Consider the first-order separable differential equation: \( \frac{dy}{dx} = f(y)g(x) \). (1)

We solve this by calculating the integrals: \( \int \frac{dy}{f(y)} = \int g(x)dx + C \). (2)

If \( y_0 \) is a value for which \( f(y_0) = 0 \), then \( y = y_0 \) will be a solution of the above differential equation (1). We call the value \( y_0 \) a critical point of the differential equation and \( y = y_0 \) (as a constant function of \( x \)) is called an equilibrium solution of the differential equation.

If there is no value of \( C \) in the solution formula (2) which yields the solution \( y = y_0 \), then the solution \( y = y_0 \) is called a singular solution of the differential equation (1).

The “general solution” of (1) consists of the solution formula (2) together with all singular solutions.

*Note: by “general solution”, I mean a set of formulae that produces every possible solution.*

**Example 1:** Solve: \( \frac{dy}{dx} = (y - 3)^2 \). (3)

**Solution:** \( \int \frac{dy}{(y - 3)^2} = \int dx \). Thus, \( \frac{-1}{y - 3} = x + C ; \quad y - 3 = \frac{-1}{x + C} ; \quad \text{and} \)

\[ y = 3 - \frac{1}{x + C} , \quad (4) \]

where \( C \) is an arbitrary constant.

Both sides of the DE (3) are zero when \( y = 3 \). No value of \( C \) in (4) gives \( y = 3 \) and thus, the solution \( y = 3 \) is a singular solution.

The general solution of (3) consists of: \( y = 3 - \frac{1}{x + C} \) (C is an arbitrary constant) *and* \( y = 3 \).

See over ⇪
Example 2: Solve:
\[ \frac{dy}{dx} = y^2 - 4. \]  
\hspace{1cm} (5)

Solution: \[ \int \frac{dy}{y^2 - 4} = \int dx. \] Using partial fractions,
\[ \int \frac{dy}{(y - 2)(y + 2)} = \int \left[ \frac{1}{4} \left( \frac{1}{y - 2} + \frac{-1}{y + 2} \right) \right] dy = \int dx. \]

Thus, \[ \int \left[ \frac{1}{y - 2} + \frac{-1}{y + 2} \right] dy = \int 4 dx. \]

Integrating, \[ \ln (y - 2) - \ln (y + 2) = 4x + C. \]

Taking exponentials, \[ \frac{y - 2}{y + 2} = e^{4x} C_1 = s \text{ (say)}. \]

Then,
\[ y - 2 = s(y + 2) = sy + 2s \]
\[ y - sy = 2 + 2s \]
\[ y(1-s) = 2 + 2s \]
\[ y = \frac{2 + 2s}{1 - s}. \]

Thus, \[ y = \frac{2 + 2C_1 e^{4x}}{1 - C_1 e^{4x}}, \]  
\hspace{1cm} (6)

where \( C_1 \) is an arbitrary constant.

Both sides of the DE (5) are zero when \( y = \pm 2 \). If we put \( C_1 = 0 \) in (6), we obtain the solution:
\( y = 2 \). However, no value of \( C_1 \) in (6) gives \( y = -2 \) and thus, the solution \( y = -2 \) is a singular solution.

The general solution of (5) consists of:
\[ y = \frac{2 + 2C_1 e^{4x}}{1 - C_1 e^{4x}} \text{ (} C_1 \text{ is an arbitrary constant)} \text{ and } y = -2. \]