Uniquely complemented lattices
The 45th Summer School on Algebra
Sept. 2–7, 2007, Hotel Partizan
Tale, Low Tatras
Slovakia

G. Grätzer

September 3, 2007

The Whitman Conditions

$\mathbf{P}(X)$ all polynomials over a set X
For p, $q \in \mathbf{P}(X)$, set $p \subseteq q$ iff it follows from the following rules:

(E)	$x \subseteq x$,	for $x \in Q$;	
$({}_\wedge\mathrm{W})$	$p = p_0 \wedge p_1$,	where	$p_0 \subseteq q$ or $p_1 \subseteq q$;
$({}_\vee\mathrm{W})$	$p = p_0 \vee p_1$,	where	$p_0 \subseteq q$ and $p_1 \subseteq q$;
$(\mathrm{W}_\wedge)$	$q = q_0 \wedge q_1$,	where	$p \subseteq q_0$ and $p \subseteq q_1$;
$(\mathrm{W}_\vee)$	$q = q_0 \vee q_1$,	where	$p \subseteq q_0$ or $p \subseteq q_1$.

The free lattice

Each polynomial A determines an element $\langle A\rangle$ of the *free lattice* $\mathsf{F}(Q)$, if we interpret $\wedge$ as the meet operation in $\mathsf{F}(Q)$ and $\vee$ as the join operation. Given A, $B \in \mathbf{P}(Q)$, we set $A \equiv B$, if $\langle A\rangle = \langle B\rangle$ in $\mathsf{F}(Q)$; equivalently, if $A \subseteq B$ and $B \subseteq A$. Let $A \leq B$, if $\langle A\rangle \leq \langle B\rangle$ in $\mathsf{F}(Q)$; $\leq$ is a quasi-ordering on $\mathbf{P}(Q)$.

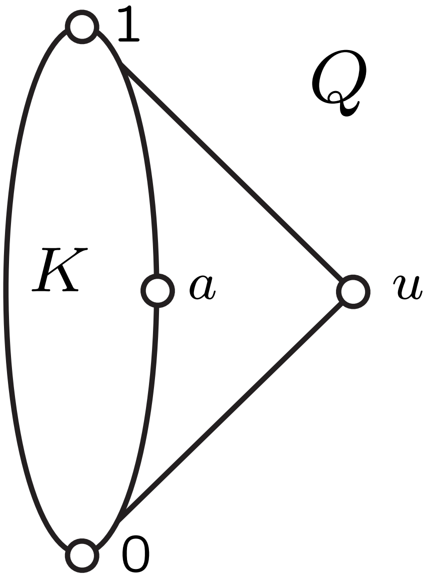
The Dilworth Conditions

$\mathbf{P}(Q)$ all polynomials over an *order* Q
For p, $q \in \mathbf{P}(Q)$, set $p \subseteq q$ iff it follows from the following rules:

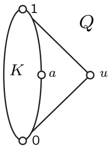
(Q)	$x \subseteq y,$	for $x, \ y \in Q$ with $x \leq y$ in Q;
$(_\wedge \mathrm{W})$	$p = p_0 \wedge p_1,$	where $p_0 \subseteq q$ or $p_1 \subseteq q$;
$(_\vee \mathrm{W})$	$p = p_0 \vee p_1,$	where $p_0 \subseteq q$ and $p_1 \subseteq q$;
$(\mathrm{W}_\wedge)$	$q = q_0 \wedge q_1,$	where $p \subseteq q_0$ and $p \subseteq q_1$;
$(\mathrm{W}_\vee)$	$q = q_0 \vee q_1,$	where $p \subseteq q_0$ or $p \subseteq q_1$.

We get the *free lattice* $\mathrm{F}(Q)$ over the order Q

The Grätzer-Lakser approach



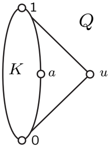
Our special structure Q



Let K be a bounded lattice. Let $a \in K - \{0, 1\}$, and let u be an element not in K. We extend the partial ordering $\leq$ of K to $Q = K \cup \{u\}$ as follows: $0 \leq u \leq 1$.

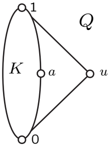
We extend the lattice operations $\wedge$ and $\vee$ to Q as *commutative partial meet and join operations*. For $x \leq y$ in Q, define $x \wedge y = x$ and $x \vee y = y$. In addition, let $a \wedge u = 0$ and $a \vee u = 1$.

The lattice $\mathrm{F}(Q)$



We now discuss the lattice $\mathrm{F}(Q)$, the lattice freely generated by Q and preserving the partial joins and meets of Q.

Covers



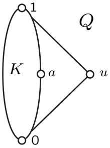
We associate, with each polynomial A,

an element A_*, which will represent the *largest* element of K *below* A

an element A^*, which will represent the *smallest* element of K *above* A

Now we give a *mutually recursive* definition of $u \subseteq A$ and A_*.

The Grätzer-Lakser Structure Theorem



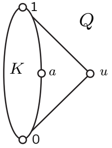
Theorem

The following statements hold:

(i) $u \subseteq u$. *If* $x \in K$, *then* $u \subseteq x$ *iff* $x = 1$.
(ii) $u_* = 0$. *If* $x \in K$, *then* $x_* = x$.
(iii) $u \subseteq A \wedge B$ *iff* $u \subseteq A$ *and* $u \subseteq B$.
(iv) $(A \wedge B)_* = A_* \wedge B_*$.
(v) $u \subseteq A \vee B$ *iff either* $u \subseteq A$, *or* $u \subseteq B$, *or* $A_* \vee B_* = 1$.
(vi)

$$(A \vee B)_* = \begin{cases} 1, & \text{if } a \subseteq A_* \vee B_* \text{ and either } u \subseteq A \text{ or } u \subseteq B; \\ A_* \vee B_*, & \text{otherwise.} \end{cases}$$

The complements in $\mathsf{F}(Q)$



Theorem

(i) *The only complement of* u *in* $\mathsf{F}(Q)$ *is* a.

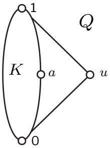
(ii) *Let* K *contain no spanning* N_5. *Let* $\langle A \rangle$, $\langle B \rangle$ *be complementary in* $\mathsf{F}(Q)$. *Then either*

$$\{\langle A \rangle, \langle B \rangle\} \subseteq K$$

or

$$\{\langle A \rangle, \langle B \rangle\} = \{u, a\}.$$

Sample proof



(i) The only complement of u in $\mathrm{F}(Q)$ is a.

Proof. Let $A \in \mathbf{P}(Q)$ be such that $\langle A \rangle$ is a complement of u in $\mathrm{F}(Q)$, that is, $A \wedge u \equiv 0 \quad \text{and} \quad A \vee u \equiv 1$.
Then

$$1 = (A \vee u)_* = \begin{cases} 1, & \text{if } a \subseteq A_* \vee u_* = A_*; \\ A_*, & \text{otherwise.} \end{cases}$$

So either $a \subseteq A_*$ or $1 = A_*$; in either case, $a \subseteq A_*$. Dually, $a \supseteq A^*$.
Thus

$$A \subseteq A^* \subseteq a \subseteq A_* \subseteq A,$$

and so $A \equiv a$.

Application: Chen and Grätzer

Theorem

Let K be a bounded, at most uniquely complemented lattice (that is, a lattice with zero and unit, in which every element has at most one complement). Then K has a $\{0,1\}$-embedding into a uniquely complemented lattice L.

Proof. Since K is at most uniquely complemented, it contains no spanning N_5. If K is uniquely complemented, there is nothing to do. If not, pick an $a \in K$ that has no complement, define $Q = K \cup \{u\}$, and form $L_1 = \mathrm{F}(Q)$. So L_1 is an at most uniquely complemented $\{0,1\}$-extension of K, and a has a complement in L_1, namely, u. By transfinite induction, we obtain an at most uniquely complemented $\{0,1\}$-extension $\overline{L}$ of K in which every element of K has a complement. Repeating this construction ω-times, we obtain the lattice L of this theorem.

Application: Dilworth

Theorem
Every lattice can be embedded into a uniquely complemented lattice.

Proof. Starting with an arbitrary lattice V, let K be the lattice we obtain by adjoining a new zero and unit to V. Then K is at most uniquely complemented, indeed, only the zero and the unit have complements. By the Chen and Grätzer result, K has a $\{0, 1\}$-embedding into a uniquely complemented lattice L.
Of course, this L will do for V.

Application: new sample result

Let $\mathfrak{m}$ be a cardinal number. A lattice K is called (at most) $\mathfrak{m}$-complemented, if K has 0 and 1, and every $x \in K - \{0, 1\}$ has (at most) $\mathfrak{m}$ complements.

Grätzer-Lakser:

Theorem

Let K be an at most $\mathfrak{m}$-complemented lattice with no spanning N_5. Then K has a $\{0, 1\}$-embedding into an $\mathfrak{m}$-complemented lattice L.

References

- M. E. Adams and J. Sichler, *Cover set lattices,* Can. J. Math. **32** (1980), 1177–1205.
- ———, *Lattices with unique complementation,* Pacific. J. Math. **92** (1981), 1–13.
- C. C. Chen and G. Grätzer, *On the construction of complemented lattices,* J. Algebra **11** (1969), 56–63.
- P. Crawley and R. P. Dilworth, *Algebraic Theory of Lattices.* Prentice-Hall, Englewood Cliffs, NJ, 1973.
- R. A. Dean, *Free lattices generated by partially ordered sets and preserving bounds,* Canad. J. Math. **16** (1964), 136–148.
- R. P. Dilworth, *Lattices with unique complements,* Trans. Amer. Math. Soc. **57** (1945), 123–154.

References

G. Grätzer, *General Lattice Theory, second edition,* new appendices by the author with B. A. Davey, R. Freese, B. Ganter, M. Greferath, P. Jipsen, H. A. Priestley, H. Rose, E. T. Schmidt, S. E. Schmidt, F. Wehrung, and R. Wille. Birkhäuser Verlag, Basel, 1998. xx+663 pp. ISBN: 0-12-295750-4, ISBN: 3-7643-5239-6. *Softcover edition,* Birkhäuser Verlag, Basel–Boston–Berlin, 2003. ISBN: 3-7643-6996-5.

______, *A reduced free product of lattices,* Fund. Math. **73** (1971/72), 21–27.

G. Grätzer and H. Lakser, *Freely adjoining a relative complement to a lattice,* to appear in Algebra Universalis.

______, *Embedding lattices into $\mathfrak{m}$-transitively complemented lattices,* manuscript.

G. Grätzer, H. Lakser, and C. R. Platt, *Free products of lattices,* Fund. Math. **69** (1970), 233–240.

References

E. V. Huntington, *Sets of independent postulates for the algebra of logic,* Trans. Amer. Math. Soc. **79** (1904), 288-309.

H. Lakser, *Free lattices generated by partially ordered sets,* Ph.D. Thesis, University of Manitoba, 1968.

V. N. Saliĭ, *Lattices with Unique Complements,* Translations of Mathematical Monographs vol. 69, Amer. Math. Soc., Providence, R.I.

P. M. Whitman, *Free lattices. I and II,* Ann. of Math. (2) **42** (1941), 325–330, (2) **43** (1942), 104–115.