

- [6] 1. Evaluate $\lim_{x \rightarrow 2} \frac{x^3 - 3x^2 + 4}{x^2 - 4x + 4}$, if it exists. Show all calculations.

- [6] 2. Find all intervals on which the function $f(x) = \frac{x+3}{x^2+3}$ is increasing.

$$f'(x) = \frac{(x+3)'(x^2+3) - (x+3)(x^2+3)'}{(x^2+3)^2} = \frac{x^2+3 - 2x(x+3)}{(x^2+3)^2} =$$

$$= \frac{x^2+3-2x^2-6x}{(x^2+3)^2} = -\frac{x^2+6x-3}{(x^2+3)^2}$$

$$x^2+6x-3=0 \Leftrightarrow x = \frac{-6 \pm \sqrt{36+4 \cdot 3}}{2} = \frac{-6 \pm \sqrt{48}}{2} = -3 \pm 2\sqrt{3}$$

i.e. $x_1 = -3 - 2\sqrt{3}$ and $x_2 = -3 + 2\sqrt{3}$ are the critical pts of $f(x)$.

$$f'(x) = -\frac{(x-x_1)(x-x_2)}{(x^2+3)^2}$$

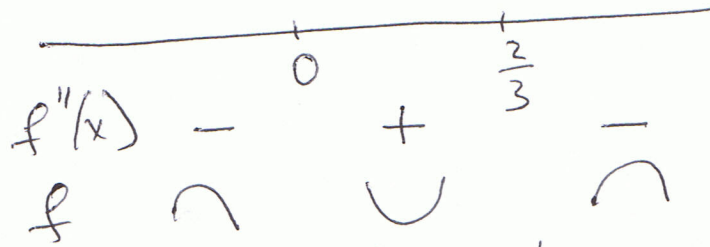
	$-3-2\sqrt{3}$	$-3+2\sqrt{3}$	
$f'(x)$	-	+	-
f	↘	↗	↘

$\therefore f$ is increasing on $[-3-2\sqrt{3}, -3+2\sqrt{3}]$.

- [6] 3. Let $f(x) = x^5 - x^6$. Determine the intervals on which $f(x)$ is concave up or concave down, and find all inflection points (if any).

$$f'(x) = 5x^4 - 6x^5, \quad f''(x) = 20x^3 - 30x^4 = 10x^3(2-3x)$$

$$f''(x) = 0 \Leftrightarrow x = 0 \text{ or } x = \frac{2}{3}$$



f is concave up on $[0, \frac{2}{3}]$ and concave down on $(-\infty, 0]$ and $[\frac{2}{3}, +\infty)$.

$$f(0) = 0, \quad f\left(\frac{2}{3}\right) = \left(\frac{2}{3}\right)^5 - \left(\frac{2}{3}\right)^6 = \left(\frac{2}{3}\right)^5 \left(1 - \frac{2}{3}\right) = \frac{2^5}{3^6} = \frac{32}{729}$$

Hence, f has inflection points at $(0, 0)$ and $(\frac{2}{3}, \frac{32}{729})$.

- [6] 4. Find the maximum value of the function $f(x) = \frac{\ln x}{x^3}$ on the interval $1 \leq x \leq 2$.

f is continuous (and continuously differentiable) on $[1, 2]$

Hence, it achieves its maximum on this interval

(which is achieved either at the endpoints or the critical pts).

$$f'(x) = \frac{(\ln x)'x^3 - \ln x \cdot (x^3)'}{x^6} = \frac{x^2 - 3x^2 \ln x}{x^6} = \frac{1 - 3 \ln x}{x^4}$$

$$f'(x) = 0 \Leftrightarrow 1 - 3 \ln x = 0 \Leftrightarrow \ln x = \frac{1}{3} \Leftrightarrow x = e^{\frac{1}{3}}$$

Hence, $x = e^{\frac{1}{3}}$ is the only critical pt of f .

Now, we could simply compare $f(1)$, $f(2)$ and $f(e^{\frac{1}{3}})$

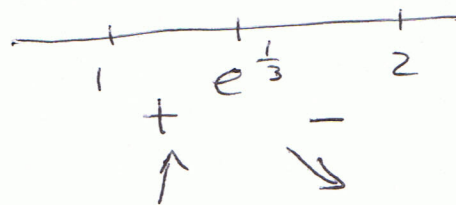
(note that $1 < e^{\frac{1}{3}} < 2$ and so $e^{\frac{1}{3}}$ is in $(1, 2)$) but this is messy without a calculator. Therefore, we use the first derivative (and the fact that $\ln x$ is increasing).

Since $f \uparrow$ on $[1, e^{\frac{1}{3}}]$

and $f \downarrow$ on $[e^{\frac{1}{3}}, 2]$,

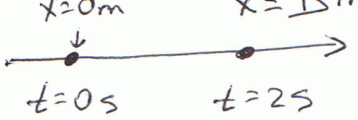
f achieves its absolute maximum at $x = e^{\frac{1}{3}}$.

\therefore Max value of f on $[1, 2]$ is $\frac{1}{3e}$.



$$f\left(e^{\frac{1}{3}}\right) = \frac{\ln e^{\frac{1}{3}}}{(e^{\frac{1}{3}})^3} = \frac{\frac{1}{3}}{e} = \frac{1}{3e}$$

- [6] 5. From a standing start, a cheetah reaches a velocity of 72 km/hr in 2 seconds. If the acceleration of the cheetah is constant, and it runs in a straight line, what distance does it cover in the 2 seconds? You are not permitted to use any physics formulas. You must derive your own.

First, note that $72 \text{ km/h} = 72 \cdot \frac{1000}{3600} \text{ m/s} = 20 \text{ m/s}$
 $x=0\text{m}$ $x=D\text{m}$

 Let x be the distance (in m) from the starting point at time t s.
 $\frac{d^2x}{dt^2} = a = \text{const} \rightarrow \text{acceleration}$

$$\frac{dx}{dt} = v(t) = \int a dt = at + C$$

Now, $v(0) = 0$ and $v(2) = 20$, i.e. $C = 0$ and $20 = a \cdot 2$
 $a = 10 \text{ (m/s}^2\text{)}$

Hence, $\frac{dx}{dt} = v(t) = 10t$

$$x(t) = \int 10t dt = 5t^2 + C_1 \text{ and } x(0) = 0, \text{ i.e. } C_1 = 0$$

Hence, $x(t) = 5t^2$ and so $x(2) = 5 \cdot 2^2 = 20 \text{ m.}$

\therefore cheetah covers 20 m in 2 seconds.

- [5] 6. A particle travels along the curve $y = x^3 - 2x^2 - x + 5$ where x and y are in metres in such a way that its x -coordinate is increasing at a constant rate of 2 metres per second. Find all points on the curve where its y -coordinate is decreasing at 4 metres per second.

$$\frac{dx}{dt} = 2 \text{ m/s}$$

$$\frac{dy}{dt} = (3x^2 - 4x - 1) \frac{dx}{dt} = 2(3x^2 - 4x - 1) \text{ (m/s)}$$

We need to find all pts on the curve s.t. $\frac{dy}{dt} = -4 \text{ (m/s)}$

$$\therefore 2(3x^2 - 4x - 1) = -4 \quad (\Rightarrow) \quad 3x^2 - 4x - 1 = -2 \quad (\Rightarrow) \quad 3x^2 - 4x + 1 = 0$$

$$\quad (\Rightarrow) \quad x = \frac{4 \pm \sqrt{16 - 4 \cdot 3}}{2 \cdot 3} = \frac{4 \pm 2}{2 \cdot 3} = \frac{2 \pm 1}{3}, \text{ i.e. } x_1 = \frac{1}{3} \text{ and } x_2 = 1$$

when $x = 1, y = 1 - 2 - 1 + 5 = 3$

$$x = \frac{1}{3}, y = \frac{1}{27} - \frac{2}{9} - \frac{1}{3} + 5 = \frac{1 - 6 - 9 + 5 \cdot 27}{27} = \frac{121}{27} = 4 \frac{13}{27}$$

Answer: two points $(1, 3)$ and $(\frac{1}{3}, 4 \frac{13}{27})$.