Introduction to Braid Groups on Surfaces – Part II

Juliana R. Theodoro de Lima
University of São Paulo, Brazil

October 26, 2015
Summary

Braid Groups on Surfaces

Relations in $B_n(M)$

Presentation of $B_n(M)$
Braid Groups on Surfaces

Relations in $B_n(M)$

Presentation of $B_n(M)$

Summary
Let \( M \) be a closed connected surface, not necessarily orientable, and let \( \mathcal{P} = \{P_1, \ldots, P_n\} \) be a set of \( n \) distinct points of \( M \). A geometric braid over \( M \) based at \( \mathcal{P} \) is an \( n \)-tuple \( \gamma = (\gamma_1, \ldots, \gamma_n) \) of paths, \( \gamma_i : [0, 1] \rightarrow M \times [0, 1] \), such that:

1. \( \gamma_i(0) = P_i \), for all \( i = 1, \ldots, n \),
2. \( \gamma_i(1) \in \mathcal{P} \), for all \( i = 1, \ldots, n \),
3. \( \{\gamma_1(t), \ldots, \gamma_n(t)\} \), are \( n \) distinct points of \( M \), for all \( t \in [0, 1] \).

For all \( i = 1, \ldots, n \), we will call \( \gamma_i \) the \( i \)-th string of \( \gamma \).
Braid Groups on Surfaces
Relations in $B_n(M)$
Presentation of $B_n(M)$
Reference

Braid Diagrams

Example of a braid on 3-strands in two different views:
Two geometric braids based at $\mathcal{P}$ are equivalent if there is a homotopy which deforms one of them into the other, provided that at anytime we always have a geometric braid based at $\mathcal{P}$.

The product of two braids is induced by the usual product of paths and it endows the set of equivalence classes of braids with a group structure $B_n(M)$.

If $\gamma_i(1) = P_i$, for all $i = 1, \ldots, n$ then we say that $\gamma$ is a pure braid. It endows the group $PB_n(M)$ which is a normal subgroup of $B_n(M)$. 
Let $M$ be a closed, orientable surface of genus $g \geq 1$. Then $B_n(M)$, admits the following presentation:

**Generators:** $\{a_{1,1}, \ldots, a_{1,2g}\} \cup \{\sigma_1, \ldots, \sigma_{n-1}\}$;
Relations:

\[(R1) \quad \sigma_i \sigma_j = \sigma_j \sigma_i, \quad |i - j| \geq 2;\]
\[(R2) \quad \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \quad 1 \leq i \leq n - 2;\]
\[(R3) \quad a_{1,1} \cdots a_{1,2g} a_{1,1}^{-1} \cdots a_{1,2g}^{-1} = \sigma_1 \cdots \sigma_{n-2} \sigma_{n-1}^2 \sigma_{n-1} \cdots \sigma_1;\]
\[(R4) \quad a_{1,r} A_{2,s} = A_{2,s} a_{1,r}, \quad 1 \leq r, s \leq 2g; \quad r \neq s;\]
\[(R5) \quad (a_{1,1} \cdots a_{1,r}) A_{2,r} = \sigma_1^2 A_{2,r} (a_{1,1} \cdots a_{1,r}), \quad 1 \leq r \leq 2g;\]
\[(R6) \quad a_{1,r} \sigma_i = \sigma_i a_{1,r}, \quad 1 \leq r \leq 2g; \quad i \geq 2.\]

Where:

\[t_{1,j} = \sigma_1 \cdots \sigma_{j-2} \sigma_{j-1}^2 \sigma_{j-2}^{-1} \cdots \sigma_1^{-1}, \quad \text{for } j = 2, \ldots, n,\]
\[A_{2,s} = \sigma_1^{-1} (a_{1,1} \cdots a_{1,s-1} a_{1,s+1}^{-1} \cdots a_{1,2g}) \sigma_1^{-1}, \quad \text{for } s = 1, \ldots, 2g.\]
Remark

In the sequel follows the calculations for $M$ closed, connected and non-orientable surface of genus $g \geq 2$. 

\[
\begin{array}{c}
\alpha_g \\
e \\
\alpha_1 \\
\alpha_g
\end{array}
\]
Generators of $B_n(M)$

From the left to the right the generators $\sigma_i$ and $a_{1,r}$. 
Summary

Braid Groups on Surfaces

Relations in $B_n(M)$

Presentation of $B_n(M)$
\begin{itemize}
  \item $\sigma_i \sigma_j = \sigma_j \sigma_i, |i - j| \geq 2$
  \item $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \quad 1 \leq i \leq n - 2,$
        holds since $B_n \subseteq B_n(M)$ when $M \neq \mathbb{RP}^2$.
  \item $a_{1,1}^2 \ldots a_{1,g}^2 = \sigma_1 \ldots \sigma_{n-2} \sigma_{n-1} \sigma_{n-2} \ldots \sigma_1$ holds since:
\end{itemize}
\[ a_{1,s} A_{2,r} = A_{2,r} a_{1,s}, \quad 1 \leq r, s \leq 2g; \quad r \neq s, \text{ since:} \]

\[ a_{1,r} \sigma_i = \sigma_i a_{1,r}, \quad 1 \leq r \leq 2g; \quad i \geq 2, \text{ since:} \]
\[ (a_{1,1}^2 \ldots a_{1,r-1}^2 a_{1,r}) A_{2,r} = \sigma_1^2 A_{2,r} (a_{1,1}^2 \ldots a_{1,r-1}^2 a_{1,r}), 1 \leq r \leq g \text{ holds since:} \]
Summary

Braid Groups on Surfaces

Relations in $B_n(M)$

Presentation of $B_n(M)$
Theorem (Gonzalez-Meneses [GM, Theorem 2.2, p.436 ])

Let $M$ be a closed, connected and orientable surface of genus $g \geq 1$. Then, $B_n(M)$ admits the following presentation:

Generators: $\sigma_1, \ldots, \sigma_{n-1}, a_1, 1, \ldots, a_1, 2g$

Relations:

(R1) $\sigma_i \sigma_j = \sigma_j \sigma_i, |i - j| \geq 2$;
(R2) $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, 1 \leq i \leq n-2$;
(R3) $a_{1,1}^2 \ldots a_{1,g}^2 = \sigma_1 \ldots \sigma_{n-2} \sigma_{n-1}^2 \sigma_{n-2} \ldots \sigma_1$;
(R4) $a_r A_{2,s} = A_{2,s} a_r, 1 \leq r, s \leq g; r \neq s$;
(R5) $(a_{1,1}^2 \ldots a_{1,r-1} a_1, r) A_{2,r} = \sigma_1^2 A_{2,r} (a_{1,1}^2 \ldots a_{1,r-1} a_1, r), 1 \leq r \leq g$;
(R6) $a_{1,r} \sigma_i = \sigma_i a_{1,r}, 1 \leq r \leq g; i \geq 2$.

where $A_{2,r} = \sigma_1^{-1} (a_{1,1}^2 \ldots a_{1,r-1} a_1^{-1} a_1^{-2} \ldots a_{1,1}^{-2}) \sigma_1$. 
The figures about braid groups on the disk can be found in [LH]. The others can be found in [GM].
Reference

Thank You!