The Congruences of a Finite Lattice

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To László Fuchs,
my thesis advisor, my teacher,
who taught me to set the bar high;

and to the coauthors of my papers,
Tomi (E. T. Schmidt),
Harry (H. Lakser),
Ervin (E. Fried),
David (D. Kelly),
Bob (R. W. Quackenbush),
Fred (Wehrung),
and more than fifty others,
who helped me raise the bar.

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Table of Notation

Symbol	Explanation	Page
$\mathrm{Atom}(U)$	set of atoms of the ideal U	84
$\mathrm{Aut}\, L$	automorphism group of L	12
B_n	boolean lattice with n atoms	4
C_n	n-element chain	4
$\mathrm{con}(a, b)$	smallest congruence under which $a \equiv b$	15
$\mathrm{con}(c)$	principal congruence for a color c	39
$\mathrm{con}(H)$	smallest congruence collapsing H	16
$\mathrm{con}(\mathfrak{p})$	principal congruence for the prime interval $\mathfrak{p}$	37
$\mathrm{Con}\, L$	congruence lattice of L	15, 48
$\mathrm{Con_J}\, L$	order of join-irreducible congruences of L	37
$\mathrm{Con_M}\, L$	order of meet-irreducible congruences of L	71
$\mathrm{Cube}\, K$	cubic extension of K	71
$\mathbf{D}$	class (variety) of distributive lattices	24
Diag	diagonal embedding of K into $\mathrm{Cube}\, K$	71
$\mathrm{Down}\, P$	order of down-sets of the (hemi)order P	4, 9, 232
$\mathrm{ext}\colon \mathrm{Con}\, K \to \mathrm{Con}\, L$	for $K \leq L$, extension map: $\Theta \mapsto \mathrm{con}_L(\Theta)$	41
$\mathrm{fil}(a)$	filter generated by the element a	14
$\mathrm{fil}(H)$	filter generated by the set H	14
$F_{\mathbf{D}}(3)$	free distributive lattice on three generators	26
$\mathrm{F}_{\mathbf{K}}(H)$	free lattice generated by H in a variety $\mathbf{K}$	26
$F_{\mathbf{M}}(3)$	free modular lattice on three generators	28
$\mathrm{Frucht}\, C$	Frucht lattice of a graph C	178
$\mathrm{hom}_{\{\vee,0\}}(X, Y)$	$\{\vee, 0\}$-homomorphism of X into Y	253

Symbol	Explanation	Page
$\mathrm{id}(a)$	ideal generated by the element a	14
$\mathrm{id}(H)$	ideal generated by the set H	14
$\mathrm{Id}\, L$	ideal lattice of L	14, 48
(Id)	condition to define ideals	14, 48
Isoform	class of isoform lattices	141
$\mathrm{J}(D)$	order of join-irreducible elements of D	19
$\mathrm{J}(\varphi)$	$\mathrm{J}(\varphi)\colon \mathrm{J}(E) \to \mathrm{J}(D)$, the "inverse" of $\varphi\colon D \to E$	32
$\mathrm{J}(a)$	set of join-irreducible elements below a	19
$\ker(\varphi)$	congruence kernel of φ	16
L	class (variety) of all lattices	25
M	class (variety) of modular lattices	25
Max	maximal elements of an order	49
$\mathbf{mcr}(n)$	minimal congruence representation function	87
$\mathbf{mcr}(n, \mathbf{V})$	**mcr** for a class **V**	87
$\mathrm{M}(D)$	order of meet-irreducible elements of D	32
M_3	five-element modular nondistributive lattice	xvii, 11, 30
$M_3[L]$	order of boolean triples of L	58
$M_3[L, a]$	interval of $M_3[L]$	63
$M_3[L, a, b]$	interval of $M_3[L]$	65
$M_3[a, b]$	order of boolean triples of the interval $[a, b]$	58
$M_3[\Theta]$	reflection of Θ^3 to $M_3[L]$	60
$M_3[\Theta, a]$	reflection of Θ^3 to $M_3[L, a]$	64
$M_3[\Theta, a, b]$	reflection of Θ^3 to $M_3[L, a, b]$	xvii, 67
N_5	five-element nonmodular lattice	xvii, 11, 30
$N_{5,5}$	seven-element nonmodular lattice	94
$N_6 = N(p, q)$	six-element nonmodular lattice	xvii, 80
$N_6[L]$	2/3-boolean triple construction	198
$N(A, B)$	lattice construction	132
$O(f)$	Landau O notation	xxvi
$\mathrm{Part}\, A$	partition lattice of A	7, 9
$\mathrm{Pow}\, X$	power set lattice of X	4
$\mathrm{Pow}^+ X$	order of nonempty subsets of X	219
$\mathrm{Prime}(L)$	set of prime intervals of L	37
$\mathrm{re}\colon \mathrm{Con}\, L \to \mathrm{Con}\, K$	reflection (restriction) map: $\Theta \mapsto \Theta\rceil K$	39
SecComp	class of sectionally complemented lattices	87
SemiMod	class of semimodular lattices	87
$\mathrm{Simp}\, K$	simple extension of K	71
$(\mathrm{SP}_\vee)$	join-substitution property	14, 48
$(\mathrm{SP}_\wedge)$	meet-substitution property	xvii, 14, 48
$\mathrm{sub}(H)$	sublattice generated by H	13
S_8	eight-element semimodular lattice	106
T	class (variety) of trivial lattices	25
Uniform	class of uniform lattices	141

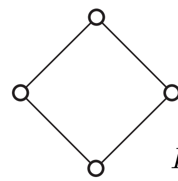
Symbol	Explanation	Page
Relations and Congruences		
A^2	set of ordered pairs of A	3
$\varrho, \tau, \pi, \ldots$	binary relations	
$\Theta, \Psi, \ldots$	congruences	
ω	zero of Part A	7
ι	unit of Part A	7
$a \equiv b \ (\pi)$	a and b in the same block of π	7
$a \mathrel{\varrho} b$	a and b in relation ϱ	3
$a \equiv b \ (\Theta)$	a and b in relation Θ	3
a/π	block containing a	6, 14
H/π	blocks represented by H	7
$\alpha \circ \beta$	product of α and β	21
$\alpha \overset{\mathrm{r}}{\circ} \beta$	reflexive product of α and β	30
$\Theta\rceil_K$	restriction of Θ to the sublattice K	14
L/Θ	quotient lattice	16
Φ/Θ	quotient congruence	16
π_i	projection map: $L_1 \times \cdots \times L_n \to L_i$	21
$\Theta \times \Phi$	direct product of congruences	21
Orders		
$\leq, <$	ordering	3
$\geq, >$	ordering, inverse notation	3
$K \leq L$	K a sublattice of L	13
$\leq_Q$	ordering of P restricted to a subset Q	4
$a \parallel b$	a incomparable with b	3
$a \prec b$	a is covered by b	5
$b \succ a$	b covers a	5
0	zero, least element of an order	4
1	unit, largest element of an order	4
$a \vee b$	join operation	9
$\bigvee H$	least upper bound of H	3
$a \wedge b$	meet operation	9
$\bigwedge H$	greatest lower bound of H	4
P^d	dual of the order (lattice) P	4, 10
$[a, b]$	interval	13
$\downarrow H$	down-set generated by H	4
$\downarrow a$	down-set generated by $\{a\}$	4
$P \cong Q$	order (lattice) P isomorphic to Q	4, 12

Symbol	Explanation	Page
Constructions		
$P \times Q$	direct product of P and Q	5, 20
$P + Q$	sum of P and Q	6
$P \dotplus Q$	glued sum of P and Q	16
$A[B]$	tensor extension of A by B	248
$A \otimes B$	tensor product of A and B	245
$U \circledast V$	modular lattice construction	120
Perpectivities		
$[a,b] \sim [c,d]$	$[a,b]$ perspective to $[c,d]$	32
$[a,b] \overset{\mathrm{u}}{\sim} [c,d]$	$[a,b]$ up-perspective to $[c,d]$	33
$[a,b] \overset{\mathrm{d}}{\sim} [c,d]$	$[a,b]$ down-perspective to $[c,d]$	33
$[a,b] \approx [c,d]$	$[a,b]$ projective to $[c,d]$	33
$[a,b] \nearrow [c,d]$	$[a,b]$ up congruence-perspective onto $[c,d]$	35
$[a,b] \searrow [c,d]$	$[a,b]$ down congruence-perspective onto $[c,d]$	35
$[a,b] \hookrightarrow [c,d]$	$[a,b]$ congruence-perspective onto $[c,d]$	35
$[a,b] \Rightarrow [c,d]$	$[a,b]$ congruence-projective onto $[c,d]$	36
$[a,b] \Leftrightarrow [c,d]$	$[a,b] \Rightarrow [c,d]$ and $[c,d] \Rightarrow [a,b]$	36
Prime intervals		
$\mathfrak{p}, \mathfrak{q}, \dots$		
$\mathrm{con}(\mathfrak{p})$	principal congruence generated by $\mathfrak{p}$	37
$\mathfrak{p} \Rightarrow \mathfrak{q}$	$\mathfrak{p}$ is congruence-projective onto $\mathfrak{q}$	36
$\mathfrak{p} \Leftrightarrow \mathfrak{q}$	$\mathfrak{p} \Rightarrow \mathfrak{q}$ and $\mathfrak{q} \Rightarrow \mathfrak{p}$	36
$\mathrm{Prime}(L)$	set of prime intervals of L	37
Miscellaneous		
$\overline{x}$	closure of x	10
$\varnothing$	empty set	4

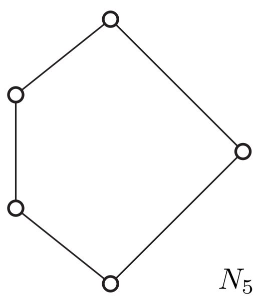
Picture Gallery

C_2

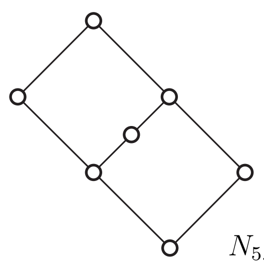
C_3



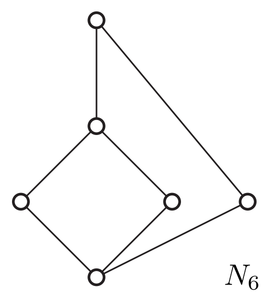
B_2



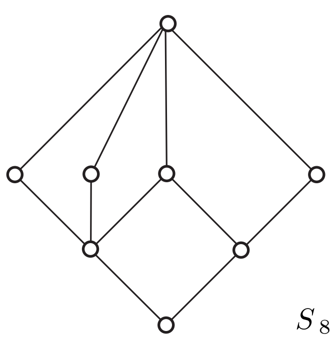
N_5



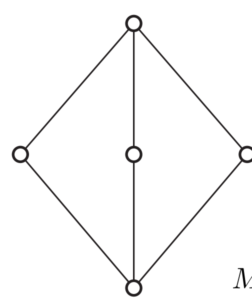
$N_{5,5}$



N_6



S_8



M_3

Acknowledgments

In 2002, I was invited to Nashville to a meeting. I wrote to E. T. Schmidt about the invitation. He could not come, but we agreed that I will give a survey lecture on finite congruence lattices. This was published in [78]. Later in the year, I was invited to the Summer School on General Algebra and Ordered Sets in Tale, Slovakia, to give a series of lectures on the same topic. Unfortunately, Schmidt did not attend.

This the book is an outgrowth of my Tale lectures. Special thanks to the organizers of that excellent meeting: Miroslav Haviar, Tibor Katriňák, and Miroslav Ploščica.

Every chapter in Parts II–V is based on one or more joint papers I wrote with my coauthors. Without their collaboration, this book could not have existed.

My seminar continues to correct errors and improve my presentation. In all matters notational and linguistic, David Kelly is my chief advisor. I got useful feedback from my fourth year honors algebra class, and especially from Mercedes Scott.

I spent two months at the Institute of Advance Study at La Trobe University, Melbourne, Australia, where I had an opportunity to present all the *Proof-by-Picture* sections of this book at the seminar of Brian Davey. I would like to thank the IAS (Gilah Leder, Director, Julia Anderson, Executive Officer and Michael Osborne, Vice Chancellor) for their wonderful hospitality and for the perfect work environment. Brian and his seminar recommended many improvements and corrected many mistakes.

Special thanks to those who read (part of) my book and sent me comments. Kira Adaricheva suggested many improvements to the first seven chapters. Fred Wehrung filled seven pages with objections minor and major; this is a much better book for his effort.

Preface

The topic

The congruences of a finite lattice L form a lattice, called the *congruence lattice of* L and denoted by $\operatorname{Con} L$. The lattice $\operatorname{Con} L$ is a finite *distributive* lattice—according to a 1942 result of Funayama and Nakayama [21]. The converse is a result of R. P. Dilworth from 1944 (see [9]):

Dilworth Theorem. *Every finite distributive lattice D can be represented as the congruence lattice of a finite lattice L.*

This result was first published in 1962 in Grätzer and Schmidt [57]. In the 60 years since the discovery of this result, a large number of papers have been published, strengthening and generalizing the Dilworth Theorem. These papers form two distinct fields:

(i) Representation theorems of finite distributive lattices as congruence lattices of lattices with special properties.

(ii) The Congruence Lattice Problem (CLP): Can congruence lattices of lattices be characterized as distributive algebraic lattices?

A nontrivial finite distributive lattice D is determined by the order $\mathrm{J}(D)$ of join-irreducible elements. So a representation of D as the congruence lattice of a finite lattice L is really a representation of a finite order P $(= \mathrm{J}(D))$, as the order of join-irreducible congruences of a finite lattice L. A join-irreducible congruence of a nontrivial finite lattice L is exactly the same as a congruence of the form $\operatorname{con}(a, b)$, where $a \prec b$ in L; that is, the smallest congruence collapsing a prime interval. Therefore, it is enough to concentrate on such congruences, and make sure that they are ordered as required by P.

The infinite case is much different. There are really only two general positive results: 1. The ideal lattice of a distributive lattice with zero is the congruence lattice of a lattice—see Schmidt [109] (also Pudlák [100]).

2. Any distributive algebraic lattice with at most $\aleph_1$ compact elements is the congruence lattice of a lattice—Huhn [92] and [93] (see also Dobbertin [13]).

The big breakthrough for negative results came in 1999 in Wehrung [122], based on the results of Wehrung [121]. There is a recent survey paper of this field by Tůma and Wehrung [119].

This book deals with the finite case.

The two types of representation theorems

The basic representation theorems are all of the same general type. We represent a finite distributive lattice D as the congruence lattice of a "nice" finite lattice L. For instance, in the 1962 paper (Grätzer and Schmidt [57]), we already proved that the finite lattice L for the Dilworth Theorem can be constructed as a *sectionally complemented lattice*.

To understand the second—the more sophisticated—type of representation theorem, we need the concept of a congruence-preserving extension.

Let L be a lattice, and let K be a sublattice of L. In general, there is not much connection between the congruence lattice of L and the congruence lattice of K. If they happen to be naturally isomorphic, we call L a *congruence-preserving extension* of K. (More formally, we require that the restriction map be an isomorphism, see Section 3.3.)

For sectionally complemented lattices, the congruence-preserving extension theorem was published in a 1999 paper, Grätzer and Schmidt [69]: *Every finite lattice K has a finite, sectionally complemented, congruence-preserving extension L.* It is difficult, reading this for the first time, to appreciate how much stronger this theorem is than the straight representation theorem. While the 1962 theorem provides, for a finite distributive lattice D, a finite sectionally complemented lattice L whose congruence lattice is isomorphic to D, the 1999 theorem starts with an *arbitrary* finite lattice K, and builds a sectionally complemented lattice L on it with the same congruence structure.

Proof-by-Picture

Trying to prove the Dilworth Theorem (unpublished at the time) in 1960 with Schmidt, we came up with the construction—more or less—as presented in Section 7.2. In 1960, we did not discover the 1968 result of Grätzer and Lakser [34] establishing that the construction of the chopped lattice solves the problem. So we translated the chopped lattice construction to a closure space, as in Section 7.4, proved that the closed sets form a sectionally complemented lattice L, and based on that, we verified that the congruence lattice of L represents the given finite distributive lattice.

When we submitted the paper [57] for publication, it had a three-page section explaining the chopped lattice construction and its translation to a closure space. The referee was strict: "You cannot have a three-page explanation for a two-page proof." I believe that in the 40 plus years since the publication of that article, few readers have developed an understanding of the idea behind the published proof.

The referee's dictum is quite in keeping with mathematical tradition and practice. When mathematicians discuss new results, they explain the constructions and the ideas with examples; when these same results are published, the motivation and the examples are largely gone. We publish definitions, constructions, and formal proofs (and conjectures, Paul Erdős would have added).

Tradition has it, when Gauss proved one of his famous results, he was not ready to publicize it because the proof gave away too much as to how the theorem was discovered. "I have had my results for a long time: but I do not yet know how I am to arrive at them", Gauss is quoted in Arber [2].

In this book I try to break with this tradition. In most chapters, after stating the main result, I include a section: *Proof-by-Picture*. This is a misnomer. A *Proof-by-Picture* is not a proof. The Pythagorean Theorem has many well known *Proofs-by-Picture*—sometimes called "Visual Proofs"; these are really proofs. My *Proof-by-Picture* is an attempt *to convey the idea of the proof.* I trust that if the idea is properly understood, the reader should be able to provide the formal proof, or should at least have less trouble reading it. Think of a *Proof-by-Picture* as a lecture to an informed audience, concluding with "the formal details now you can provide." I converted many of these sections into lectures; the transparencies for these can be found on my Web site:
`http://www.maths.umanitoba.ca/homepages/gratzer.html`
see the directory
`/MathBooks/lectures.html`

I will use the same Web site to post corrections to this book, problems solved, and so on.

Outline

In the last paragraph, I call an audience "informed" if they are familiar with the basic concepts and techniques of lattice theory. Part I provides this. I am very selective as to what to include. Also, there are no proofs in this part—with a few exceptions—they are easy enough for the reader to work them out on his own. For proofs, lots of exercises, and a more detailed exposition, I refer the reader to my book [26]. (See also Davey and Priestley [11].)

Most of the research in this book deals with representation theorems; lattices with certain properties are constructed with prescribed congruence structures. The constructions are *ad hoc*. Nevertheless, there are three basic techniques to prove representation theorems:

- chopped lattices, used in almost every chapter;
- boolean triples, used in Chapters 10, 12, and 16, and generalized in Chapter 19; also used in some papers that did not make it in this book, for instance, Grätzer and Schmidt [71];
- cubic extensions, used in most chapters of Part IV.

These are presented in Part II with proofs.

Actually, there are two more basic techniques. *Multi-coloring* is used in three relevant papers: Grätzer, Lakser, and Schmidt [46], [49] and Grätzer and Schmidt [68]; however, it appears in the book only in Chapter 17, so we introduce it there. *Pruning* is utilized in Chapters 11 and 14—it would seem to qualify for Part II; however, there is no theory of pruning, just concrete uses, so there is no general theory to discuss in Part II.

Part III contains the representation theorems, requiring only chopped lattices from Part II. I cover the following topics:

- The Dilworth Theorem and the representation theorem for *sectionally complemented* lattices in Chapter 7 (Grätzer and Schmidt [57], Crawley and Dilworth [10]; see also [9]).
- *Minimal representations* in Chapter 8; that is, for a given $|\mathrm{J}(D)|$, we minimize the size of L representing D (Grätzer, Lakser, and Schmidt [45], Grätzer, Rival, and Zaguia [54]).
- The *semimodular* representation theorem in Chapter 9 (Grätzer, Lakser, and Schmidt [48]).
- The representation theorem for *modular* lattices in Chapter 10 (Schmidt [106] and Grätzer and Schmidt [74]); we are forced to represent with a countable lattice L, since the congruence lattice of a finite modular lattice is always boolean.
- The representation theorem for *uniform* lattices (that is, lattices in which any two congruence classes of a congruence are of the same size) in Chapter 11 (Grätzer, Schmidt, and Thomsen [79]).

Part IV is mostly about congruence-preserving extension. I present the congruence-preserving extension theorem for

- *sectionally complemented* lattices in Chapter 12 (Grätzer and Schmidt [69]);
- *semimodular* lattices in Chapter 13 (Grätzer and Schmidt [72]);
- *isoform* lattices (that is, lattices in which any two congruence classes of a congruence are isomorphic) in Chapter 14 (Grätzer, Quackenbush, and Schmidt [53]).

These three constructions are based on cubic extensions, introduced in Part II.

In Chapter 15, I present the congruence-preserving extension version of the Baranskiĭ-Urquhart Theorem (Baranskiĭ [3], [4] and Urquhart [120]) on the independence of the congruence lattice and the automorphism group of a finite lattice (Grätzer and Schmidt [66]).

Finally, in Chapter 16, I discuss two congruence "destroying" extensions, which we call "magic wands." It is hoped that these can be used to construct new classes of algebraic distributive lattices as congruence lattices of lattices (Grätzer and Schmidt [75], Grätzer, Greenberg, and Schmidt [32]).

What happens if we consider the congruence lattices of two lattices? I take up three variants of this question in Part V.

Let L be a finite lattice, and let K be a sublattice of L. As we discuss it in Section 3.3, there is a map ext from $\operatorname{Con} K$ into $\operatorname{Con} L$: For a congruence relation Θ of K, let the image $\operatorname{ext} \Theta$ be the congruence relation $\operatorname{con}_L(\Theta)$ of L generated by Θ. The map ext is a $\{0\}$-separating join-homomorphism.

Chapter 17 proves the converse, a 1974 result of Huhn [91] and a stronger form due to Grätzer, Lakser, and Schmidt [46].

In Chapter 18, we deal with ideals. Let I be an ideal of a lattice L. Then the restriction map $\operatorname{re}\colon \operatorname{Con} L \to \operatorname{Con} I$ (which assigns to a congruence Θ of L, the restriction $\Theta\rceil_K$ of Θ to K) is a $\{0,1\}$-homomorphism. We prove the corresponding representation theorem for finite lattices—Grätzer and Lakser [35].

We also prove two variants. The first is by Grätzer and Lakser [42] that this result also holds for *sectionally complemented* lattices. The second is by Grätzer and Lakser [40] that this result also holds for *planar* lattices.

The final chapter is a first contribution to the following class of problems. Let $\circledast$ be a construction for finite distributive lattices (that is, if D and E are finite distributive lattices, then so is $D \circledast E$). Find a construction $\circledcirc$ of finite lattices (that is, if K and L are finite lattices, then so is $K \circledcirc L$) satisfying $\operatorname{Con}(K \circledcirc L) \cong \operatorname{Con} K \circledast \operatorname{Con} L$.

If $\circledast$ is the direct product, the answer is obvious since $\operatorname{Con}(K \times L) \cong \operatorname{Con} K \times \operatorname{Con} L$.

In Chapter 19, we take up the construction $D \circledast E = D[E]$, defined as the distributive lattice of all isotone maps from $\mathrm{J}(E)$ to D.

In Grätzer and Greenberg [29], we introduced a construction: the *tensor extension* $A[B]$, for nontrivial finite lattices A and B. In Chapter 19, we prove that $\operatorname{Con}(A[B]) \cong (\operatorname{Con} A)[\operatorname{Con} B]$. The background of this result is described in Chapter 19.

Each chapter in Parts III–V concludes with an extensive discussion section, giving the background for the topic, further results, and open problems. This book lists almost 80 open problems, hoping to convince the reader that we have hardly started. There are also more than 120 references and a detailed index of about 1500 entries.

This book is, as much as possible, visually oriented. I cannot stress too much the use of diagrams as a major research tool in lattice theory. I did not include in the book the list of figures because there is not much use to it; it lists over 110 figures.

Notation and terminology

Lattice-theoretic terminology and notation evolved from the three editions of G. Birkhoff's *Lattice Theory;* [8], by way of my books, [22]–[26], and McKenzie, McNulty, and Taylor [97], changing quite a bit in the process.

Birkhoff's notation for the congruence lattice and ideal lattice of a lattice changed from $\Theta(L)$ and $I(L)$ to $\operatorname{Con} L$ and $\operatorname{Id} L$, respectively. The advent of LaTeX promoted the use of operators for lattice constructions. I try to be consistent: I use an operator when a new structure is constructed; so I use $\operatorname{Con} L$, $\operatorname{Id} L$, $\operatorname{Aut} L$, and so on, without parentheses, unless required for readability, for instance, $\mathrm{J}(D)$ and $\operatorname{Con}(\operatorname{Id} L)$. I use functional notation when sets are constructed, as in $\operatorname{Atom}(L)$ and $\mathrm{J}(a)$. "Generated by" uses the same letters as the corresponding lattice construction, but starting with a lower case letter: $\operatorname{con}(H)$ is the congruence generated by H and $\operatorname{id}(H)$ is the ideal generated by H.

I reversed the arrows for projectivities as compared with my book [26]. I think the new notation is easier to remember: congruences spread as the arrows point. This is also more consistent with the use in universal algebra.

New concepts introduced in more recent research papers exhibit the usual richness in notation and terminology. I use this opportunity, with the wisdom of hindsight, to make their use consistent. The reader will often find different notation and terminology when reading the original papers. The detailed Table of Notation and Index may help.

In combinatorial results, I will use Landau's O notation: for the functions f and g, we write $f = O(g)$ to mean that $|f| \leq C|g|$, for a suitable constant C.

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Part I

A Brief Introduction to Lattices

Chapter

1

Basic Concepts

In this chapter we introduce the most basic order theoretic concepts: orders, lattices, diagrams, and the most basic algebraic concepts: sublattices, congruences, products.

1.1. Ordering

1.1.1 Orders

A *binary relation* ϱ on a nonempty set A is a subset of A^2, that is, a set of ordered pairs $\langle a, b\rangle$, with a, $b \in A$. For $\langle a, b\rangle \in \varrho$, we will write $a \mathrel{\varrho} b$ or $a \equiv b \pmod{\varrho}$.

A binary relation $\leq$ on a set P is called an *ordering* if it is *reflexive* ($a \leq a$, for all $a \in P$), *antisymmetric* ($a \leq b$ and $b \leq a$ imply that $a = b$, for all a, $b \in P$), and *transitive* ($a \leq b$ and $b \leq c$ imply that $a \leq c$, for all a, b, $c \in P$). An *order* $\langle P, \leq\rangle$ consists of a nonempty set P and an ordering $\leq$.

$a < b$ means that $a \leq b$ and $a \neq b$. We also use the "inverse" relations: $a \geq b$ defined as $b \leq a$ and $a > b$ for $b < a$. If more than one ordering is being considered, we write $\leq_P$ for the ordering of $\langle P, \leq\rangle$; on the other hand if the ordering is understood, we will say that P (rather than $\langle P, \leq\rangle$) is an order.

An order P is *trivial* if P has only one element.

The elements a and b of the order P are *comparable* if $a \leq b$ or $b \leq a$. Otherwise, a and b are *incomparable*, in notation, $a \parallel b$.

Let $H \subseteq P$ and $a \in P$. Then a is an *upper bound* of H iff $h \leq a$, for all $h \in H$. An upper bound a of H is the *least upper bound* of H iff, $a \leq b$, for any upper bound b of H; in this case, we will write $a = \bigvee H$. If $a = \bigvee H$

exists, then it is unique. By definition, $\bigvee \varnothing$ exist ($\varnothing$ is the empty set) iff P has a smallest element, *zero*, denoted by 0. The concepts of *lower bound* and *greatest lower bound* are similarly defined; the latter is denoted by $\bigwedge H$. Note that $\bigwedge \varnothing$ exists iff P has a largest element, *unit*, denoted by 1. A *bounded order* is one that has both 0 and 1.

The adverb "similarly" (in "similarly defined") in the previous paragraph can be given concrete meaning. Let $\langle P, \leq \rangle$ be an order. Then $\langle P, \geq \rangle$ is also an order, called the *dual* of $\langle P, \leq \rangle$. The dual of the order P will be denoted by P^d. Now if Φ is a "statement" about orders, and if in Φ we replace all occurrences of $\leq$ by $\geq$, then we get the *dual* of Φ.

Duality Principle for Orders. *If a statement Φ is true for all orders, then its dual is also true for all orders.*

A *chain* (*linear order*) is an order with no incomparable elements. An *antichain* is one in which $a \parallel b$, for all $a \neq b$.

Let $\langle P, \leq \rangle$ be an order and let Q be a nonempty subset of P. Then there is a natural order $\leq_Q$ on Q induced by $\leq$: for a, $b \in Q$, let $a \leq_Q b$ iff $a \leq b$; we call $\langle Q, \leq_Q \rangle$ (or simply, $\langle Q, \leq \rangle$, or even simpler, Q) a *suborder* of $\langle P, \leq \rangle$.

A *chain C in an order P* is a nonempty subset, which, as a suborder, is a chain. An *antichain C in an order P* is a nonempty subset which, as a suborder, is an antichain.

The *length* of a finite chain C, $\operatorname{length} C$, is $|C| - 1$. An order P is said to be *of length n* (in formula, $\operatorname{length} P = n$), where n is a natural number iff there is a chain in P of length n and all chains in P are of length $\leq n$.

The orders P and Q *are isomorphic* (in formula, $P \cong Q$) and the map $\varphi\colon P \to Q$ is an *isomorphism* iff φ is one-to-one and onto and

$$a \leq b \text{ in } P \quad \text{iff} \quad a\varphi \leq b\varphi \text{ in } Q.$$

Let C_n denote the set $\{0, \dots, n-1\}$ ordered by $0 < 1 < 2 < \cdots < n-1$. Then C_n is an n-element chain. Observe that $\operatorname{length} C_n = n - 1$. If $C = \{x_0, \dots, x_{n-1}\}$ is an n-element chain and $x_0 < x_1 < \cdots < x_{n-1}$, then $\varphi\colon i \mapsto x_i$ is an isomorphism between C_n and C. Therefore, the n-element chain is unique up to isomorphism.

Let B_n denote the set of all subsets of the set $\{0, \dots, n-1\}$ ordered by containment. Observe that the order B_n has 2^n elements and $\operatorname{length} B_n = n$. In general, for a set X, we denote by $\operatorname{Pow} X$ the *power set* of X, that is, the set of all subsets of X ordered by set inclusion.

For an order P, call $A \subseteq P$ a *down-set* iff $x \in A$ and $y \leq x$ imply that $y \in A$. For $H \subseteq P$, there is a smallest down-set containing H, namely, $\{\, x \mid x \leq h, \text{ for some } h \in H \,\}$; we use the notation $\downarrow H$ for this set. If $H = \{a\}$, we write $\downarrow a$ for $\downarrow \{a\}$. Let $\operatorname{Down} P$ denote the set of all down-sets ordered by set inclusion. If P is an antichain, then $\operatorname{Down} P \cong B_n$, where $n = |P|$.

The map $\varphi\colon P_1 \to P_2$ is an *isotone map* (resp., *antitone map*) of the order P_1 into the order P_2 iff $a \leq b$ in P_1 implies that $a\varphi \leq b\varphi$ (resp., $a\varphi \geq b\varphi$)

in P_2. Then $P_1\varphi$ is a suborder of P_2. Even if φ is one-to-one, the orders P_1 and $P_2\varphi$ need not be isomorphic.

1.1.2 Diagrams

In the order P, the element a *is covered by* b or b *covers* a (in formula, $a \prec b$ or $b \succ a$) iff $a < b$ and $a < x < b$, for no $x \in P$. The binary relation $\prec$ is called the *covering relation*. The covering determines the ordering:

Let P be a finite order. Then $a \leq b$ iff $a = b$ or if there exists a finite sequence of elements x_1, x_2, ..., x_n such that

$$a = x_1 \prec x_2 \prec \cdots \prec x_n = b.$$

A *diagram* of an order P represents the elements with small circles ○; the circles representing two elements x, y are connected by a line segment iff one covers the other; if x is covered by y, then the circle representing x is placed lower than the circle representing y.

The diagram of a finite order determines the order up to isomorphism.

In a diagram the intersection of two line segments does not indicate an element. A diagram is *planar* if no two line segments intersect. An order P is *planar* if it has a diagram that is planar. Figure 1.1 shows three diagrams of the same order P. Since the third diagram is planar, P is a planar order.

1.1.3 Order constructions

Given the orders P and Q, we can form the *direct product* $P \times Q$, consisting of all ordered pairs $\langle x_1, x_2 \rangle$, with $x_1 \in P$ and $x_2 \in Q$, ordered componentwise, that is, $\langle x_1, x_2 \rangle \leq \langle y_1, y_2 \rangle$ iff $x_1 \leq y_1$ and $x_2 \leq y_2$. If $P = Q$, then we write P^2 for $P \times Q$. Similarly, we use the notation P^n for $P^{n-1} \times P$, for $n > 2$. Figure 1.2 shows a diagram of $C_2 \times P$, where P is the order with diagrams in Figure 1.1.

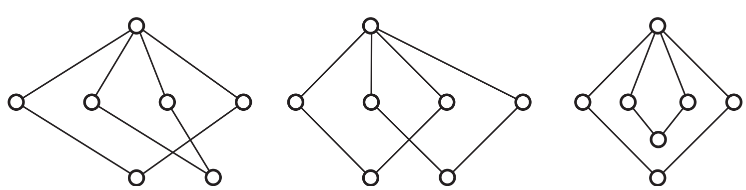

Figure 1.1: Three diagrams of an order.

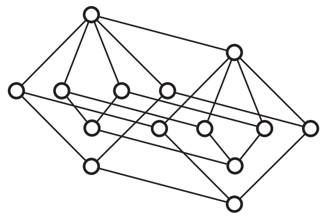

Figure 1.2: A diagram of $C_2 \times P$.

Another often used construction is the (ordinal) *sum* $P + Q$ of P and Q, defined on the (disjoint) union $P \cup Q$ and ordered as follows:

$$x \le y \quad \text{iff} \quad \begin{cases} x \le_P y, \text{ for } x,\ y \in P; \\ x \le_Q y, \text{ for } x,\ y \in Q; \\ x \in P,\ \ y \in Q. \end{cases}$$

Figure 1.3 shows diagrams of $C_2 + P$ and $P + C_2$, where P is the order with diagrams in Figure 1.1. In both diagrams, the elements of C_2 are black-filled. Figure 1.3 also shows diagram of $P + C_2$.

A variant of this is the *glued sum*, $P \dotplus Q$, applied to an order P with largest element 1_P and an order Q with smallest element 0_Q; then $P \dotplus Q$ is $P + Q$ in which 1_P and 0_Q are identified (that is, $1_P = 0_Q$ in $P \dotplus Q$).

1.1.4 Partitions

We now give a nontrivial example of an order. A *partition* of a nonempty set A is a set π of nonempty pairwise disjoint subsets of A whose union is A. The members of π are called the *blocks* of π. The block containing $a \in A$ will be denoted by a/π. A singleton as a block is called *trivial*. If the elements a and

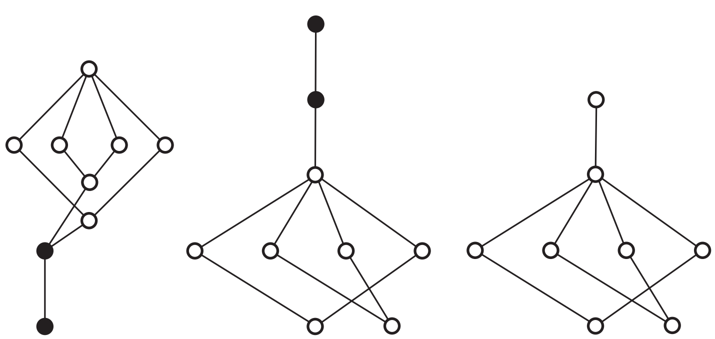

Figure 1.3: Diagrams of $C_2 + P$, $P + C_2$, and $P \dotplus C_2$.

b of A belong to the same block, we write $a \equiv b\ (\pi)$ or $a\ \pi\ b$ or $a/\pi = b/\pi$. In general, for $H \subseteq A$,

$$H/\pi = \{\, a/\pi \mid a \in H \,\}.$$

An *equivalence relation* ε on the set A is a reflexive, symmetric ($a \varepsilon b$ implies that $b \varepsilon a$, for all a, $b \in A$), and transitive binary relation. Given a partition π, we can define an equivalence relation ε by $\langle x, y \rangle \in \varepsilon$ iff $x/\pi = y/\pi$. Conversely, if ε is an equivalence relation, then $\pi = \{\, a/\varepsilon \mid a \in A \,\}$ is a partition of A. There is a one-to-one correspondence between partitions and equivalence relations; we will use the two terms interchangeably.

Part A will denote the set of all partitions of A ordered by

$$\pi_1 \leq \pi_2 \quad \text{iff} \quad x \equiv y\,(\pi_1) \text{ implies that } x \equiv y\,(\pi_2).$$

We draw a picture of a partition by drawing the boundary lines of the (nontrivial) blocks. Then $\pi_1 \leq \pi_2$ iff the boundary lines of π_2 are also boundary lines of π_1 (but π_1 may have some more boundary lines). Equivalently, the blocks of π_2 are unions of blocks of π_1; see Figure 1.4.

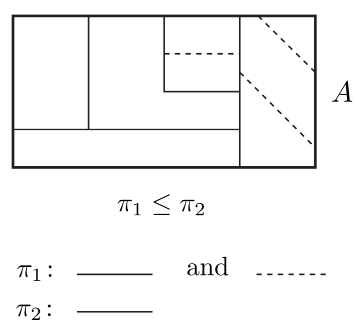

Figure 1.4: Drawing a partition.

Part A has a zero and a unit, denoted by ω and ι, respectively, defined by

$$\begin{aligned} x \equiv y \quad (\omega) \qquad & \text{iff } x = y; \\ x \equiv y \quad (\iota), \qquad & \text{for all } x,\ y \in A. \end{aligned}$$

Figure 1.5 shows the diagrams of Part A, for $|A| \leq 4$. The partitions are labeled by listing the nontrivial blocks.

A *preorder* is a nonempty set Q with a binary relation $\leq$ that is reflexive and transitive. Let us define the binary relation $a \approx b$ on Q as $a \leq b$ and $b \leq a$. Then $\approx$ is an equivalence relation. Define the set P as $Q/{\approx}$, and on P define the binary relation $\leq$:

$$a/{\approx} \leq b/{\approx} \quad \text{iff} \quad a \leq b \text{ in } Q.$$

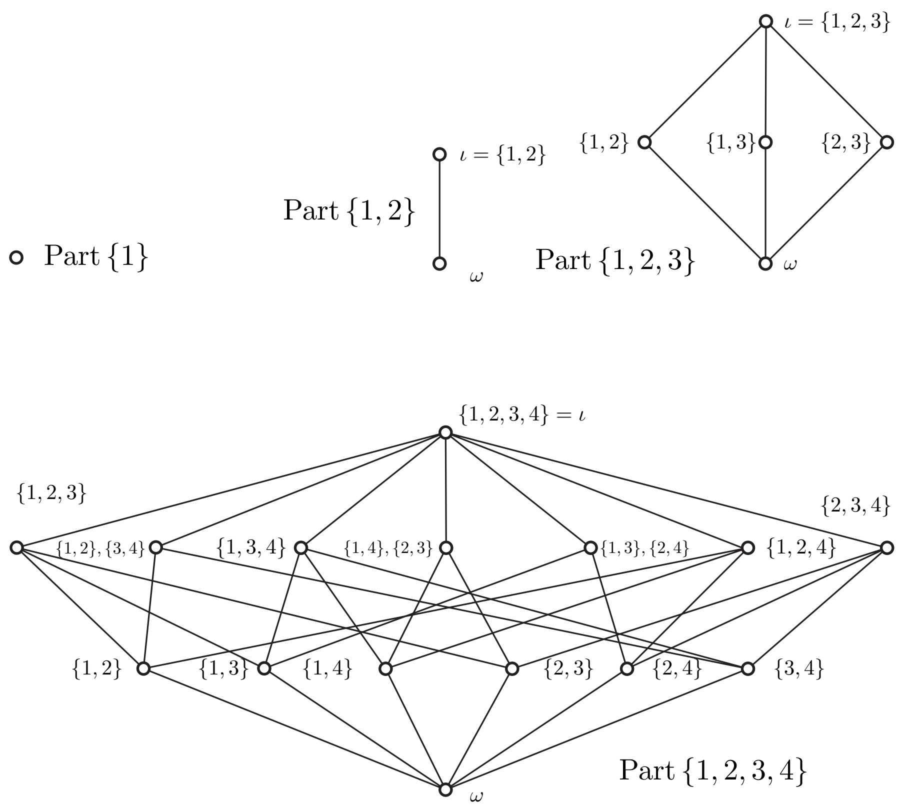

Figure 1.5: Part A, for $|A| \leq 4$.

It is easy to see that the definition of $\leq$ on P is well-defined and that P is an order. We will call P the *order associated with the preorder* Q.

Starting with a binary relation $\prec$ on the set Q, we can define the *reflexive-transitive closure* $\leq$ of $\prec$ by the formula: for a, $b \in Q$, let $a \leq b$ iff $a = b$ or if $a = x_0 \prec x_1 \prec \cdots \prec x_n = b$, for elements $x_1, \ldots, x_{n-1} \in Q$. Then $\leq$ is a preordering on Q. A *cycle* on Q is a sequence $x_1, \ldots, x_n \in Q$ satisfying $x_1 \prec x_2 \cdots \prec x_n \prec x_1$ $(n > 1)$. The preordering $\leq$ is an ordering iff there are no cycles.

1.2. Lattices and semilattices

1.2.1 Lattices

An order $\langle L, \leq \rangle$ is a *lattice* if $\bigvee\{a, b\}$ and $\bigwedge\{a, b\}$ exist, for all a, $b \in L$. A lattice L is *trivial* if it has only one element; otherwise, it is *nontrivial*.

We will use the notations

$$a \vee b = \bigvee \{a, b\},$$
$$a \wedge b = \bigwedge \{a, b\},$$

and call $\vee$ the *join* and $\wedge$ the *meet*. They are both *binary operations* that are *idempotent* ($a \vee a = a$ and $a \wedge a = a$), *commutative* ($a \vee b = b \vee a$ and $a \wedge b = b \wedge a$), *associative* ($(a \vee b) \vee c = a \vee (b \vee c)$ and $(a \wedge b) \wedge c = a \wedge (b \wedge c)$), and *absorptive* ($a \vee (a \wedge b) = a$ and $a \wedge (a \vee b) = a$). These properties of the operations are also called the *idempotent identities, commutative identities*, *associative identities*, and *absorption identities*, respectively. (Identities, in general, are introduced in Section 2.3.) As always in algebra, associativity makes it possible to write $a_1 \vee a_2 \vee \cdots \vee a_n$ without using parentheses (and the same for $\wedge$).

For instance, for A, $B \in \operatorname{Pow} X$, we have $A \vee B = A \cup B$ and $A \wedge B = A \cap B$. So $\operatorname{Pow} X$ is a lattice.

For Θ, $\Phi \in \operatorname{Part} A$, if we regard Θ and Φ as equivalence relations, then the meet formula is trivial: $\Theta \wedge \Phi = \Theta \cap \Phi$, but the formula for joins is a bit more complicated:

$x \equiv y \;(\Theta \vee \Phi)$ *iff there is a sequence* $x = z_0, z_1, \ldots, z_n = y$ *of elements of* A *such that* $z_i \equiv z_{i+1} \;(\Theta)$ *or* $z_i \equiv z_{i+1} \;(\Phi)$, *for each* $0 \leq i < n$.

So $\operatorname{Part} A$ is a lattice; it is called the *partition lattice* on A.

For an order P, the order $\operatorname{Down} P$ is a lattice: $A \vee B = A \cup B$ and $A \wedge B = A \cap B$, for A, $B \in \operatorname{Down} P$.

An (n-ary) *operation* on a nonempty set A is a map from A^n to A. For $n = 2$, we call the operation *binary*. An *algebra* is a nonempty set A with operations defined on A.

To treat lattices as algebras, define an algebra $\langle L, \vee, \wedge \rangle$ a *lattice* iff L is a nonempty set, $\vee$ and $\wedge$ are binary operations on L, both $\vee$ and $\wedge$ are idempotent, commutative, and associative, and they satisfy the two absorption identities. A lattice as an algebra and a lattice as an order are "equivalent" concepts: Let the order $\mathfrak{L} = \langle L, \leq \rangle$ be a lattice. Then the algebra $\mathfrak{L}^a = \langle L, \vee, \wedge \rangle$ is a lattice. Conversely, let the algebra $\mathfrak{L} = \langle L, \vee, \wedge \rangle$ be a lattice. Define $a \leq b$ iff $a \vee b = b$. Then $\mathfrak{L}^p = \langle L, \leq \rangle$ is an order, and the order $\mathfrak{L}^p$ is a lattice. For an order $\mathfrak{L}$ that is a lattice, we have $\mathfrak{L}^{ap} = \mathfrak{L}$; for an algebra $\mathfrak{L}$ that is a lattice, we have $\mathfrak{L}^{pa} = \mathfrak{L}$.

Note that for lattices as algebras, the Duality Principle takes on the following very simple form.

Duality Principle for Lattices. *Let* Φ *be a statement about lattices expressed in terms of* $\vee$ *and* $\wedge$. *The dual of* Φ *is the statement we get from* Φ *by interchanging* $\vee$ *and* $\wedge$. *If* Φ *is true for all lattices, then the dual of* Φ *is also true for all lattices.*

If the operations are understood, we will say that L (rather than $\langle L, \vee, \wedge \rangle$) is a lattice. The dual of the lattice L will be denoted by L^d; the order L^d is also a lattice.

A finite lattice L is *planar* if it is planar as an order (see Section 1.1.2). We have quite a bit of flexibility to construct a planar diagram for an order, but for a lattice, we are much more constrained because L has a zero, which must be the lowest element and a unit, which must be the highest element—contrast this with Figure 1.1. All lattices with five or fewer elements are planar; all but the five chains are shown in the first two rows of Figure 1.6 (see next page).

The third row of Figure 1.6 provides a good example of "good" and "bad" lattice diagrams; the two diagrams represent the same lattice, C_3^2. Planar diagrams are the best. Diagrams in which meets and joins are hard to figure out are not of much value.

In the last row of Figure 1.6 there are two more diagrams. The one on the left is not planar; nevertheless, it is very easy to work with: meets and joins are easy to see (the notation $M_3[C_3]$ will be explained in Section 5.1). The one on the right is not a lattice: the two black-filled elements have no join.

In this book we deal almost exclusively with finite lattices. Some concepts, however, are more natural to introduce in a more general context. An order $\langle L, \leq \rangle$ is a *complete lattice* if $\bigvee X$ and $\bigwedge X$ exist, for all $X \subseteq L$. All finite lattices are complete, of course.

1.2.2 Semilattices and closure systems

A *join-semilattice* $\langle S, \vee \rangle$ is an algebra: a nonempty set S with an idempotent, commutative, and associative binary operation $\vee$. In a join-semilattice $\langle S, \vee \rangle$, we can define an ordering: $a \leq_\vee b$ iff $a \vee b = b$. In the order $\langle S, \leq_\vee \rangle$, we have $\bigvee \{a, b\} = a \vee b$.

Similarly, a *meet-semilattice* $\langle S, \wedge \rangle$ is an algebra: a nonempty set S with an idempotent, commutative, and associative binary operation $\wedge$. In a meet-semilattice $\langle S, \wedge \rangle$, we can define an ordering: $a \leq_\wedge b$ iff $a \wedge b = a$. In the order $\langle S, \leq_\wedge \rangle$, we have $\bigwedge \{a, b\} = a \wedge b$.

If the operation is understood, we will say that S (rather than $\langle S, \vee \rangle$) is a join-semilattice; similarly, for a meet-semilattice.

If $\langle L, \vee, \wedge \rangle$ is a lattice, then $\langle L, \vee \rangle$ is a join-semilattice and $\langle L, \wedge \rangle$ is a meet-semilattice; moreover, the orderings $\leq_\vee$ and $\leq_\wedge$ agree. The converse also holds.

Let L be a lattice and let C be a nonempty subset of L with the property that for every $x \in L$, there is a smallest element $\overline{x}$ of C with $x \leq \overline{x}$. We call C a *closure system* in L, and $\overline{x}$ the *closure* of x in C.

Obviously, C, as a suborder of L, is a lattice: For x, $y \in C$, the meet in C is the same as the meet in L, and the join is

$$x \vee_C y = \overline{x \vee_L y}.$$

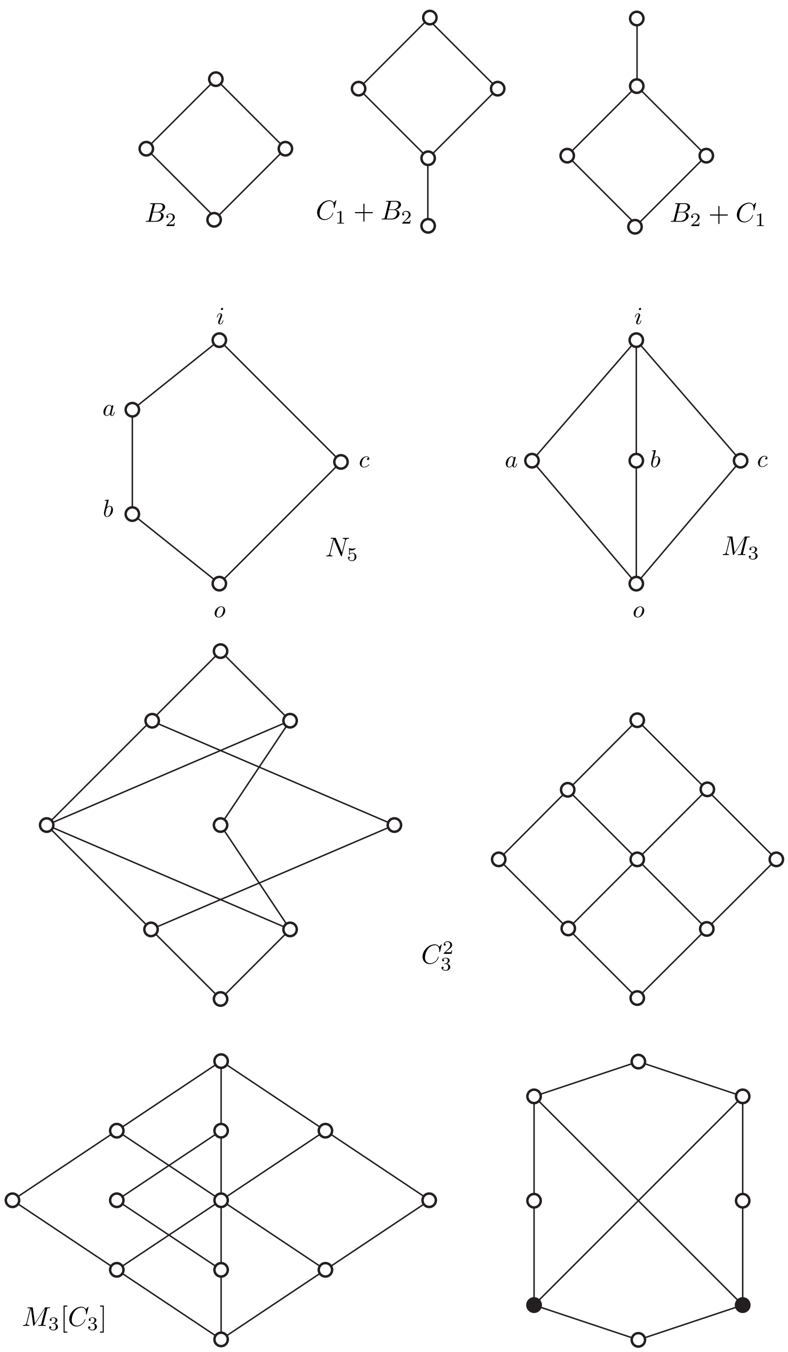

Figure 1.6: More diagrams.

Let L be a complete lattice and let C be $\bigwedge$-closed subset of L, that is, if $X \subseteq C$, then $\bigwedge X \in C$. (Since $\bigwedge \varnothing = 1$, such a subset is nonempty and contains the 1 of L.) Then C is a closure system in L, and for every $x \in L$,

$$\overline{x} = \bigwedge(\, y \in C \mid x \leq y \,).$$

1.3. Some algebraic concepts

1.3.1 Homomorphisms

The lattices $\mathfrak{L}_1 = \langle L_1, \vee, \wedge \rangle$ and $\mathfrak{L}_2 = \langle L_2, \vee, \wedge \rangle$ are *isomorphic* as algebras (in symbols, $\mathfrak{L}_1 \cong \mathfrak{L}_2$), and the map $\varphi \colon L_1 \to L_2$ is an *isomorphism* iff φ is one-to-one and onto and

(1) $$(a \vee b)\varphi = a\varphi \vee b\varphi,$$

(2) $$(a \wedge b)\varphi = a\varphi \wedge b\varphi,$$

for a, $b \in L_1$.

A map, in general, and a homomorphism, in particular, is called a *surjection* if it is onto, and a *bijection* if it is also one-to-one.

An isomorphism of a lattice with itself is called an *automorphism.* The automorphisms of a lattice L form a group $\operatorname{Aut} L$ under composition. A lattice L is *rigid* if the identity map is the only automorphism of L, that is, if $\operatorname{Aut} L$ is the one-element group.

It is easy to see that two lattices are isomorphic as orders iff they are isomorphic as algebras.

Let us define a *homomorphism* of the join-semilattice $\langle S_1, \vee \rangle$ into the join-semilattice $\langle S_2, \vee \rangle$ as a map $\varphi \colon S_1 \to S_2$ satisfying (1); similarly, for meet-semilattices, we require (2). A *lattice homomorphism* (or simply, *homomorphism*) φ of the lattice L_1 into the lattice L_2 is a map of L_1 into L_2 satisfying both (1) and (2). A homomorphism of a lattice into itself is called an *endomorphism.* A one-to-one homomorphism will also be called an *embedding.*

Note that meet-homomorphisms, join-homomorphisms, and (lattice) homomorphisms are all isotone.

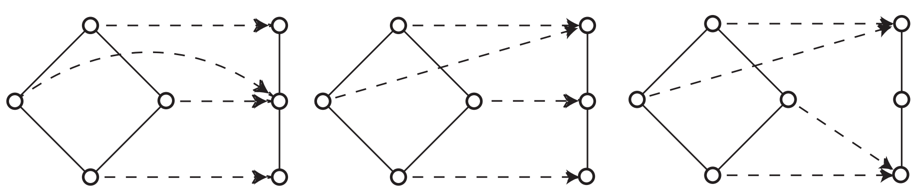

Figure 1.7: Morphisms.

Figure 1.7 shows three maps of the four-element lattice B_2 into the three-element chain C_3. The first map is isotone but it is neither a meet- nor a join-homomorphism. The second map is a join-homomorphism but is not a meet-homomorphism, thus not a homomorphism. The third map is a (lattice) homomorphism.

Various versions of homomorphisms and embeddings will be used. For instance, for lattices and join-semilattices, there are also $\{\vee, 0\}$-homomorphism, and so on, with obvious meanings. An onto homomorphism φ is also called *surjective*, while a one-to-one homomorphism is called *injective*; it is the same as an *embedding*. For bounded lattices, we often use $\{0, 1\}$-homomorphisms and $\{0, 1\}$-embeddings.

It should always be clear from the context what kind of homomorphism we are considering. If we say, "let φ be a homomorphism of K into L", where K and L are lattices, then φ is a lattice homomorphism, unless otherwise stated.

1.3.2 Sublattices

A *sublattice* $\langle K, \vee, \wedge \rangle$ of the lattice $\langle L, \vee, \wedge \rangle$ is defined on a nonempty subset K of L with the property that a, $b \in K$ implies that $a \vee b$, $a \wedge b \in K$ (the operations $\vee$, $\wedge$ are formed in $\langle L, \vee, \wedge \rangle$), and the $\vee$ and the $\wedge$ of $\langle K, \vee, \wedge \rangle$ are restrictions to K of the $\vee$ and the $\wedge$ of $\langle L, \vee, \wedge \rangle$, respectively. Instead of "$\langle K, \vee, \wedge \rangle$ is a sublattice of $\langle L, \vee, \wedge \rangle$", we will simply say that "K is a sublattice of L"—in symbols, $K \leq L$. Of course, a sublattice of a lattice is again a lattice. If K is a sublattice of L, then we call L an *extension* of K—in symbols, $L \geq K$.

For a bounded lattice L, the sublattice K is a $\{0, 1\}$-sublattice if $K \leq L$ and the 0 and 1 of L are in K. Similarly, we can define a $\{0\}$-sublattice, and so on.

For every $H \subseteq L$, $H \neq \varnothing$, there is a smallest sublattice $\operatorname{sub}(H) \subseteq L$ containing H called the *sublattice of* L *generated by* H. We say that H is a *generating set* of $\operatorname{sub}(H)$.

The subset K of the lattice L is called *convex* iff a, $b \in K$, $c \in L$, and $a \leq c \leq b$ imply that $c \in K$. We can add the adjective "convex" to sublattices, extensions, and embeddings. A sublattice K of the lattice L is *convex* if it a convex subset of L. Let L be an extension of K; then L is a *convex extension* if K is a convex sublattice. An embedding is *convex* if the image is a convex sublattice.

For a, $b \in L$, $a \leq b$, the *interval*

$$[a, b] = \{\, x \mid a \leq x \leq b \,\}$$

is an important example of a convex sublattice. An interval $[a, b]$ is *trivial* if $a = b$. The smallest nontrivial intervals are called *prime*; that is, $[a, b]$ is *prime* iff $a \prec b$. Another important example of a convex sublattice is an ideal. A nonempty subset I of L is an *ideal* iff it is a down-set with the property:

(Id) a, $b \in I$ implies that $a \vee b \in I$.

An ideal I of L is *proper* if $I \neq L$. Since the intersection of any number of ideals is an ideal, unless empty, we can define $\mathrm{id}(H)$, the *ideal generated by a subset* H of the lattice L, provided that $H \neq \varnothing$. If $H = \{a\}$, we write $\mathrm{id}(a)$ for $\mathrm{id}(\{a\})$, and call it a *principal ideal*. Obviously, $\mathrm{id}(a) = \{\, x \mid x \leq a \,\} = \downarrow a$. So instead of $\mathrm{id}(a)$, we could use $\downarrow a$; many do, who work in categorical aspects of lattice theory—and use id for the identity map.

The set $\mathrm{Id}\, L$ of all ideals of L is an order under set inclusion, and as an order it is a lattice. In fact, for I, $J \in \mathrm{Id}\, L$, the lattice operations in $\mathrm{Id}\, L$ are $I \vee J = \mathrm{id}(I \cup J)$ and $I \wedge J = I \cap J$. So we obtain the formula for the ideal join:

$x \in I \vee J$ *iff* $x \leq i \vee j$, *for some* $i \in I$, $j \in J$.

We call $\mathrm{Id}\, L$ the *ideal lattice* of L. Now observe the formulas: $\mathrm{id}(a) \vee \mathrm{id}(b) = \mathrm{id}(a \vee b)$, $\mathrm{id}(a) \wedge \mathrm{id}(b) = \mathrm{id}(a \wedge b)$. Since $a \neq b$ implies that $\mathrm{id}(a) \neq \mathrm{id}(b)$, these yield:

The map $a \mapsto \mathrm{id}(a)$ *embeds* L *into* $\mathrm{Id}\, L$.

Since the definition of an ideal uses only $\vee$ and $\leq$, it applies to any join-semilattice S. The order $\mathrm{Id}\, S$ is a join-semilattice and the same join formula holds as the one for lattices. Since the intersection of two ideals could be empty, $\mathrm{Id}\, S$ is not a lattice, in general. However, for a $\{\vee, 0\}$-semilattice (a join-semilattice with zero), $\mathrm{Id}\, S$ is a lattice.

For (join-semi-) lattices S and T, let $\varepsilon \colon S \to T$ be an embedding. We call ε an *ideal-embedding* if $S\varepsilon$ is an ideal of T. Then, of course, for any ideal I of S, we have that $I\varepsilon$ is an ideal of T. Ideal-embeddings play a major role in Chapter 18.

By dualizing, we get the concepts of *filter*, $\mathrm{fil}(H)$, the *filter generated by a subset* H of the lattice L, provided that $H \neq \varnothing$, *principal filter* $\mathrm{fil}(a)$, and so on.

1.3.3 Congruences

An equivalence relation Θ on a lattice L is called a *congruence relation*, or *congruence*, of L iff $a \equiv b \pmod{\Theta}$ and $c \equiv d \pmod{\Theta}$ imply that

$$(\mathrm{SP}_\vee) \qquad a \wedge c \equiv b \wedge d \quad (\Theta),$$

$$(\mathrm{SP}_\wedge) \qquad a \vee c \equiv b \vee d \quad (\Theta)$$

(*Substitution Properties*). Trivial examples are ω and ι (introduced in Section 1.1.4). As in Section 1.1.4, for $a \in L$, we write a/Θ for the *congruence class* containing a; observe that a/Θ is a convex sublattice.

If L is a lattice, $K \leq L$, and Θ a congruence on L, then $\Theta\rceil K$, the *restriction of* Θ *to* K, is a congruence of K. Formally, for x, $y \in K$,

$$x \equiv y \,(\Theta\rceil K) \quad \text{iff} \quad x \equiv y \,(\Theta) \text{ in } L.$$

We call Θ *discrete* on K if $\Theta\rceil K = \omega$.

Sometimes it is tedious to compute that a binary relation is a congruence relation. Such computations are often facilitated by the following lemma (Grätzer and E. T. Schmidt [56] and Maeda [96]):

Lemma 1.1. *A reflexive binary relation Θ on a lattice L is a congruence relation iff the following three properties are satisfied, for x, y, z, $t \in L$:*

(i) $x \equiv y \pmod{\Theta}$ *iff* $x \wedge y \equiv x \vee y \pmod{\Theta}$.

(ii) $x \leq y \leq z$, $x \equiv y \pmod{\Theta}$, *and* $y \equiv z \pmod{\Theta}$ *imply that* $x \equiv z \pmod{\Theta}$.

(iii) $x \leq y$ *and* $x \equiv y \pmod{\Theta}$ *imply that* $x \wedge t \equiv y \wedge t \pmod{\Theta}$ *and* $x \vee t \equiv y \vee t \pmod{\Theta}$.

Let $\operatorname{Con} L$ denote the set of all congruence relations on L ordered by set inclusion (remember that we can view $\Theta \in \operatorname{Con} L$ as a subset of L^2).

Theorem 1.2. $\operatorname{Con} L$ *is a lattice. For* Θ, $\Phi \in \operatorname{Con} L$,

$$\Theta \wedge \Phi = \Theta \cap \Phi.$$

The join, $\Theta \vee \Phi$, *can be described as follows:*
$x \equiv y \pmod{\Theta \vee \Phi}$ *iff there is a sequence*

$$z_0 = x \wedge y \leq z_1 \leq \cdots \leq z_n = x \vee y$$

of elements of L *such that* $z_i \equiv z_{i+1} \pmod{\Theta}$ *or* $z_i \equiv z_{i+1} \pmod{\Phi}$, *for every* i *with* $0 \leq i < n$.

Remark. For the binary relations α and β on a set A, we define the binary relation $\alpha \circ \beta$, the *product of* α *and* β, as follows: for a, $b \in A$, the relation $a \; \alpha \circ \beta \; b$ holds iff $a \; \alpha \; x$ and $x \; \beta \; b$, for some $x \in A$. The relation $\Theta \vee \Phi$ is formed by repeated products. Theorem 1.2 strengthens this statement.

The integer n in Theorem 1.2 can be restricted for some congruence joins. We call the congruences Θ and Φ *permutable* if $\Theta \vee \Phi = \Theta \circ \Phi$. A lattice L is *congruence permutable* if any pair of congruences of L are permutable. The chain C_n is congruence permutable iff $n \leq 2$.

$\operatorname{Con} L$ is called the *congruence lattice* of L. Observe that $\operatorname{Con} L$ is a sublattice of $\operatorname{Part} L$; that is, the join and meet of congruence relations as congruence relations and as equivalence relations (partitions) coincide.

If L is nontrivial, then $\operatorname{Con} L$ contains the two-element sublattice $\{\omega, \iota\}$. If $\operatorname{Con} L = \{\omega, \iota\}$, we call the lattice L *simple*. All the nontrivial lattices of Figure 1.5 are simple. Of the many lattices of Figure 1.6, only M_3 is simple.

Given a, $b \in L$, there is a smallest congruence $\operatorname{con}(a, b)$—called a *principal congruence*—under which $a \equiv b$. The formula

$$\Theta = \bigvee(\, \operatorname{con}(a, b) \mid a \equiv b \pmod{\Theta}\,) \tag{3}$$

is trivial but important. For $H \subseteq L$, the smallest congruence under which H is in one class is $\operatorname{con}(H) = \bigvee(\operatorname{con}(a,b) \mid a,\ b \in H)$.

Homomorphisms and congruence relations express two sides of the same phenomenon. Let L be a lattice and let Θ be a congruence relation on L. Let $L/\Theta = \{ a/\Theta \mid a \in L \}$. Define $\wedge$ and $\vee$ on L/Θ by $a/\Theta \wedge b/\Theta = (a \wedge b)/\Theta$ and $a/\Theta \vee b/\Theta = (a \vee b)/\Theta$. The lattice axioms are easily verified. The lattice L/Θ is the *quotient lattice* of L modulo Θ.

Lemma 1.3. *The map*

$$\varphi_\Theta \colon x \mapsto x/\Theta, \qquad \textit{for } x \in L,$$

is a homomorphism of L onto L/Θ.

The lattice K is a *homomorphic image* of the lattice L iff there is a homomorphism of L *onto* K. Theorem 1.4 (illustrated in Figure 1.8) states that any quotient lattice is a homomorphic image. To state it, we need one more concept: Let $\varphi\colon L \to L_1$ be a homomorphism of the lattice L into the lattice L_1, and define the binary relation Θ on L by $x \mathrel{\Theta} y$ iff $x\varphi = y\varphi$; the relation Θ is a congruence relation of L, called the *kernel* of φ, in notation, $\ker(\varphi) = \Theta$.

Theorem 1.4 (Homomorphism Theorem)**.** *Let L be a lattice. Any homomorphic image of L is isomorphic to a suitable quotient lattice of L. In fact, if $\varphi\colon L \to L_1$ is a homomorphism of L onto L_1 and Θ is the kernel of φ, then $L/\Theta \cong L_1$; an isomorphism (see Figure 1.8) is given by $\psi\colon x/\Theta \mapsto x\varphi$, for $x \in L$.*

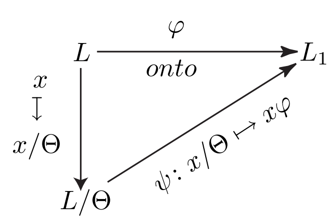

Figure 1.8: The Homomorphism Theorem.

We also know the congruence lattice of a homomorphic image:

Theorem 1.5 (Second Isomorphism Theorem)**.** *Let L be a lattice and let Θ be a congruence relation of L. For any congruence Φ of L such that $\Phi \geq \Theta$, define the relation Φ/Θ on L/Θ by*

$$x/\Theta \equiv y/\Theta \quad (\Phi/\Theta) \qquad \textit{iff} \qquad x \equiv y \quad (\Phi).$$

Then Φ/Θ *is a congruence of* L/Θ*. Conversely, every congruence* Ψ *of* L/Θ *can be (uniquely) represented in the form* $\Psi = \Phi/\Theta$*, for some congruence* $\Phi \geq \Theta$ *of* L*. In particular, the congruence lattice of* L/Θ *is isomorphic with the interval* $[\Theta, \iota]$ *of the congruence lattice of* L*.*

Let L be a bounded lattice. A congruence Θ of L *separates* 0 if $0/\Theta = \{0\}$, that is, $x \equiv 0\ (\Theta)$ implies that $x = 0$. Similarly, a congruence Θ of L *separates* 1 if $1/\Theta = \{1\}$, that is, $x \equiv 1\ (\Theta)$ implies that $x = 1$. We call the lattice L *non-separating* if 0 and 1 are not separated by any congruence $\Theta \neq \omega$.

Similarly, a homomorphism φ of the lattices L_1 and L_2 with zero is 0-*separating* if $0\varphi = 0$, but $x\varphi \neq 0$, for $x \neq 0$.

Chapter 7

The Dilworth Theorem

In this book we discuss a subfield of Lattice Theory that started with the following result—a converse of the Funayama-Nakayama [21] result, Theorem 3.3 (page 37).

Theorem 7.1 (Dilworth Theorem). *Every finite distributive lattice D can be represented as the congruence lattice of a finite lattice L.*

Our presentation is based on Grätzer and Schmidt [57], where the first proof appeared. In his book (Crawley and Dilworth [10]), Dilworth reproduces the proof from [57]. It is clear from his recollections in [9] that his thinking was very close to ours.

In this chapter we follow Grätzer and Lakser [34] (published in [26]), and prove this result based on the discussion of chopped lattices in Chapter 4, a simpler proof than the one in [57]. We will also prove that L can be constructed as a sectionally complemented lattice, as stated in [57].

7.1. The representation theorem

By Theorem 4.6, to prove the Dilworth Theorem, it is sufficient to verify the following:

Theorem 7.2. *Let D be a finite distributive lattice. Then there exists a chopped lattice M such that* $\operatorname{Con} M$ *is isomorphic to D.*

Using the equivalence of nontrivial finite distributive lattices and finite orders (see Section 2.5.2) and using the notation $\operatorname{Con}_{\mathrm{J}} M$ (see Section 3.2) for the order of join-irreducible congruences, we can rephrase Theorem 7.2 as follows:

Theorem 7.3. *Let P be a finite order. Then there exists a chopped lattice M such that* $\mathrm{Con}_{\mathrm{J}}\, M$ *is isomorphic to* P.

We are going to prove the Dilworth Theorem in this form.

7.2. *Proof-by-Picture*

The basic gadget for the construction is the lattice $N_6 = N(p, q)$ of Figure 7.1. The lattice $N(p, q)$ has three congruence relations, namely, ω, ι, and Θ, where Θ is the congruence relation with congruence classes $\{0, q_1, q_2, q\}$ and $\{p_1, p(q)\}$, indicated by the dashed line. Thus $\mathrm{con}(p_1, 0) = \iota$. In other words, $p_1 \equiv 0$ "implies" that $q_1 \equiv 0$, but $q_1 \equiv 0$ "does not imply" that $p_1 \equiv 0$. We will use the "gadget" $N_6 = N(p, q)$ to achieve such congruence-forcing.

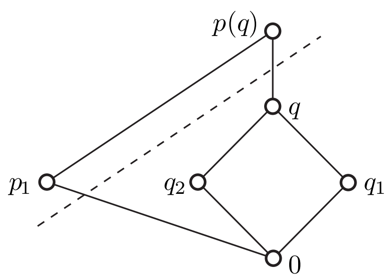

Figure 7.1: The lattice $N_6 = N(p, q)$ and the congruence Θ.

To convey the idea how to prove Theorem 7.3, we present three small examples in which we construct the chopped lattice M from copies of $N(p, q)$.

Example 1: The three-element chain. Let $P = \{a, b, c\}$ with $c \prec b \prec a$. We take two copies of the gadget, $N(a, b)$ and $N(b, c)$; they share the ideal $I = \{0, b_1\}$; see Figure 7.2. So we can merge them (in the sense of Section 4.1) and form the chopped lattice

$$M = \mathrm{Merge}(N(a, b), N(b, c))$$

as shown in Figure 7.2.

The congruences of M are easy to find. The isomorphism $P \cong \mathrm{Con}_{\mathrm{J}}\, M$ is given by $x_1 \mapsto \mathrm{con}(0, x)$, for $x \in P$.

The congruences of M can be described by a compatible congruence vector $\langle \Theta_{a,b}, \Theta_{b,c} \rangle$ (see Section 4.3), where $\Theta_{a,b}$ is a congruence of the lattice $N(a, b)$ and $\Theta_{b,c}$ is a congruence of the lattice $N(b, c)$, subject to the condition that $\Theta_{a,b}$ and $\Theta_{b,c}$ agree on I. Looking at Figure 7.1, we see that if the shared congruence on I is ω $(= \omega_I)$, then we must have $\Theta_{a,b} = \omega$ $(= \omega_{N(a,b)})$ and $\Theta_{b,c} = \omega$ $(= \omega_{N(b,c)})$ or $\Theta_{b,c} = \Theta$ on $N(b, c)$. If the shared congruence on I is ι $(= \iota_I)$, then we must have $\Theta_{a,b} = \Theta$ or $\Theta_{a,b} = \iota$ $(= \iota_{N(a,b)})$ on $N(a, b)$

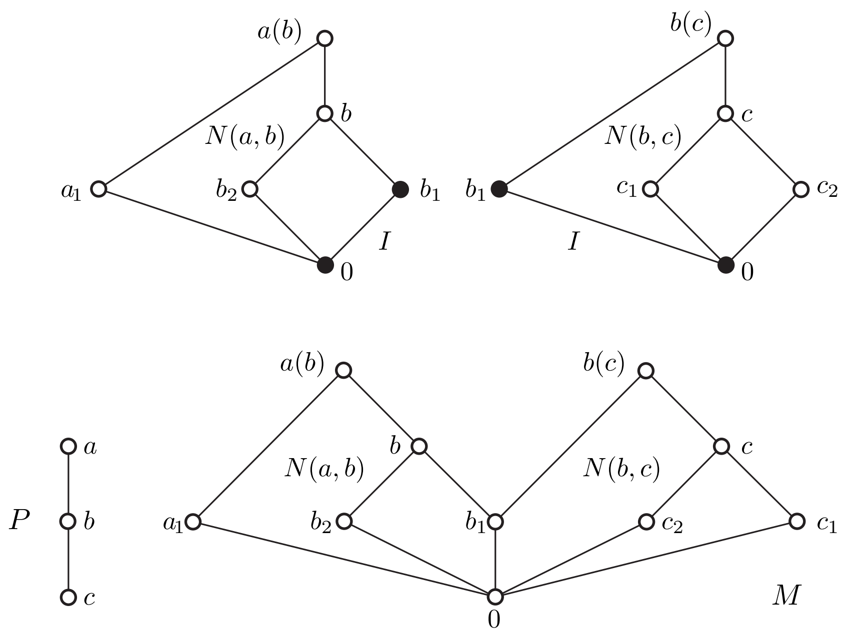

Figure 7.2: The chopped lattice M for $P = C_3$.

and $\Theta_{b,c} = \iota$ $(= \iota_{N(b,c)})$ on $N(b, c)$. So there are three congruences distinct from ω: $\langle \omega, \Theta \rangle$, $\langle \Theta, \iota \rangle$, $\langle \iota, \iota \rangle$. Thus all the join-irreducible congruences form the three-element chain.

Example 2: The three-element order P_V of Figure 7.3. (We call P_V the "order V.") We take two copies of the gadget, $N(b, a)$ and $N(c, a)$; they share the ideal $J = \{0, a_1, a_2, a\}$; we merge them to form the chopped lattice

$$M_V = \mathrm{Merge}(N(b, a), N(c, a)),$$

see Figure 7.3. Again, the isomorphism $P_V \cong \mathrm{Con_J}\, M_V$ is given by $x_1 \mapsto \mathrm{con}(0, x)$, for $x \in P_V$.

Example 3: The three-element order P_H of Figure 7.4. (We call P_H the "order hat.") We take two copies of the gadget, $N(a, b)$ and $N(a, c)$; they share the ideal $J = \{0, a_1\}$; we merge them to form the chopped lattice

$$M_V = \mathrm{Merge}(N(a, b), N(a, c)),$$

see Figure 7.4. Again, the isomorphism $P_H \cong \mathrm{Con_J}\, M_H$ is given by $x_1 \mapsto \mathrm{con}(0, x)$, for $x \in P_V$.

The reader should now be able to picture the general proof: instead of the few atoms in these examples, we start with enough atoms to reflect the

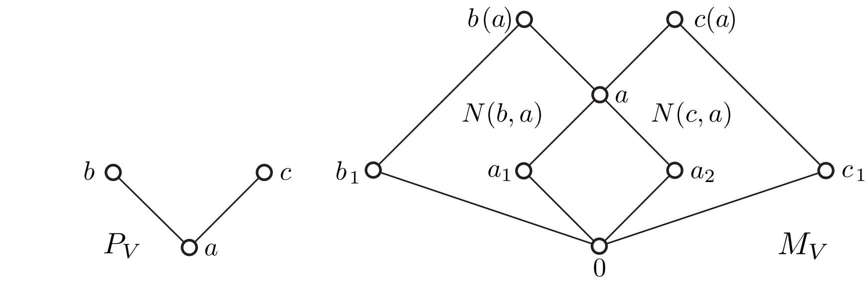

Figure 7.3: The chopped lattice for the order V.

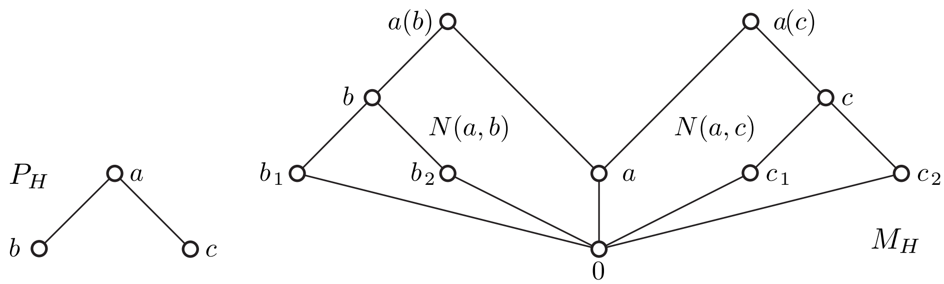

Figure 7.4: The chopped lattice for the order hat.

structure of P, see Figure 7.5. Whenever $b \prec a$ in P, we build a copy of $N(a,b)$, see Figure 7.6.

7.3. Computing

For a finite order P, let Max be the set of maximal elements in P. We form the set

$$M_0 = \{0\} \cup \{\, p_1 \mid p \in \text{Max}\,\} \cup \bigcup(\,\{a_1, a_2\} \mid a \in P - \text{Max}\,)$$

consisting of 0, the maximal elements of P indexed by 1, and two copies of the nonmaximal elements of P, indexed by 1 and 2. We make M_0 a meet-semilattice by defining $\inf\{x, y\} = 0$ if $x \neq y$, as illustrated in Figure 7.5. Note that $x \equiv y \pmod{\Theta}$ and $x \neq y$ imply that $x \equiv 0 \pmod{\Theta}$ and $y \equiv 0 \pmod{\Theta}$ in M_0; therefore, the congruence relations of M_0 are in one-to-one correspondence with subsets of P. Thus $\text{Con}\, M_0$ is a boolean lattice whose atoms are associated with atoms of M_0; the congruence Φ_x associated with the atom x has only one nontrivial block $\{0, x\}$.

We construct an extension M of M_0 as follows:

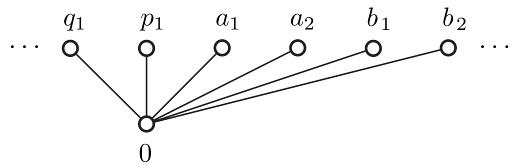

Figure 7.5: The chopped lattice M_0.

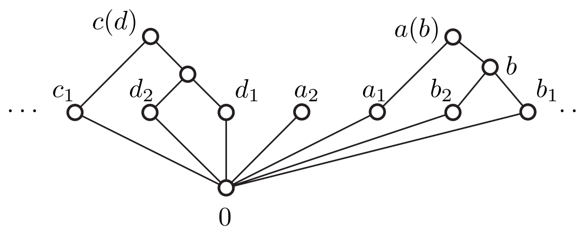

Figure 7.6: The chopped lattice M.

The chopped lattice M consists of four kinds of elements: (i) the zero, 0; (ii) for all maximal elements p of P, the element p_1; (iii) for any nonmaximal element p of P, three elements: p, p_1, p_2; (iv) for each pair p, $q \in P$ with $p \succ q$, a new element, $p(q)$. For p, $q \in P$ with $p \succ q$, we set $N_6 = N(p, q) = \{0, p_1, q, q_1, q_2, p(q)\}$.

For x, $y \in M$, let us define $x \leq y$ to mean that, for some p, $q \in P$ with $p \succ q$, we have x, $y \in N(p, q)$ and $x \leq y$ in the lattice $N(p, q)$. It is easily seen that $x \leq y$ does not depend on the choice of p and q, and that $\leq$ is an ordering. Since, under this ordering, all $N(p, q)$ and $N(p, q) \cap N(p', q')$ ($p \succ q$ and $p' \succ q'$ in P) are lattices and x, $y \in M$, $x \in N(p, q)$, and $y \leq x$ imply that $y \in N(p, q)$, we conclude that M is a chopped lattice; in fact, it is a union of the ideals $N(p, q)$ with $p \succ q$ in P, and two such distinct ideals intersect in a one-, two-, or four-element ideal.

Since the chopped lattice M is atomistic, Corollary 3.8 applies. If $p_i \Rightarrow q_j$ in M, for p, $q \in P$ and i, $j \in \{1, 2\}$, then $p \geq q$ in P, and conversely. So the equivalence classes of the atoms under the preordering $\Rightarrow$ form an order isomorphic to $\operatorname{Down} P$. This completes the verification that $\operatorname{Con_J} M \cong P$, and therefore, of Theorem 7.3.

7.4. Sectionally complemented lattices

Let M be the chopped lattice described in Section 7.3. Since all $N(p, q)$-s are sectionally complemented, so is M.

In Grätzer and Schmidt [57] the following result is proved:

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