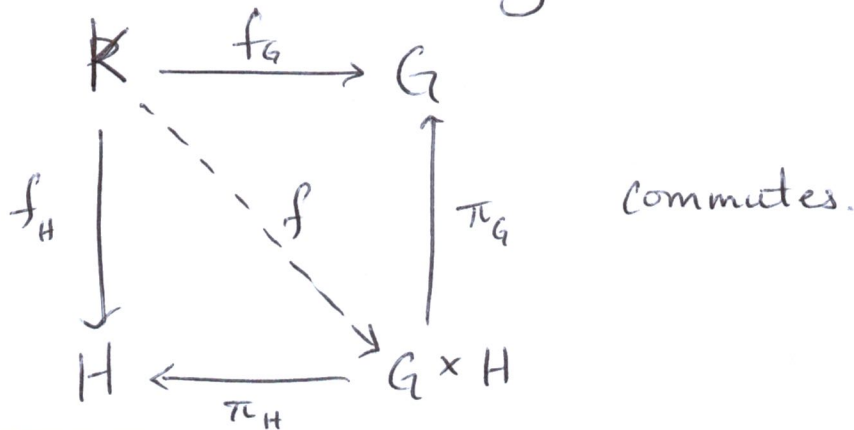
The direct product of groups can also be described in a more sophisticated way, using what is called "its universal property". Here is the universal property of the direct product stated for the case of two groups G, H and their product $G \times H$:

The group $G \times H$ is the unique group (up to isomorphism) satisfying the following: There are homomorphisms $G \times H \xrightarrow{\pi_G} G$ and $G \times H \xrightarrow{\pi_H} H$ such that for any group K and any homomorphisms $f_G: K \rightarrow G$, $f_H: K \rightarrow H$ there exists a unique homomorphism $f: K \rightarrow G \times H$ such that

$$\pi_G \circ f = f_G$$

$$\text{and } \pi_H \circ f = f_H$$

in other words, we say



It should be rather obvious that $G \times H$ has this property, because given f_G, f_H as above, the map $f: K \rightarrow G \times H$ with $f(k) = (f_G(k), f_H(k))$ certainly makes $\pi_G \circ f = f_G$ true.

$$\pi_H \circ f = f_H$$

It is also easy to see that there can be no other map: Suppose $f': K \rightarrow G \times H$ were another map with $\pi_G \circ f' = f_G$ and $\pi_H \circ f' = f_H$.

Then we must have, $\forall k \in K$:

$$\pi_G f'(k) = f_G(k), \text{ so } f'(k) = (f_G(k), ?)$$

$$\pi_H f'(k) = f_H(k), \text{ so } f'(k) = (?, f_H(k)).$$

$$\text{Overall, } f'(k) = (f_G(k), f_H(k)) = f(k).$$

How do we know $G \times H$ is "the unique group" with this property?

Theorem: Suppose some other group X with homomorphisms⁵

$X \xrightarrow{\pi'_G} G$, $X \xrightarrow{\pi'_H} H$ satisfied: For every K with $f_G: K \rightarrow G$ and $f_H: K \rightarrow H$ there is a unique $f: K \rightarrow X$ with $\pi'_G \circ f = f_G$, $\pi'_H \circ f = f_H$. Then X is isomorphic to $G \times H$.

Proof: By the universal property applied to $G \times H$, there exists a unique map $h: X \rightarrow G \times H$ such that

$$\begin{array}{ccc}
 X & \xrightarrow{\pi'_G} & G \\
 \pi'_H \downarrow & \searrow h & \uparrow \pi_G \\
 H & \xleftarrow{\pi_H} & G \times H
 \end{array}$$

commutes, i.e. $\pi_G \circ h = \pi'_G$ and $\pi_H \circ h = \pi'_H$.

By the universal property applied to X , there's a unique map $h': G \times H \rightarrow X$ such that

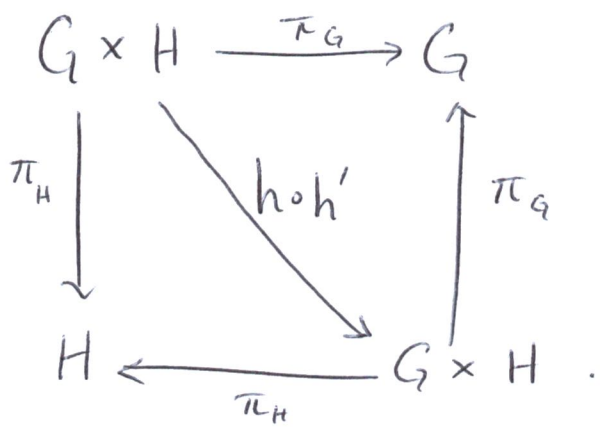
$$\begin{array}{ccc}
 G \times H & \xrightarrow{\pi_G} & G \\
 \pi_H \downarrow & \searrow h' & \uparrow \pi'_G \\
 H & \xleftarrow{\pi'_H} & X
 \end{array}$$

commutes: $\pi'_G \circ h' = \pi_G$ and $\pi'_H \circ h' = \pi_H$.

So we can calculate: $\pi'_G \circ h' = \pi_G \Rightarrow \pi'_G \circ h \circ h' = \pi_G$

$\pi'_H \circ h' = \pi_H \Rightarrow \pi'_H \circ h \circ h' = \pi_H$.

But then we have:



By the universal property of $G \times H$, there's a unique map making the diagram above commute, and we know it's the identity $1_{G \times H}: G \times H \rightarrow G \times H$. So we must have $h \circ h' = id$, and similarly we can show $h' \circ h = id$. Thus $X \cong G \times H$.

Of course this is true for our direct product over a general index set as well:

Theorem: Let $\{G_i \mid i \in I\}$ be a family of groups, and $\{\varphi_i: K \rightarrow G_i \mid i \in I\}$ a family of homomorphisms.

Then there is a unique homomorphism $\varphi: K \rightarrow \prod_{i \in I} G_i$ such that $\pi_i \circ \varphi = \varphi_i$ for all $i \in I$.

Moreover, $\prod_{i \in I} G_i$ is the only group which has this property (ie any other group with this property must be isomorphic to $\prod_{i \in I} G_i$).

We can make a slight variation on the direct product:

Definition: The external (weak) direct product of groups $\{G_i \mid i \in I\}$ is denoted $\prod_{i \in I}^w G_i$ and is the set of all $f: I \rightarrow \bigcup_{i \in I} G_i$ with $f(i) \in G_i$ for all i and $f(i) = e_i$ (e_i the identity in G_i) for all but finitely many i . If all G_i are abelian, then $\prod_{i \in I}^w G_i$ is typically written $\sum_{i \in I} G_i$ or $\bigoplus_{i \in I} G_i$.

Example: If $\{\mathbb{Z}_i \mid i \in \mathbb{N}\}$ is a collection of copies of \mathbb{Z} , then $\bigoplus_{i \in \mathbb{N}} \mathbb{Z}_i$ consists of "tuples"

$$(n_1, n_2, n_3, n_4, \dots)$$

where all but finitely many of the n_i 's are zero. Addition is still componentwise, as in $\prod_{i \in \mathbb{N}} \mathbb{Z}_i$.

The additional requirement that $f(i)$ is not the identity for all but finitely many i may seem arbitrary and/or artificial.

In fact it behaves much like the product (direct):

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Theorem: If $\{G_i \mid i \in I\}$ is a family of groups, then

(i) $\prod_{i \in I}^w G_i$ is a normal subgroup of $\prod_{i \in I} G_i$

(ii) For each $k \in I$ the map $i_k: G_k \rightarrow \prod_{i \in I}^w G_i$ given by $i_k(g) = f$, where $f(i) = e_i$ (identity) for $i \neq k$ and $f(k) = g$, is a one-to-one homomorphism.

(iii) For each $k \in I$, $i_k(G_k)$ is a normal subgroup of $\prod_{i \in I} G_i$.

Proof: To be done on assignment.

Remarks: ① Note that this is very similar to a theorem we wrote concerning direct products. In fact, it turns out we can also come up with a universal property for this construction.

② When the index set I is finite, direct products $\prod_{i \in I} G_i$ and external weak direct products $\prod_{i \in I}^w G_i$ are the same thing.

③ The maps $i_k: G_k \rightarrow \prod_{i \in I}^w G_i$ are called canonical injections.

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It is, in fact, fairly natural since the external weak direct product satisfies a universal property (as the direct product did), but we "reverse arrows"

Theorem (Universal property of the weak direct product).

Let $\{A_i \mid i \in I\}$ be a family of abelian groups, and B any abelian group. For any family $\{\psi_i: A_i \rightarrow B \mid i \in I\}$ of group homomorphisms there is a unique homomorphism

$$\psi: \bigoplus_{i \in I} A_i \longrightarrow B$$

such that $\psi \circ \iota_k = \psi_k$ for all $k \in I$, and this property determines $\bigoplus_{i \in I} A_i$ up to isomorphism.

Remark: If we omit the word "abelian" the theorem fails. So the ^{external} weak direct product is a construction which satisfies a universal property for abelian groups, but not for general groups.

Proof: All groups are abelian, so we write additively.

Let $\{A_i \mid i \in I\}$ be as above, and B some abelian group with homomorphisms $\{\psi_i: A_i \rightarrow B \mid i \in I\}$

To define $\Psi: \bigoplus_{i \in I} A_i \longrightarrow B$, let $a \in \bigoplus_{i \in I} A_i$

be an arbitrary element. Then $a(j)$ is nonzero for only finitely many $j \in I$, say it's nonzero for the indices j_1, j_2, \dots, j_n and that $a(j_k) = a_{j_k} \in A_{j_k}$.

Define $\Psi(a) = \Psi_{j_1}(a_{j_1}) + \Psi_{j_2}(a_{j_2}) + \dots + \Psi_{j_n}(a_{j_n})$.

Since B is abelian and each Ψ_j is a homomorphism, it's easy to check that Ψ is a homomorphism, too.

It also follows from the definition of Ψ that

$$\Psi \circ i_k = \Psi_k \quad \text{for all } k \in I.$$

We only need to check that Ψ is unique. So suppose $\varphi: \bigoplus_{i \in I} A_i \rightarrow B$ were some other map with

$$\varphi \circ i_k = \Psi_k \quad \forall k \in I.$$

Suppose we plug in an arbitrary element $a \in \bigoplus_{i \in I} A_i$ as above. Then $a = i_{j_1}(a_{j_1}) + i_{j_2}(a_{j_2}) + \dots + i_{j_n}(a_{j_n})$, so

$$\varphi(a) = \varphi\left(\sum_{k=1}^n i_{j_k}(a_{j_k})\right)$$

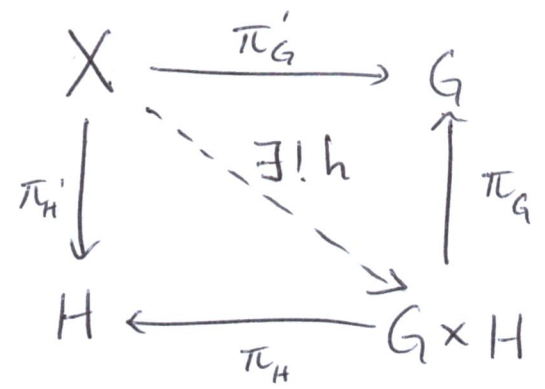
$$= \sum_{k=1}^n \varphi \circ i_{j_k}(a_{j_k})$$

$$= \sum_{k=1}^n \Psi_{j_k}(a_{j_k})$$

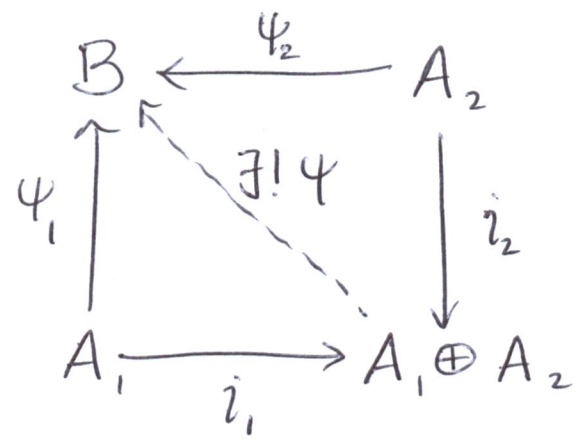
$$= \Psi(a), \quad \text{so that } \varphi = \Psi.$$

Uniqueness of $\bigoplus_{i \in I} A_i$ follows from this universal property by an argument similar to the case of direct products.

In pictures, recall that if G and H are groups then $G \times H$ was characterized by the diagram:



If we translate the universal property of $A_1 \oplus A_2$ into pictures it becomes:



Note that this is "the same picture with arrows reversed".

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A natural question to ask at this point is: Given a group G , how do we know if G is secretly a direct product or external weak direct product?

Theorem: Let $\{N_i \mid i \in I\}$ be a family of normal subgroups of a group G , such that

(i) $G = \langle \bigcup_{i \in I} N_i \rangle$ (here, $\langle - \rangle$ means "subgroup generated by")

(ii) For each $k \in I$, $N_k \cap \langle \bigcup_{i \neq k} N_i \rangle = \langle e \rangle$
↑ identity..

Then $G \cong \prod_{i \in I}^w N_i$.

Remark: When $|I| < \infty$, since $\prod_{i \in I} G_i = \prod_{i \in I}^w G_i$, this also gives a way of recognizing groups as finite direct products.

Sketch of proof: Let $a \in \prod_{i \in I}^w N_i$ be an arbitrary element. Then $a: I \longrightarrow \bigcup_{i \in I} N_i$ satisfies $a(i) \in N_i \forall i$ and $a(i) = e_i$ (identity) for all i but finitely many $i \in I$. Say that $j_1, j_2, \dots, j_n \in I$ are the elements for which $a(j_k)$ is not the identity,

write $a(j_k) = a_{j_k} \in N_{j_k}$. Then consider the (13)

product $b = a_{j_1} \cdot a_{j_2} \cdot \dots \cdot a_{j_n} \in G$.

Problem: The element b might not be well-defined, because if we list the a_{j_k} 's in a different order then we get a different expression for b (the a_{j_k} 's are multiplied in a different order).

The following lemma saves us:

Lemma: If N_1 and N_2 are normal in G and $N_1 \cap N_2 = \langle e \rangle$, then $n_1 n_2 = n_2 n_1$ for all $n_2 \in N_2$ and $n_1 \in N_1$.

By this lemma, the element b above ~~is well defined~~ depends only on $a \in \prod_{i \in I}^w N_i$, and not on any ordering of $a_{j_1}, a_{j_2}, \dots, a_{j_n}$. So call b " $\varphi(a)$ ". This

defines a homomorphism $\varphi: \prod_{i \in I}^w N_i \rightarrow G$.

With some work, we can show this is an isomorphism.

(Hungerford Theorem 8.6 Ch 1).

Thus we make the following definition:

(4)

Definition: Let $\{N_i \mid i \in I\}$ be a family of normal subgroups of G such that $G = \langle \bigcup_{i \in I} N_i \rangle$ and for each $k \in I$, $N_k \cap \langle \bigcup_{i \neq k} N_i \rangle = \langle e \rangle$. Then G is called the internal weak direct product of $\{N_i \mid i \in I\}$ or the direct sum (internal) if $\{N_i \mid i \in I\}$ are all abelian.