Short Contents

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Preface

My book, *General Lattice Theory,* was published in 1978. Its goal: "to discuss in depth the basics of general lattice theory". Each chapter concluded with a section, *Further Topics and References,* providing brief outlines of, and references to, related topics. Each chapter contained a long list of open problems.

The *second edition* appeared twenty years later, in 1998. It included the material of the first edition, and a series of appendices. The first, *Retrospective,* reviewed developments of the 20 years between the two editions, especially, solutions of the open problems proposed in the first edition. The other seven appendices surveyed new fields. They were written by the best experts available. Obviously, I could no longer command an overview of all of lattice theory. The book provided foundation, the appendices surveyed contemporary research.

The explosive growth of the field continued. While the nineteen sixties provided under 1,500 papers and books, the seventies 2,700, the eighties over 3,200, the nineties almost 3,600, and the first decade of this century about 4,000. As a result, it became almost inevitable that we split the book into two volumes.

This book, *Lattice Theory: Foundation,* lays the foundations of the field. There are no *Retrospectives* and no lists of open problems. Its companion volume, *Lattice Theory: Special Topics and Applications*, completes the picture; it is written by experts in the various topics covered.

To help the readers of this book to acquire a wider view, almost a thousand exercises are provided. And there are over forty *diamond sections*, brief sections marked by the symbol $\diamondsuit$, that provide brief glimpses into research fields beyond the horizon of this book.

Contributors

The following mathematicians contributed diamond sections:

- Kira Adaricheva (Sections VI.2.7 and VII.2.7);
- Gábor Czédli (Section V.1.9);
- Brian A. Davey and Miroslav Haviar (Section VI.2.8);
- Kalle Kaarli (Section II.2.5);
- Jimmie D. Lawson (Section I.3.16);
- Joseph P. S. Kung (Section V.5.13);
- Tibor Katriňák (Section V.1.8);
- J. B. Nation (Section IV.4.4);
- Hilary A. Priestley (Section II.5.6);
- Aleš Pultr (Section II.5.7);
- Manfred Stern (Section V.2.6);
- Friedrich Wehrung (Sections III.4.4, IV.5.5, V.5.11, and V.5.12).

I am deeply appreciative to all of them.

Acknowledgements

To keep this *Preface* short, I put the history of this book and the very extensive acknowledgements into the *Afterword*. But let me repeat one point made there. I started writing this book in 1968. In the forty plus years of this endeavor, I received help from hundreds of mathematicians. I am forever grateful.

Foreword

In the first half of the nineteenth century, George Boole's attempt to formalize propositional logic led to the concept of boolean algebras. While investigating the axiomatics of boolean algebras at the end of the nineteenth century, Charles S. Pierce and Ernst Schröder found it useful to introduce the lattice concept. Independently, Richard Dedekind's research on ideals of algebraic numbers led to the same discovery. In fact, Dedekind also introduced modularity, a weakened form of distributivity. Although some of the early results of these mathematicians and of Edward V. Huntington are very elegant and far from trivial, they did not attract the attention of the mathematical community.

It was Garrett Birkhoff's work in the mid-1930s that started the general development of lattice theory. In a brilliant series of papers, he demonstrated the importance of lattice theory and showed that it provides a unifying framework for hitherto unrelated developments in many mathematical disciplines. Birkhoff himself, Valère Glivenko, Karl Menger, John von Neumann, Oystein Ore, and others had developed enough of this new field for Birkhoff to attempt to "sell" it to the general mathematical community, which he did with astonishing success in the first edition of his *Lattice Theory.* The further early development of the subject matter can best be followed by comparing the first, second, and third editions of his book: G. Birkhoff [65] (1940), [70] (1948), and [71] (1967).

The goal of the present volume can be stated very simply: to discuss in depth the foundation of lattice theory. I tried to include the most important results and research methods that form the basis of all the work in this field.

Special topics and applications of lattice theory are presented in the companion volume. As I mentioned in the *Preface,* over forty *diamond sections* whet the appetite of the reader by providing brief glimpses into areas not covered in this volume.

In my view, distributive lattices have played a many-faceted role in the development of lattice theory. Historically, lattice theory started with (boolean) distributive lattices; as a result, the theory of distributive lattices is one of the most extensive and most satisfying chapters of lattice theory. Distributive lattices have provided the motivation for many results in general lattice theory. Several conditions on lattices and on elements and ideals of lattices are weakened forms of distributivity. Therefore, a thorough understanding of distributive lattices is indispensable for work in lattice theory.

This viewpoint moved me to break with the traditional approach to lattice theory, which proceeds from orders to general lattices, semimodular lattices, modular lattices, and, finally, to distributive lattices. My approach has the added advantage that the reader reaches interesting and deep results early in the book.

Chapter I develops the basic concepts of orders and lattices. Diagrams are emphasized because I believe that an important part of learning lattice theory is the acquisition of skill in drawing diagrams. This point of view is stressed throughout the book by about 130 diagrams (heeding Alice's advice: "and what is the use of a book without pictures", L. Carroll [1865]); the reader would be well advised to draw lots more while reading the book.

A special feature of this chapter is a detailed development of free lattices generated by a partial lattice over an arbitrary variety; this is one of the most important research tools of lattice theory.

Diamond section topics include tolerances, continuous lattices, the characterization theorem of congruence lattices of universal algebras, finitely presented lattices, and various axiom systems for lattices.

Chapter II develops distributive lattices including representation theorems, congruences, boolean algebras, and topological representations. The last section is a brief introduction to the theory of distributive lattices with pseudocomplementation. While the theory of distributive lattices is developed in detail, the reader should keep in mind that the purpose of this chapter is, basically, to serve as a model for the rest of lattice theory.

Diamond section topics include polynomial completeness, Priestley spaces, frames (a lattice theoretic approach to topology), and generalizations of Stone algebras.

In **Chapter III**, we discuss congruences and ideals of general lattices. The various types of ideals discussed all imitate to some extent the behavior of ideals in distributive lattices.

There is only one *diamond section*, discussing infinite direct decompositions of complete lattices.

Lattice constructions play a central role in lattice theory. **Chapter IV** discusses a construction of old: gluing (1941) and the newer One-Point Extension (1992), the crucial chopped lattices (from the 1970s), and the newest construction (1999): boolean triples.

Diamond section topics include generalized gluing constructions, congruence lattices of (i) finite lattices, (ii) finite lattices in special classes, (iii) more than one finite lattice, (iv) general lattices, (v) complete lattices; furthermore, independence results, tensor products, and congruence-permutable, congruence-preserving extensions.

After presenting the basic facts concerning modular and semimodular lattices, **Chapter V** investigates in detail the connection between lattice theory and geometry. We develop the theory of geometric lattices, in particular, direct decompositions and geometric lattices arising out of geometries and graphs. As an important example, we investigate partition lattices. The last section deals with complemented modular lattices and projective geometries, including the Coordinatization Theorem and Frink's Embedding Theorem.

Diamond section topics include pseudocomplementation in modular lattices, identities of submodule lattices, consistency (a generalization of modularity different from semimodularity), type 2 and 3 congruence lattices for universal algebras, special topics on partition lattices, coordinatization results of sectionally complemented modular lattices, the dimension monoid of a lattice (a precursor of the congruence lattice), and Dilworth's covering theorem.

Chapter VI deals with varieties of lattices. It covers the basic properties, including Jónsson's Lemma, the lattice of varieties of lattices, equational bases, and the Amalgamation Property.

Diamond section topics include products of varieties, lattices of (quasi-) equational theories, and modified Priestley dualities.

Chapter VII presents free products of lattices, including the Structure Theorem, the Common Refinement Property, sublattices of a free lattice, reduced free products, and hopfian lattices.

Diamond section topics include amalgamated free products, the word problem for modular lattices, transferable lattices and finite sublattices of a free lattice, semidistributive lattices, and Dean's Lemma.

The exercises form an integral part of the book; do not leave a section without doing a good number of them.

The Bibliography contains about 700 entries; it is not a comprehensive bibliography of this field. With a few exceptions, it contains only items referred to in the text. To find the references for a topic, use the AMS online database, *MathSciNet,* or turn to *Zentralblatt.*

A very detailed *Index* and the *Glossary of Notation* should help the reader in finding where a concept or notation is first introduced. For names and concepts, such as "Jónsson, B." and "Priestley space", use the *Index*; symbols, such as $\operatorname{Con} L$, $\operatorname{rank}(p)$, should be looked up in the *Glossary.*

I assume a rudimentary knowledge of basic set theory and algebra.

Notation

More difficult exercises are marked by *. Theorems (lemmas) presented without proofs are marked by the diamond symbol $\diamondsuit$.

Section 5 refers to a section in the chapter you are reading, whereas Section II.5 refers to a section in Chapter II. Exercise 5.2 refers to the second exercise in Section 5 of the chapter you are in, while Exercise V.5.2 refers to the second exercise in Section 5 of Chapter V. Finally, Lemma 403(ii) refers to the second statement of Lemma 403 and Definition 41(i) to the first condition of Definition 41.

If you are curious how the mathematical notational system used in this book developed, consult the *Afterword*.

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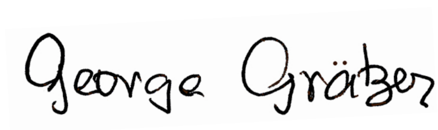

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Glossary of Notation

Symbol	Explanation	Page
$\mathfrak{A}$, $(A; F)$	universal algebra	12
$(A, ^{-})$	geometry	349
(A, L)	projective plane	393
$\mathbf{Amal}(\mathbf{K})$	amalgamation class of $\mathbf{K}$	461
(Assoc)	associativity condition for a binary operation	10
(ASym)	antisymmetry condition for binary relations	1
$\mathrm{Atom}(L)$	set of atoms of the lattice L	101
$\mathrm{Aut}\, L$	automorphism group of L	29
$\mathrm{breadth}(P)$	breadth of an order	8
$(B; \vee, \wedge, ', 0, 1)$	boolean algebra	99
B_1	boolean lattice with 2 elements	99
B_n	boolean lattice with n atoms	99
$\mathrm{BR}\, L$	generalized boolean lattice R-generated by L	152
$\mathrm{con}(a, b)$	smallest congruence under which $a \equiv b$	39
$\mathrm{con}(H)$	smallest congruence collapsing H	39
$\mathrm{con}(\mathfrak{p})$	principal congruence for the prime interval $\mathfrak{p}$	213
(C)	covering condition for free product	472
C_n	n-element chain	4
$\mathrm{Cen}\, L$	center of the lattice L	250
CEP	Congruence Extension Property	42
(CID)	Complete Infinite Distributive Identity	164
(Com)	commutativity condition for a binary operation	10
$\mathrm{Con}\, \mathfrak{A}$	congruence lattice of an algebra $\mathfrak{A}$	57
$\mathrm{Con}\, L$	congruence lattice of a lattice L	38
$\mathrm{Con}(\varphi)$	Con applied to a homomorphism φ	42
$\mathrm{Con}_{\mathrm{Ji}}\, L$	order of join-irreducible congruences of L	213
$\mathrm{Comp}(A)$	set of complementary pairs in the lattice A	513

Symbol	Explanation	Page
Cov P	covering graph of order P	7
$\mathbf{D}$	class (variety) of distributive lattices	15, 75
$\dim(P)$	order-dimension of an order P	9
Distr L	set of all distributive elements of L	223
$\mathrm{Distr}^{\delta} L$	set of all dually distributive elements of L	224
Dns L	dense set of L	101, 194
Down P	order of down-sets of the order P	7
ext	for $K \leq L$, extension map: $\boldsymbol{\alpha} \mapsto \mathrm{con}_L(\boldsymbol{\alpha})$	216
End L	endomorphism monoid of L	122, 515
$\mathrm{End}_{\{0,1\}} L$	$\{0,1\}$-endomorphism monoid of L	122, 515
Equ A	lattice of all equivalences on A	3
(Ex)	condition for the existence of a free lattice	77
$\mathrm{fil}(a)$	filter generated by the element a	34
$\mathrm{fil}(H)$	filter generated by the set H	34
(Fil)	a condition for partial lattices	88
Fil L	filter lattice of a lattice L	34
$\mathrm{Fil}_0 L$	augmented filter lattice of a lattice L	34, 88
$\mathrm{Free}(\mathfrak{m})$	free lattice on $\mathfrak{m}$ generators	76
Free P	free lattice over the order P	76
Free $\mathfrak{A}$	free lattice over the partial lattice $\mathfrak{A}$	90
$\mathrm{Free}_{\mathbf{D}}(3)$	free distributive lattice on three generators	82
$\mathrm{Free}_{\mathbf{K}} \mathfrak{A}$	free lattice over $\mathfrak{A}$ in a variety $\mathbf{K}$	89
$\mathrm{Free}_{\mathbf{K}} P$	free lattice over the order P in a variety $\mathbf{K}$	76
$\mathrm{Free}_{\mathbf{M}}(3)$	free modular lattice on three generators	83
$\mathrm{Free}(P; \mathcal{J}, \mathcal{M})$	free lattice in Dean's Lemma	517
(GC)	compactness condition on open sets	170
$(G; E)$	graph on set G with edges E	7
$\mathrm{height}(a)$	height of an element	4
$\mathbf{H}(\mathbf{K})$	class of homomorphic images of members of $\mathbf{K}$	413
$\mathrm{id}(a)$	principal ideal generated by the element a	32, 88
$\mathrm{id}(H)$	ideal generated by the set H	32
$\mathbf{I}(\mathbf{K})$	class of isomorphic copies of members of $\mathbf{K}$	413
Id L	ideal lattice of L	33, 52, 270
$\mathrm{Id}_0 L$	augmented ideal lattice of L	33, 88
(Id)	for ideals in chopped lattices	270
(Idem)	idempotency condition for a binary operation	10
$\mathrm{Iden}(\mathbf{K})$	set of identities holding in the class $\mathbf{K}$	409
(Idl)	a condition for partial lattices	88
$\inf H, \bigwedge H$	greatest lower bound of H	5
Ji L	order of join-irreducible elements of L	102
(JID)	Join Infinite Distributive Identity	154

Symbol	Explanation	Page
$\mathrm{Ker}(\varphi)$	congruence kernel of φ	41
$\mathbf{L}$	class (variety) of all lattices	75
$\mathbf{\Lambda}$	lattice of all varieties of lattices	423
$\mathrm{Lat}\,G$	lattice representation of the graph G	515
$\mathfrak{L}^{\mathrm{alg}}$	the (order) lattice L as an algebra	13
$\mathfrak{L}^{\mathrm{ord}}$	the (algebra) lattice L as an order	13
$\mathrm{len}(P)$	length of a finite order P	4
(Lin)	linearity condition for binary relations	1
$\mathbf{M}$	class (variety) of modular lattices	15, 75
M_3	five-element modular nondistributive lattice	23
$\mathbf{M}_3$	variety generated by M_3	425
M_4	a modular lattice with four atoms	425
$\mathbf{M}_4$	variety generated by M_4	425
$\mathsf{M}_{3,3}$	two copies of M_3 glued together	425
$\mathbf{M}_{3,3}$	variety generated by $\mathsf{M}_{3,3}$	425
$\mathrm{Max}(M)$	maximal elements of a chopped lattice	270
$\mathrm{Merge}(C, D)$	merging of lattices C and D	269
$\mathrm{Mi}\,L$	order of meet-irreducible elements of a lattice L	102
(MID)	Meet Infinite Distributive Identity	154
$\mathbf{Mod}(\Sigma)$	class of all lattice models of Σ	409
$\mathsf{M}_3[L]$	order of boolean triples of the lattice L	295
$\mathsf{M}_3[\boldsymbol{\alpha}]$	congruence on $\mathsf{M}_3[L]$	297
N_5	five-element nonmodular lattice	23
$\mathbf{N}_5$	the variety generated by N_5	423
$\mathsf{N}_6 = \mathsf{N}(p, q)$	six-element nonmodular lattice	277
$\mathrm{Neutr}\,L$	set of all neutral elements of L	226
(OP)	condition for One-Point Extension	256
$\mathfrak{p}, \mathfrak{q}$	prime intervals	35
P^{δ}	dual of P	5
$P^{\max}$	a partial lattice formed from the order P	90
$P^{\min}$	a partial lattice formed from the order P	90
$\mathrm{Part}\,A$	partition lattice of A	3, 359
$\mathrm{Part}_{\mathrm{fin}}\,A$	set of finite partitions	361
$\mathrm{PG}(D, \mathfrak{m})$	$\mathfrak{m}$-dimensional projective geometry over D	378
$\mathrm{Pow}\,X$	power set lattice of X	4
$\mathbf{P}(\mathbf{K})$	class of direct products of members of $\mathbf{K}$	413
$\mathbf{P_s}(\mathbf{K})$	class of subdirect products of members of $\mathbf{K}$	414
$\mathbf{P_u}(\mathbf{K})$	class of ultraproducts of members of $\mathbf{K}$	416
$\mathrm{PrInt}(L)$	set of prime intervals of L	213
$\mathbb{Q}$	chain of rational numbers	158

Symbol	Explanation	Page
rank(p)	rank of a term p	68
re	for $K \leq L$, restriction map: $\boldsymbol{\alpha} \mapsto \boldsymbol{\alpha}\rceil K$	214
rep(p)	terms equivalent to p in a free product	474
(Refl)	reflexivity condition for binary relations	1
sub(H)	sublattice generated by H	31
$\sup H, \bigvee H$	least upper bound of H	5
spec(a)	spectrum of an element a	112, 118
($\mathrm{SD}_\vee$)	join-semidistributive law	479
($\mathrm{SD}_\wedge$)	meet-semidistributive law	479
$\mathbf{S}(\mathbf{K})$	class of subalgebras of members of $\mathbf{K}$	413
$\mathbf{Si}(\mathbf{K})$	class of subdirectly irreducible members of $\mathbf{K}$	418
S_1	three-element Stone algebra	201
$\mathcal{S}^{\mathrm{B}}$	booleanization of the Stone space $\mathcal{S}$	176
$\mathrm{Spec}\, L$	spectrum of a distributive lattice L	116
$\mathrm{Skel}\, L$	skeleton of L	99
($\mathrm{SP}_\vee$)	join-substitution property	36, 43, 269
($\mathrm{SP}_\wedge$)	meet-substitution property	36, 43, 269
$\mathrm{Sub}\, \mathfrak{A}$	subalgebra lattice of an algebra $\mathfrak{A}$	57
$\mathrm{Stand}\, L$	set of all standard elements of L	224
(Stone1)–(Stone3)	Stone conditions on a topological space	171, 174
$\mathrm{Sub}\, L$	sublattice lattice (including $\varnothing$) of a lattice L	51
$\mathbf{T}$	class (variety) of trivial lattices	92, 423
T_n	Tamari lattice	27
Term(n)	n-ary lattice terms	66
$\mathrm{Term}_{\mathbf{B}}(n)$	n-ary lattice terms in $\mathbf{B}$	129
$\mathrm{Term}_{\mathbf{D}}(n)$	n-ary lattice terms in $\mathbf{D}$	126
tran(ϱ)	transitive closure for binary relation ϱ	3
(Trans)	transitivity condition for binary relations	1
(W)	Whitman condition for free product	479
$\mathbf{Var}(\mathbf{K})$	smallest variety containing the class $\mathbf{K}$	414
W_i	Whitney number	353

Symbol	Explanation	Page
Relations and Congruences		
A^2	set of ordered pairs of A	2
ε, ϱ, τ, π, ...	binary relations	2
$\mathrm{equ}(\pi)$	binary relation from a partition π	3
$\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, ..., $\boldsymbol{\theta}$	congruences	36
$\mathbf{0}, \mathbf{1}$	zero and unit of $\mathrm{Part}\, A$ and $\mathrm{Con}\, L$	36
$(a, b) \in \varepsilon$	a and b are in relation ε	3
$a \mathrel{\varepsilon} b$	a and b are in relation ε	3
$a \equiv b \pmod{\varepsilon}$	a and b are in relation ε	3
$\mathfrak{p} \in \boldsymbol{\alpha}$	prime interval $\mathfrak{p}$ collapsed by $\boldsymbol{\alpha}$	213
a/π	block containing a	3, 36
A/π	set of all block of π	3, 40
$L/\boldsymbol{\alpha}$	quotient lattice	40
$\boldsymbol{\beta}/\boldsymbol{\alpha}$	quotient congruence, tolerance	42, 43, 198
$\boldsymbol{\alpha}\rceil_K$	restriction of $\boldsymbol{\alpha}$ to the sublattice K	41
π_i	projection map	46
$\boldsymbol{\alpha} \times \boldsymbol{\beta}$	direct product of congruences	46
Orders		
$\leq$, $<$	ordering	2
$\geq$, $>$	ordering, inverse notation	2
$K \leq L$	K is a sublattice of L	31
$L \geq K$	L is an extension of K	31
$\leq_Q$	ordering of P restricted to a subset Q	4
$a \parallel b$	a incomparable with b	4
$a \prec b$	a is covered by b	6
$a \preceq b$	$a \prec b$ or $a = b$	6
$b \succ a$	b covers a	6
$b \succeq a$	$b \succ a$ or $b = a$	6
$0, 1$	zero and unit of order	5
$a \vee b$	join operation	10
$\bigvee H$	least upper bound of H	5
$a \wedge b$	meet operation	10
$\bigwedge H$	greatest lower bound of H	5
$[a, b]$	interval	35
$\mathrm{down}(H)$	down-set generated by H	7
$\mathrm{down}(a)$, $\downarrow a$	down-set generated by a	7
$P \cong Q$	order (lattice) P isomorphic to Q	4, 12, 28

Symbol	Explanation	Page
Constructions		
$P \times Q$	direct product of P and Q	1, 7, 45
P^2	$P \times P$	7
$\prod(L_i \mid i \in I)$	direct product of $\{ L_i \mid i \in I \}$	46
$\prod_{\mathcal{D}}(L_i \mid i \in I)$	ultraproduct of $\{ L_i \mid i \in I \}$	416
P^δ	dual of an order (lattice) P	5, 14
$P + Q$	sum of P and Q	8
$P \dot{+} Q$	glued sum of P and Q	8
$\downarrow H$	down-set generated by H	7
$\uparrow H$	up-set generated by H	7
$A \otimes B$	tensor product of A and B	300
$L_{\mathbf{I}}$	One-Point Extension of the lattice L	255
$L[I]$	interval doubling	259
$\mathsf{M}_3[L]$	order of boolean triples of the lattice L	295
$A * B$	free product of the lattices A and B	467
Perspectivities		
$[a,b] \sim [c,d]$	perspective intervals	35
$[a,b] \overset{\text{up}}{\sim} [c,d]$	up-perspective intervals	35
$[a,b] \overset{\text{dn}}{\sim} [c,d]$	down-perspective intervals	35
$[a,b] \approx [c,d]$	projective intervals	35
$[a,b] \overset{n}{\approx} [c,d]$	projective intervals in n steps	35
$[a,b] \twoheadrightarrow [c,d]$	congruence-perspective intervals	208
$[a,b] \overset{\text{up}}{\twoheadrightarrow} [c,d]$	congruence-perspective up intervals	208
$[a,b] \overset{\text{dn}}{\twoheadrightarrow} [c,d]$	congruence-perspective down intervals	208
$[a,b] \Rightarrow [c,d]$	congruence-projective intervals	208
$[a,b] \Leftrightarrow [c,d]$	$[a,b] \Rightarrow [c,d]$ and $[c,d] \Rightarrow [a,b]$	208
$x \sim y$	x is perspective to y	239
$x \lesssim y$	x is subperspective to y	239
$x \leq^{\oplus} y$	x has a relative complement in $[0,y]$	239
ProjRep(p)	projective representative of p	344
ProjCl(p)	projective closure of p	344
unit(p)	unit of projective closure of p	344

Symbol	Explanation	Page
Miscellaneous		
a^*	pseudocomplement of a	99
a_*	unique lower cover of a	102
a'	complement of a	99
$\overline{x}, \overline{X}$	closure of x and X	47, 49
$\varnothing$	empty set	5
$\equiv_{\mathbf{B}}$	equivalence of boolean terms	128
$p/\mathbf{B}$	block of boolean terms	129
$\equiv_{\mathbf{D}}$	equivalence of distributive lattice terms	126
$p/\mathbf{D}$	block of distributive lattice terms	126
$x + y$	symmetric difference	132
$\alpha \circ \beta$	product of binary relations	2
$\mathbf{V} \circ \mathbf{W}$	product of the varieties $\mathbf{V}$ and $\mathbf{W}$	430
$a\,M\,b$	modular pair	335
$p^{(i)}, p_{(i)}$	upper and lower i-cover of p in a free product	471
$\overline{p}, \underline{p}$	upper and lower cover of p in P	517
A^b	lattice A with two new bounds	471