Introduction

Braid Groups on Surfaces

Relations in $B_n(M)$

Presentation of $B_n(M)$

Reference

Introduction to Braid Groups on Surfaces

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Artin Braid Groups

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A geometric braid in n strings $\beta$ is a system of $n$ embedded arcs $A = \{A_1, \ldots, A_n\}$ in $\mathbb{E}^3$, where the $i$-th arc $A_i$ connects the point $P_i$ on the upper plane to the point $P'_{\tau(i)}$ on the lower plane, for some permutation $\tau \in \{1, \ldots, n\}$, satisfying:

- Each arc $A_i$ intersects each intermediate parallel plane between the upper and the lower plane exactly once;
- The arcs $\{A_1, \ldots, A_n\}$ intersect each intermediate parallel plane between the upper and the lower plane in exactly $n$ different points.

![Diagram of geometric braid in n strings](image-url)
Artin’s Presentation Theorem ([A])

The braid group $B_n$ admits the following presentation:

- **Generators**: $\sigma_1, \ldots, \sigma_{n-1}$.
- **Relations**:

$$
\sigma_i \sigma_j = \sigma_j \sigma_i, \quad |i - j| \geq 2, \quad 1 \leq i, j \leq n - 1,
$$

$$
\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \quad 1 \leq i \leq n - 2.
$$
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\[ \sigma_i \cdot \sigma_j \]

\[ \sigma_j \cdot \sigma_i \]

\[ \sigma_1 \cdot \sigma_{i+1} \cdot \sigma_i \]

\[ \sigma_{j+1} \cdot \sigma_j \cdot \sigma_{j+1} \]

\[ \mathbb{Z} \]
Summary

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Relations in $B_n(M)$

Presentation of $B_n(M)$
Let $M$ be a closed connected surface, not necessarily orientable, and let $\mathcal{P} = \{P_1, \ldots, P_n\}$ be a set of $n$ distinct points of $M$. A geometric braid over $M$ based at $\mathcal{P}$ is an $n$-tuple $\gamma = (\gamma_1, \ldots, \gamma_n)$ of paths, $\gamma_i : [0, 1] \to M \times [0, 1]$, such that:

1. $\gamma_i(0) = P_i$, for all $i = 1, \ldots, n$,
2. $\gamma_i(1) \in \mathcal{P}$, for all $i = 1, \ldots, n$,
3. $\{\gamma_1(t), \ldots, \gamma_n(t)\}$, are $n$ distinct points of $M$, for all $t \in [0, 1]$.

For all $i = 1, \ldots, n$, we will call $\gamma_i$ the $i$-th string of $\gamma$. 
Braid Diagrams

Example of a braid on 3-strands in two different views:
Two geometric braids based at $P$ are equivalent if there is a homotopy which deforms one of them into the other, provided that at anytime we always have a geometric braid based at $P$.

The product of two braids is induced by the usual product of paths and it endows the set of equivalence classes of braids with a group structure $B_n(M)$.

If $\gamma_i(1) = P_i$, for all $i = 1, \ldots, n$ then we say that $\gamma$ is a pure braid. It endows the group $PB_n(M)$ which is a normal subgroup of $B_n(M)$.
Remark

In the sequel follows the calculations for $M$ closed, connected and orientable surface of genus $g \geq 1$. 
Generators of $B_n(M)$

From the left to the right: $a_{1,2k+1}$, $a_{1,2k}$ and $\sigma_i$. 
\[ \sigma_i \sigma_j = \sigma_j \sigma_i, |i - j| \geq 2 \]
\[ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \quad 1 \leq i \leq n - 2, \]
holds since \( B_n \subseteq B_n(M) \) when \( M \neq S^2 \).
\[ a_{1,1} \cdots a_{1,2g} a_{1,1}^{-1} \cdots a_{1,2g}^{-1} = \sigma_1 \cdots \sigma_{n-2} \sigma_{n-1}^2 \sigma_{n-2} \cdots \sigma_1 \]
holds since:
\[ a_{1,r}A_{2,s} = A_{2,s}a_{1,r}, \quad 1 \leq r, s \leq 2g; \quad r \neq s, \quad \text{since:} \]

\[ a_{1,r}\sigma_i = \sigma_i a_{1,r}, \quad 1 \leq r \leq 2g; \quad i \geq 2, \quad \text{since:} \]
\((a_{1,1} \cdots a_{1,r})A_{2,r} = \sigma_1^2 A_{2,r}(a_{1,1} \cdots a_{1,r}),\ 1 \leq r \leq 2g\) holds since:
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Theorem (Gonzalez-Meneses [GM, Theorem 2.1, p.435 ])

Let \( M \) be a closed, connected and orientable surface of genus \( g \geq 1 \). Then, \( B_n(M) \) admits the following presentation:

Generators: \( \sigma_1, \ldots, \sigma_{n-1}, a_{1,1}, \ldots, a_{1,2g} \)

Relations:

(R1) \( \sigma_i \sigma_j = \sigma_j \sigma_i \), \hspace{1cm} |i − j| \geq 2.

(R2) \( \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \), \hspace{1cm} 1 \leq i \leq n − 2.

(R3) \( a_{1,1} \cdots a_{1,2g} a_{1,1}^{-1} \cdots a_{1,2g}^{-1} = \sigma_1 \cdots \sigma_{n-2} \sigma_{n-1}^2 \sigma_{n-2} \cdots \sigma_1 \).

(R4) \( a_{1,r} A_{2,s} = A_{2,s} a_{1,r} \), \hspace{1cm} 1 \leq r, s \leq 2g; \ r \neq s.

(R5) \( (a_{1,1} \cdots a_{1,r}) A_{2,r} = \sigma_1^2 A_{2,r} (a_{1,1} \cdots a_{1,r}) \), \hspace{1cm} 1 \leq r \leq 2g.

(R6) \( a_{1,r} \sigma_i = \sigma_i a_{1,r} \), \hspace{1cm} 1 \leq r \leq 2g; \ i \geq 2,

where \( A_{2,r} = \sigma_1^{-1}(a_{1,1} \cdots a_{1,r} a_{1,r+1}^{-1} \cdots a_{1,2g}^{-1}) \sigma_1^{-1} \).
The figures about braid groups on the disk can be found in [LH]. The others can be found in [GM].

In the same paper, the author found the presentation for $M$ a closed, connected and non-orientable surface of genus $g \geq 2$. 
Reference


Thank You!