VARIATION OF NEUMANN AND GREEN FUNCTIONS UNDER HOMOTOPIES OF THE BOUNDARY

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Abstract. We develop variational formulas for certain Neumann and Green functions of multiply connected planar domains, valid for any smooth homotopy of the boundary. The modified Neumann and Green functions under consideration arise in the study of holomorphic functions with single-valued primitives.

1. Introduction

1.1. **Statement and motivation of results.** In this paper we derive first-order variational formulas for Green and Neumann functions of multiply connected planar domains, which are valid for arbitrary smooth homotopies of the boundary curves. Variational formulas are given for the Neumann function of the domain, as well as for certain modified Green and Neumann functions which play a central role in the theory of complex analytic functions with primitives in multiply connected domains [3] [5] [7]. The variational formula for the ordinary Green function was proven in [8].

The formulas obtained in this paper are improvements of the classical Hadamard variational formulas. The Hadamard formulas are valid for variations of the boundary curves along the normal (that is, they are of the form $z(s)+\Delta n(s)n(s)$ where $z(s)$ traces the original boundary curve, $n(s)$ is the unit outward normal and $\Delta n(s)$ is an arbitrary smooth quantity). In brief, the improvement is that while the Hadamard variation can be used to provide necessary conditions for validity of certain functional inequalities, the formulas provided here are often *sufficient*.

More precisely, assume that one wants to show that some quantity I depending on intrinsic domain functions (such as the Green or the Neumann function and its derivatives) increases as the domain increases set-theoretically. That is, if $D_1 \subset D_2$ then $I(D_1) < I(D_2)$. In particular, I must increase under a Hadamard variation, so taking the functional derivative of I with respect to a Hadamard variation one obtains a necessary condition for monotonicity. However, this does not rigorously establish the result for *arbitrary* pairs of domains D_1 and D_2 such that $D_1 \subset D_2$, since 1) it is not possible to reach an arbitrary domain by a normal variation and 2) one cannot even establish that the quantity I is increasing on a small interval. Note that given a normal variation $z_t(s) = z_{t_0}(s) + \Delta n_{t_0}(s)n(s)$ it is usually not possible to write $z_t(s) = z_{t_1}(s)+\Delta n_{t_1}(s)n_{t_1}(s)$ for $t_0 < t_1 < t$ for any function $\Delta n_{t_1}(s)$. Nevertheless, Hadamard variations are generic enough that one suspects that the necessary conditions *should* be sufficient for monotonicity.

The solution presented here is based on the intuitive idea that a generic smooth variation of a curve is a normal variation up to first order. We give precise formulas for the first-order variations of various domain functions under general homotopies. Using these formulas, for any homotopy of domains D_t we may simply differentiate the quantity $I(D_t)$ with respect to t, and if it is positive then by elementary calculus $I(D_t)$ must in fact be increasing. This establishes that $I(D_1) < I(D_2)$ for any pair of domains $D_1 \subset D_2$ with homotopic boundary curves. For examples of such applications see [8].

The assumption that the boundaries are smooth is not an obstacle in proving inequalities, since one can easily extend inequalities to the non-smooth case by using exhaustions by smoothly bounded domains (see [8, Remark 2]). This can be contrasted with the variational methods of Schiffer and others, which handle the non-smooth case directly and provide necessary conditions that a particular domain be extremal, but often are not sufficient to prove that an inequality holds in general. The variational techniques presented here and in [8] are designed for sufficient conditions, and thus can be seen as complementary to Schiffer variation.

In this paper we will focus on the establishment of such variational formulas for other domain functions besides the Green function. In particular, we have devoted considerable effort into establishing the formulas for certain modified Green and Neumann functions considered in for example [3] [5] [7], with an eye to future applications. These domain functions are important to the study of the class of exact holomorphic one-forms.

Our results are the following: (1) Theorem 3.5 establishes the variational formula for the modified Green function under an arbitrary homotopy. As described above, this generalizes the Hadamard variational formula; (2) Theorem 3.1 yields a variational formula under an arbitrary homotopy for a modified Neumann function H defined by Schiffer and Hawley [7]. The formula was proven in the special case of Schiffer variations in [7] and therefore it has not even been established for arbitrary Hadamard variations; (3) In Theorem 3.7 we establish the formula for variation of the ordinary Neumann function N under an arbitrary homotopy.

1.2. **Preliminaries.** Let $\mathcal{G}$ be a domain in the complex plane bounded by m smooth curves.

Definition 1.1. Let $\mathcal{D}(\mathcal{G})$ denote the set of harmonic functions on $\mathcal{G}$ with finite Dirichlet energy.

Two facts concerning $\mathcal{D}(\mathcal{G})$ will be crucial. First, by a result of Privalov, Plessner, Marcinkiewicz, Zygmund and Spencer, see e.g. [11], for almost every point $p \in \partial\mathcal{G}$, any element of $\mathcal{D}(\mathcal{G})$ has non-tangential limit at p. See also [10] for detailed proofs and generalizations to all dimensions and domains with various boundary regularities. Second, it is known that if $\mathcal{G}$ is a smoothly bounded domain and u is a harmonic function with finite Dirichlet energy then the boundary value of u is an $L^2(\partial\mathcal{G})$ function with respect to the boundary measure of the domain, see Aronszajn [1] and Sobolev [9].

Now let $z = z(s)$ be the parametric representation of the boundary curves in terms of the arc length s. Denote differentiation with respect to s with a dot, e.g. $\dot{z}$. Assume that the boundaries are traced with positive orientation with respect to the interior. We will use the following conventions in this paper: (1) n denotes the unit *outward* normal; (2) the curvature κ of $z(s)$ is a signed curvature, so that the curvature is positive if $\ddot{z}$ points in the direction of the outward normal n (i.e. $\ddot{z} = \kappa n$). With these conventions, we have that

$$n = -i\dot{z} \tag{1.1}$$

and

$$\kappa(s) = i\frac{\ddot{z}}{\dot{z}}. \tag{1.2}$$

Following Schiffer and Hawley [7], we define the subspace $\mathcal{D}_0(\mathcal{G})$ of $\mathcal{D}(\mathcal{G})$ consisting of functions which satisfy

$$\int_{\partial\mathcal{G}} f(z(s))\kappa(s)ds = 0. \tag{1.3}$$

By the Cauchy-Schwarz inequality and the preceding discussion the integral $\int_{\partial\mathcal{G}} f\kappa$ exists.

Besides the ordinary Neumann function, we will consider a modified Neumann and Green function on $\mathcal{D}_0(\mathcal{G})$. These functions were considered by Schiffer and Hawley [7] in the study of connections on planar domains, and play a central role in the construction of kernel functions in $\mathcal{D}_0(\mathcal{G})$ and holomorphic functions of finite Dirichlet energy satisfying the same normalization. $\mathcal{D}_0(\mathcal{G})$ and the corresponding space of holomorphic functions are in one-to-one correspondence with exact harmonic and holomorphic one-forms on $\mathcal{G}$ respectively via application of the exterior derivative, which accounts for their importance. The non-standard normalization is a key part of the machinery developed in [7] for the construction of intrinsic differentials on $\mathcal{G}$.

We now define the Neumann and Green functions considered in this paper. First we define the ordinary Neumann function N of a domain $\mathcal{G}$ in the plane. As is well-known this function is defined by the properties that $N(z,\zeta) = -\log|z-\zeta| + N_1(z,\zeta)$ where N_1 is harmonic in z on $\mathcal{G}$,

$$\frac{\partial N}{\partial n_z}(z,\zeta) = \frac{2\pi}{L}, \quad z \in \partial\mathcal{G}, \tag{1.4}$$

where $\int_{\partial\mathcal{G}} ds = L$ and the normalization

$$\int_{\partial\mathcal{G}} N(z,\zeta)\, ds_z = 0. \tag{1.5}$$

Because of its logarithmic pole, N satisfies Green's third identity, see [6], for any u which is harmonic on $\mathcal{G}$:

$$u(\zeta) = \frac{1}{2\pi}\int_{\partial\mathcal{G}} \left(N(z,\zeta)\frac{\partial u}{\partial n}(z) - u(z)\frac{\partial N}{\partial n}(z,\zeta) \right) ds. \tag{1.6}$$

Of course if

$$\int_{\partial \mathcal{G}} u(z) ds_z = 0$$

then

$$u(\zeta) = \frac{1}{2\pi} \int_{\partial \mathcal{G}} N(z,\zeta) \frac{\partial u}{\partial n}(z) ds_z.$$

Next we define the "modified Neumann function" H of Schiffer and Hawley [7]. Given a planar domain $\mathcal{G}$ bounded by m smooth closed curves, $H(z,\zeta)$ is the unique function with the following properties:

$$H(z,\zeta) = -m \log|z-\zeta| + H_1(z,\zeta) \tag{1.7}$$

where H_1 is harmonic in z,

$$\frac{\partial H}{\partial n}(z,\zeta) = -\kappa(z) \tag{1.8}$$

where once again n denotes the unit outward normal and

$$\int_{\partial \mathcal{G}} \kappa(z) H(z,\zeta) ds_z = 0. \tag{1.9}$$

Remark 1.2 (Comparison with the sign conventions in [7]). Schiffer and Hawley [7] choose the inward normal for n and also choose curvature to have opposite sign κ as that used here. Unfortunately their choices of sign are inconvenient in the present setting. Thus with these two changes, the condition

$$\frac{\partial H}{\partial n} = -\kappa$$

defines precisely the same function H as in [7].

Because of the logarithmic pole, H has the property that for any harmonic function u on $\mathcal{G}$

$$\begin{aligned} u(\zeta) &= \frac{1}{2\pi m} \int_{\partial \mathcal{G}} \left(H(z,\zeta) \frac{\partial u}{\partial n}(z) - u(z) \frac{\partial H}{\partial n}(z,\zeta) \right) ds_z \\ &= \frac{1}{2\pi m} \int_{\partial \mathcal{G}} \left(H(z,\zeta) \frac{\partial u}{\partial n}(z) + u(z)\kappa(z) \right) ds_z \end{aligned} \tag{1.10}$$

In particular, if u satisfies the normalization

$$\int_{\partial \mathcal{G}} \kappa(z) u(z) ds_z = 0$$

then

$$u(\zeta) = \frac{1}{2\pi m} \int_{\partial \mathcal{G}} H(z,\zeta) \frac{\partial u}{\partial n}(z)\, ds_z.$$

Thus H is the "Neumann function" for the class of harmonic functions with this normalization.

Finally, we define the modified Green function G introduced in [7]. Again, note that some signs differ from [7] because of our choice that n denotes the outward rather than inward normal. Let $g(z,\zeta)$ denote the ordinary Green function defined to be the unique function in ζ satisfying

$$g(z,\zeta) = -\log|z-\zeta| + g_1(z,\zeta)$$

where g_1 is harmonic in ζ and

$$g(z,\zeta) = 0$$

for any $\zeta \in \mathcal{G}$ and $z \in \partial\mathcal{G}$. Let $\partial\mathcal{G} = \cup_{\mu=1}^m C_\mu$,

$$\omega_\mu(\zeta) = -\frac{1}{2\pi}\int_{C_\mu} \frac{\partial g}{\partial n_z}(z,\zeta)\,ds_z$$

denote the harmonic measures of $\mathcal{G}$, and

$$P_{\mu\nu} = \frac{1}{2\pi}\int_{C_\mu} \frac{\partial \omega_\nu}{\partial n}\,ds$$

denote the period matrix. Let $c_{\nu\mu}$ be the symmetric matrix defined by the conditions

$$\delta_{\mu\rho} - \frac{1}{m} = 2\pi \sum_{\nu=1}^m c_{\nu\mu} P_{\nu\rho}$$

where $\delta_{\mu\rho}$ is the Kronecker delta and

$$\sum_{\nu=1}^m c_{\nu\mu} = 0 \qquad \mu = 1,\ldots,m.$$

The modified Green function G is defined by

$$G(z,\zeta) = g(z,\zeta) + 2\pi \sum_{\mu,\nu=1}^m c_{\nu\mu}\omega_\nu(z)\omega_\mu(\zeta). \tag{1.11}$$

G so defined, is harmonic for $z \neq \zeta$ and has a logarithmic pole for $z = \zeta$:

$$G(z,\zeta) = -\log|z-\zeta| + G_1(z,\zeta) \tag{1.12}$$

where G_1 is harmonic in ζ. Of course G_1 is also symmetric.

From the representation above, it follows that $G(z,\zeta)$ is symmetric in z and ζ. It can be shown that G satisfies the following relations:

$$\frac{1}{2\pi}\int_{C_\mu} \frac{\partial G(z,\zeta)}{\partial n_z}\,ds_z = -\frac{1}{m}, \;\; \mu = 1,\ldots,m. \tag{1.13}$$

On each boundary curve C_μ, $G(z,\zeta)$ takes a constant value in the sense that

$$G(z,\zeta) = k_\mu(\zeta), \;\; z \in C_\mu, \tag{1.14}$$

with

$$\sum_{\mu=1}^{m} k_\mu(\zeta) = 0. \tag{1.15}$$

Furthermore

$$\int_{\partial \mathcal{G}} \kappa(s_z) G(z,\zeta) ds_z = 0. \tag{1.16}$$

See [7] for the details and proofs of these statements.

2. Homotopies of the boundary curves

In this section we carefully define the kind of variation under consideration, establish some notation, and collect some crucial technical lemmas.

2.1. **Definition of admissible homotopies and notation.**

Definition 2.1. Let $\mathcal{G}_t$ be a family of domains for $t \in (a,b)$, which are bounded by m C^p simple closed curves for each fixed t. A collection of functions $F^i : (a,b) \times [0,2\pi] \to \mathbb{C}$, $i = 1, \ldots, m$ is called an *admissible C^p homotopy for $\mathcal{G}_t$* if

(1) F^i is injective for each i, except that $F^i(t,0) = F^i(t,2\pi)$
(2) F^i is C^p for each i; furthermore, the right-hand derivatives in the second variable up to order p at $(t,0)$ match the left-hand derivatives at $(t,2\pi)$ for all $t \in (a,b)$
(3) for each fixed t the curves $F^i(t,\cdot)$ are the boundary curves of $\mathcal{G}_t$, and
(4) $\mathcal{G}_{t'} \subset \mathcal{G}_t$ whenever $t' < t$.

We will refer to the collection of functions F^i as "the homotopy F".

The following definition fixes notation regarding the infinitesimal variation corresponding to an admissible homotopy.

Definition 2.2. Let $\mathcal{G}_t$ be a collection of domains for $t \in (a,b)$ and F be an admissible homotopy for $\mathcal{G}_t$. For $t_0 \in (a,b)$ denote the outward unit normal to $F^i(t_0,\cdot)$ at $F^i(t_0,\tau)$ by $n_{t_0}(\tau)$ as above. Let $\Delta n^i_{t_0}(t,\tau)$ denote the distance from $F^i(t_0,\tau)$ to the curve $F(t,\cdot)$ along the normal line at $F^i(t_0,\tau)$ (see Figure 2.1). Furthermore denote

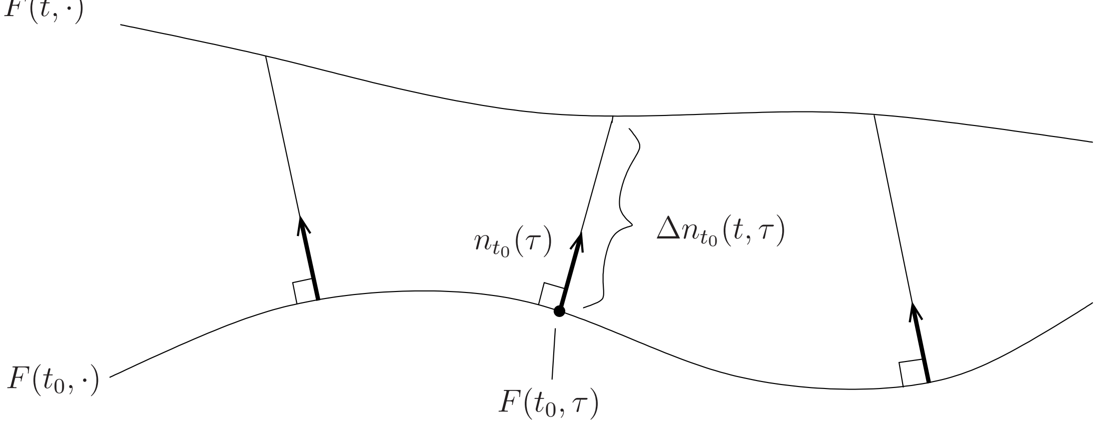
$$\nu^i_{t_0}(\tau) = \left.\frac{d}{dt}\right|_{t=t_0} \Delta n^i_{t_0}(t,\tau).$$

If F is suitably regular, the definitions of $\Delta n^i_{t_0}(\tau)$ and $\nu^i_{t_0}(t,\tau)$ make sense for some interval $(t_0-\epsilon, t_0+\epsilon) \times [0,2\pi]$ by Lemma 2.5 ahead.

Remark 2.3. We will often drop the index i to reduce clutter.

Remark 2.4. Note that it is usually not true that $F(t_0,\tau) + \Delta n_{t_0}(t,\tau)\, n_{t_0}(\tau) = F(t,\tau)$.

FIGURE 1. Definition of Δn



2.2. **Technical lemmas regarding homotopies.** Here we collect some crucial lemmas established in [8], as well as an elementary formula for the variation of curvature κ.

Given such a homotopy, fix $t_0 \in (a,b)$. It is intuitively clear that for small $t-t_0$ the curve $F(t,\cdot)$ can be obtained from $F(t_0,\cdot)$ by variation along the normal line. That is, for small $t-t_0$ a general homotopy is in fact a normal variation (that is, up to reparametrization of the curve $F(t,\cdot)$; see Remark 2.4). For our applications it is necessary to have control over the interval on which this is true. The following two lemmas proved in [8] are for this purpose.

Lemma 2.5. *Let F be an admissible C^2 homotopy, and $[c,d]$ be a compact subinterval of (a,b). There is a fixed $\varepsilon > 0$, such that for every $t_0 \in [c,d]$ and $t \in (t_0-\varepsilon, t_0+\varepsilon)$, the curve $\tau \mapsto F^i(t,\tau)$ intersects the normal line $r \mapsto F^i(t_0,\tau_0) + rn^i_{t_0}(\tau_0)$ once and only once for every $\tau_0 \in [0,2\pi]$ and $i = 1,\ldots,m$.*

Clearly $\Delta n_{t_0}(t,\tau)$ is approximately $(t-t_0)\nu_{t_0}(\tau)$. We will need a uniform version of this approximation:

Lemma 2.6. *The ε of Lemma 2.5 can be chosen so that*

$$\Delta n_{t_0}(t,\tau) = (t-t_0)\nu_{t_0}(\tau) + O(|t-t_0|^2)$$

for $|t-t_0| < \varepsilon$, where the remainder is uniform for $(t_0,\tau) \in [c,d] \times [0,2\pi]$.

We will also need the following elementary formula for the variation of curvature κ.

Lemma 2.7. *Let F be an admissible C^3 homotopy corresponding to domains $\mathcal{G}_t$. Let $z_{t_1}(s)$ parametrize the curve $\tau \mapsto F(t_1,\tau)$ with respect to arc length, and let $z_{t_2}(s) = z_{t_1}(s) + \Delta n_{t_1}(t_2,\tau(s))\, n(s)$ parametrize the curve $\tau' \mapsto F(t_2,\tau')$. Let $\kappa_{t_i}(s)$ denote the curvature of these two curves as a function of s for $i = 1,2$ respectively. Let $\nu_{t_1}(s) = \nu_{t_1}(\tau(s))$. Then denoting differentiation with respect to s with a dot,*

$$\kappa_{t_2}(s) = \kappa_{t_1}(s) + (t_2-t_1)\left(\kappa^2_{t_1}(s)\nu_{t_1}(s) + \ddot{\nu}_{t_1}(s)\right) + O(|t_2-t_1|^2).$$

The remainder term is uniformly bounded by $K|t_2 - t_1|^2$ on the entire curve $F(t_1, \cdot)$ for some K.

Proof. For simplicity denote $u(t_2) = \Delta n_{t_1}(t_2)\, n = -i\Delta n_{t_1}(t_2)\dot{z}$, (suppressing the dependence on s). Then we have

$$\kappa_{t_2} = -\frac{1}{|\dot{z}_{t_1} + \ddot{u}(t_2)|}\mathrm{Im}\left(\frac{\ddot{z}_{t_1} + \ddot{u}(t_2)}{\dot{z}_{t_1} + \dot{u}(t_2)}\right).$$

Note that $\Delta n(t_1)$, $\dot{\Delta n}(t_1)$ and $\ddot{\Delta n}(t_1)$ are all identically zero. It follows that $u(t_1)$, $\dot{u}(t_1)$ and $\ddot{u}(t_1)$ are also identically zero. Denoting differentiation with respect to t_2 with a $'$, it is then easy to compute that

$$\left.\frac{d}{dt_2}\,\frac{\ddot{z}_{t_1} + \ddot{u}(t_2)}{\dot{z}_{t_1} + \dot{u}(t_2)}\right|_{t_2=t_1} = \frac{\ddot{u}'(t_1)}{\dot{z}_{t_1}} - \frac{\dot{u}'(t_1)}{\dot{z}_{t_1}}\frac{\ddot{z}_{t_1}}{\dot{z}_{t_1}}$$

and

$$\left.\frac{d}{dt_2}\,\frac{1}{|\dot{z}_{t_1} + \dot{u}'(t_2)|}\right|_{t_2=t_1} = -\mathrm{Re}\left(\frac{\dot{u}'(t_1)}{\dot{z}_{t_1}}\right).$$

Thus

$$\begin{aligned}\left.\frac{d}{dt_2}\kappa_{t_2}(s)\right|_{t_2=t_1} &= \mathrm{Re}\left(\frac{\dot{u}'(t_1)}{\dot{z}_{t_1}}\right)\mathrm{Im}\left(\frac{\ddot{z}_{t_1}}{\dot{z}_{t_1}}\right) - \mathrm{Im}\left(\frac{\ddot{u}'(t_1)}{\dot{z}_{t_1}} - \frac{\dot{u}'(t_1)}{\dot{z}_{t_1}}\frac{\ddot{z}_{t_1}}{\dot{z}_{t_1}}\right) \\ &= -2\mathrm{Re}\left(\frac{\dot{u}'(t_1)}{\dot{z}_{t_1}}\right)\kappa_{t_1} - \mathrm{Im}\left(\frac{\ddot{u}'(t_1)}{\dot{z}_{t_1}}\right).\end{aligned} \tag{2.1}$$

In the second equality we used equation (1.2); note that $\ddot{z}_{t_1}/\dot{z}_{t_1}$ is pure imaginary since $z_{t_1}(s)$ is parametrized by arc length.

Again using equations (1.1) and (1.2)

$$\dot{u}(t_2) = -i\dot{\Delta n}(t_2)\dot{z}_{t_1} - \Delta n(t_2)\kappa_{t_1}\dot{z}_{t_1}.$$

Therefore since $\Delta n'(t_1) = \nu_{t_1}$

$$2\mathrm{Re}\left(\frac{\dot{u}'(t_1)}{\dot{z}_{t_1}}\right) = -2\kappa_{t_1}\nu_{t_1}. \tag{2.2}$$

Differentiating $\dot{u}$, we obtain

$$\ddot{u}(t_2) = -i\ddot{\Delta n}(t_2)\dot{z}_{t_1} - 2\kappa_{t_1}\dot{\Delta n}(t_2)\dot{z}_{t_1} - \Delta n(t_2)\dot{\kappa}_{t_1}\dot{z}_{t_1} + i\Delta n(t_2)\kappa_{t_1}^2\dot{z}_{t_1},$$

so

$$\ddot{u}'(t_1) = i\left(\kappa_{t_1}^2\nu_{t_1} - \ddot{\nu}_{t_1}\right)\dot{z}_{t_1} - \left(2\kappa_{t_1}\dot{\nu}_{t_1} + \dot{\kappa}_{t_1}\nu_{t_1}\right)\dot{z}_{t_1}.$$

It then follows that

$$\mathrm{Im}\left(\frac{\ddot{u}'(z_1)}{\dot{z}_{t_1}}\right) = \kappa_{t_1}^2\nu_{t_1} - \ddot{\nu}_{t_1}. \tag{2.3}$$

The lemma now follows from equations (2.1), (2.2) and (2.3). □

3. Main theorems: variational formulas for domain functions

3.1. **Variational formula for the modified Neumann function H.** We now state and prove the variational formula for H. The same formula was derived by Schiffer and Hawley [7] in the special case of a Schiffer variation. Since not every normal variation of a curve arises from a Schiffer variation, even the Hadamard variational result has not been proven for H. Here we extend the formula to arbitrary homotopies of the boundary, in the case that the boundaries are smooth, significantly generalizing Schiffer and Hawley's result for the smoothly bounded case. On the other hand, Schiffer and Hawley gave formulas for domains whose boundaries are not smooth.

H satisfies certain useful identities. Let $z(s)$ parametrize $\partial\mathcal{G}$ with respect to arc length, and let $\dot{z}$ denote the unit tangent vector. Then

$$2\frac{\partial H}{\partial z}\dot{z} = \frac{\partial H}{\partial s} + i\frac{\partial H}{\partial n} \tag{3.1}$$

and so

$$\mathrm{Re}\left(4\frac{\partial H}{\partial z}(z,\zeta)\frac{\partial H}{\partial z}(z,\eta)\dot{z}^2\right) = \frac{\partial H}{\partial s}(z,\zeta)\frac{\partial H}{\partial s}(z,\eta) - \frac{\partial H}{\partial n}(z,\zeta)\frac{\partial H}{\partial n}(z,\eta). \tag{3.2}$$

Finally we have the identity

$$2\mathrm{Re}\left(\frac{\partial^2 H}{\partial z^2}\dot{z}^2\right) = \frac{\partial^2 H}{\partial s^2} + \kappa^2. \tag{3.3}$$

To see this, an elementary computation yields for $z = x + iy$,

$$\frac{\partial^2 H}{\partial s^2} = \dot{x}^2\frac{\partial^2 H}{\partial x^2} + \dot{y}^2\frac{\partial^2 H}{\partial y^2} + 2\dot{x}\dot{y}\frac{\partial^2 H}{\partial x\partial y} + \ddot{x}\frac{\partial H}{\partial x} + \ddot{y}\frac{\partial H}{\partial y}.$$

Now since H is harmonic we have that (suppressing z and ζ dependence)

$$\begin{aligned}\frac{\partial^2 H}{\partial s^2} &= \frac{\partial^2 H}{\partial s^2} - \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right)\left(\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2}\right)\\ &= \frac{1}{2}\left(\dot{x}^2 - \dot{y}^2\right)\left(\frac{\partial^2 H}{\partial x^2} - \frac{\partial^2 H}{\partial y^2}\right) + 2\dot{x}\dot{y}\frac{\partial^2 H}{\partial x\partial y} + \ddot{x}\frac{\partial H}{\partial x} + \ddot{y}\frac{\partial H}{\partial y}\\ &= 2\mathrm{Re}\left(\frac{\partial^2 H}{\partial z^2}\dot{z}^2\right) - \kappa^2\end{aligned}$$

In the last step we have used the fact that

$$\ddot{x}\frac{\partial H}{\partial x} + \ddot{y}\frac{\partial H}{\partial y} = 2\mathrm{Re}\left(\left(\ddot{x} + i\ddot{y}\right)\frac{\partial H}{\partial z}\right) = -2\mathrm{Re}\left(\kappa i\dot{z}\frac{\partial H}{\partial z}\right) = \kappa\frac{\partial H}{\partial n} = -\kappa^2$$

which follows from equations (1.2) and (1.8).

We now state the variational formula for H. In the following, we drop the index i on the quantities $\nu^i_{t_0}$, $\Delta n^i_{t_0}$ to reduce clutter. Where the integrals are taken over the entire boundary $\partial\mathcal{G}_{t_0}$ for example, it is understood that n_{t_0}, Δn_{t_0} and ν_{t_0} stand for the appropriate function along each boundary component. Also, we write $\nu_{t_0}(z)$ for $\nu_{t_0}(\tau)$ where $F(t_0,\tau)=z$.

Theorem 3.1. *Let $\mathcal{G}_t$, $t\in(a,b)$ be a collection of domains bounded by m simple closed C^3 curves and F an admissible C^3 homotopy for $\mathcal{G}_t$. Denoting arc length by s we have that*

$$H_t(\zeta,\eta)-H_{t_0}(\zeta,\eta)=\frac{t-t_0}{2\pi m}\int_{\partial\mathcal{G}_{t_0}}(\mathrm{Re}(R_{t_0}(z,\zeta,\eta)\dot{z}^2)+2\kappa_{t_0}^2)\nu_{t_0}(z)\,ds_z+\mathcal{R}_1(t_0,\eta;t,\zeta)$$

where

$$R_{t_0}(z,\zeta,\eta)=-4\frac{\partial H_{t_0}}{\partial z}(z,\zeta)\frac{\partial H_{t_0}}{\partial z}(z,\eta)-2\frac{\partial^2 H_{t_0}}{\partial z^2}(z,\eta)-2\frac{\partial^2 H_{t_0}}{\partial z^2}(z,\eta). \tag{3.4}$$

The remainder term $\mathcal{R}_1$ is harmonic in ζ and satisfies $\mathcal{R}_1(t_0,\eta;t,\zeta)=O(|t-t_0|^2)$ uniformly in ζ on any compact subset of $\mathcal{G}_{t_0}$.

Remark 3.2. More precisely, by "uniformly in ζ on any compact set in $\mathcal{G}_{t_0}$" we mean that for any compact subset K of $\mathcal{G}_{t_0}$ and t large enough so that $K\subset\mathcal{G}_t$ the remainder term is bounded by $C|t-t_0|^2$ for a constant C which depends on K but not $\zeta\in K$.

Remark 3.3. Note that since $H_t(\zeta,\eta)=H_t(\eta,\zeta)$ for all t and the first order variation is also symmetric, the remainder terms must be symmetric and hence we know that the estimate is also uniform in compact sets in η in the same sense.

Proof. First assume that $t<t_0$. By equations (1.9) and (1.10)

$$\begin{aligned}H_t(\zeta,\eta)-H_{t_0}(\zeta,\eta)=&\frac{1}{2\pi m}\int_{\partial\mathcal{G}_t}\left(\frac{\partial H_t}{\partial n}(z,\eta)-\frac{\partial H_{t_0}}{\partial n}(z,\eta)\right)H_t(z,\zeta)\,ds_z\\&-\frac{1}{2\pi m}\int_{\partial\mathcal{G}_t}H_{t_0}(z,\eta)\kappa_t(z)\,ds_z.\end{aligned} \tag{3.5}$$

We first claim that $H_t(\zeta,\eta)-H_{t_0}(\zeta,\eta)\le C|t-t_0|$ where the constant C is independent of $\zeta\in\mathcal{G}_{t_0}$. (More precisely, for some $\varepsilon>0$, given any fixed t such that $|t-t_0|<\varepsilon$ this inequality holds for any $\zeta\in\mathcal{G}_t$). Let I and II denote the first and second integrals respectively in (3.5). In Lemma 2.7 set $t_1=t$ and $t_2=t_0$ (so that in (3.5) $z=z_t$). Denote the normal to the curve $F(t,\cdot)$ by n_t (so that in the above formula $n=n_t$). If $\varepsilon>0$ is small enough, then

$$\frac{\partial H_{t_0}}{\partial n_t}(z,\eta)-\frac{\partial H_{t_0}}{\partial n_{t_0}}(z_{t_0},\eta)=O(|t-t_0|)$$

uniformly for all z in the strips $\cup_i F^i((t_0-\varepsilon,t]\times[0,2\pi])$. Thus the first integral is

$$I=\frac{1}{2\pi m}\int_{\partial\mathcal{G}_t}[\kappa_t(s)-\kappa_{t_0}(z(s)+\Delta n_t(t_0,\tau(s))n_t)]\,ds_t+O(|t-t_0|).$$

Since F is C^3 it follows that κ_t^2, ν_t and $\ddot{\nu}_t$ are uniformly bounded in s and $t\in(t_0-\varepsilon,t_0]$. Thus I is $O(|t-t_0|)$ by Lemma 2.7.

In the same manner, approximating $H_{t_0}(z,\eta)$ with $H_{t_0}(z_{t_0},\eta)$, we have that the second integral is

$$II = \frac{1}{2\pi m}\int_{\partial\mathcal{G}_t} H_{t_0}(z_{t_0},\eta)\kappa_{t_0}(z_{t_0})ds + O(|t-t_0|)$$

and, using equation (1.9) and the fact that $ds = ds_{t_0} + O(|t-t_0|)$ we have that $II = O(|t-t_0|)$. This proves the first claim.

Next we derive an expression for $H_t - H_{t_0}$ involving Dirichlet energy which will be useful in computing the first-order variation. Denoting $H_t^\zeta(z) = H_t(z,\zeta)$ and the Dirichlet inner product on a domain $\mathcal{G}$ by

$$(f,g)_{\mathcal{G}} = \iint_{\mathcal{G}} \nabla f\cdot\nabla g\, dA,$$

we have using Green's identity that

$$\frac{1}{2\pi m}\left(H_t^\zeta - H_{t_0}^\zeta, H_t^\eta - H_{t_0}^\eta\right)_{\mathcal{G}_t} = \frac{1}{2\pi m}\int_{\partial\mathcal{G}_t}(H_t(z,\zeta)-H_{t_0}(z,\zeta))\frac{\partial}{\partial n_z}(H_t-H_{t_0})(z,\eta)ds_z.$$

Next, using Green's identity and the fact that H satisfies (1.8) and (1.9) it follows that

$$\begin{aligned}\frac{1}{2\pi m}\left(H_{t_0}^\zeta, H_{t_0}^\eta\right)_{\mathcal{G}_{t_0}\setminus\mathcal{G}_t} &= -\frac{1}{2\pi m}\int_{\partial\mathcal{G}_t} H_{t_0}(z,\zeta)\frac{\partial H_{t_0}}{\partial n}(z,\eta)\,ds_z\\ &= -\frac{1}{2\pi m}\int_{\partial\mathcal{G}_t} H_{t_0}(z,\zeta)\frac{\partial}{\partial n}(H_{t_0}-H_t)(z,\eta)\,ds_z\\ &\quad+\frac{1}{2\pi m}\int_{\partial\mathcal{G}_t} H_{t_0}(z,\zeta)\kappa_t(z)\,ds_z.\end{aligned}$$

Combining these two equations with (3.5) we have that

$$\begin{aligned}H_t(\zeta,\eta)-H_{t_0}(\zeta,\eta) &= \frac{1}{2\pi m}\left(H_t^\zeta - H_{t_0}^\zeta, H_t^\eta - H_{t_0}^\eta\right)_{\mathcal{G}_t} + \frac{1}{2\pi m}\left(H_{t_0}^\zeta, H_{t_0}^\eta\right)_{\mathcal{G}_{t_0}\setminus\mathcal{G}_t}\\ &\quad-\frac{1}{2\pi m}\int_{\partial\mathcal{G}_t}\left(H_{t_0}(z,\zeta)+H_{t_0}(z,\eta)\right)\kappa_t(z)\,ds_z.\end{aligned}\tag{3.6}$$

We will compute each of these terms up to first order in $t-t_0$. The first term is $O(|t-t_0|^2)$. To see this, first note that since the difference $H_t(\zeta,\eta)-H_{t_0}(\zeta,\eta)$ is harmonic in ζ and $H_t - H_{t_0}\to 0$ uniformly on compact sets as $t\to t_0$, the derivative

$$\frac{\partial H_t}{\partial\zeta} - \frac{\partial H_{t_0}}{\partial\zeta}$$

is holomorphic and converges to zero uniformly on compact sets. The same is true for the derivative with respect to $\bar\zeta$. From this we can conclude that the estimate

$$\nabla_\zeta H_t(\zeta,\eta) - \nabla_\zeta H_{t_0}(\zeta,\eta) = O(|t-t_0|)$$

holds uniformly on $\mathcal{G}_t$ in the same sense as above. Thus

$$\left(H_t^\zeta - H_{t_0}^\zeta, H_t^\zeta - H_{t_0}^\zeta\right)_{\mathcal{G}_t} = O(|t-t_0|^2)$$

as claimed.

Next we compute the second term. We will parametrize the region $\mathcal{G}_{t_0}\backslash\mathcal{G}_t$ as follows. Let $z_0(s)$ parametrize $\partial\mathcal{G}_{t_0}$ with respect to arc length. Let $n_{t_0}(s) \equiv n_{t_0}(\tau(s))$ denote the unit outward normal to $\partial\mathcal{G}_{t_0}$ parametrized with respect to arc length. Then for $\Delta n_{t_0}(t,\tau(s)) \leq r \leq 0$ the lines $r \mapsto z_0(s) + rn_{t_0}(s)$ sweep out the region $\mathcal{G}_{t_0}\backslash\mathcal{G}_t$. We have by an easy computation of the Jacobian determinant $\partial(x,y)/\partial(s,r)$ (for $z = x+iy$) that

$$dA = (1 + r\kappa_{t_0}(s))\, ds\, dr = ds\, dr + O(|t-t_0|).$$

The remainder term is uniform on $\mathcal{G}_{t_0}\backslash\mathcal{G}_t$ since $\kappa_{t_0}(s)$ is uniformly bounded since all the functions F^i are C^3, and by Lemma 2.6 $\Delta n_{t_0}(t,\tau(s)) = O(|t-t_0|)$ since $|\nu_{t_0}(\tau(s))|$ is uniformly bounded on $\partial\mathcal{G}_{t_0}$. Thus $|r| = O(|t-t_0|)$ uniformly for the specified range of s and r, and the claim follows.

Since

$$\nabla H(z,\zeta)\cdot\nabla H(z,\eta) = \frac{\partial H}{\partial s}(z,\zeta)\frac{\partial H}{\partial s}(z,\eta) + \frac{\partial H}{\partial n}(z,\zeta)\frac{\partial H}{\partial n}(z,\eta),$$

using the identity (3.2) we have that

$$\begin{aligned}
\frac{1}{2\pi m}\left(H_{t_0}^{\zeta}, H_{t_0}^{\eta}\right) &= \frac{2}{\pi m}\int_{\partial\mathcal{G}_{t_0}}\int_{\Delta n_{t_0}(t,\tau(s))}^{0} \mathrm{Re}\left[\frac{\partial H_{t_0}}{\partial z}(z,\zeta)\frac{\partial H_{t_0}}{\partial z}(z,\eta)\dot{z}^2\right] dr\, ds_z \\
&\quad + \frac{1}{\pi m}\int_{\partial\mathcal{G}_{t_0}}\int_{\Delta n_{t_0}(t,\tau(s))}^{0} \frac{\partial H_{t_0}}{\partial n}(z,\zeta)\frac{\partial H_{t_0}}{\partial n}(z,\eta) dr ds_z + O(|t-t_0|) \\
&= -\frac{2}{\pi m}\int_{\partial\mathcal{G}_{t_0}} \mathrm{Re}\left[\frac{\partial H_{t_0}}{\partial z}(z,\zeta)\frac{\partial H_{t_0}}{\partial z}(z,\eta)\dot{z}^2\right]\Delta n_{t_0}(t,\tau(t)) ds_z \\
&\quad - \frac{1}{\pi m}\int_{\partial\mathcal{G}_{t_0}} \kappa_{t_0}(z)^2\,\Delta n_{t_0}(t,\tau(s)) ds_z + O(|t-t_0|^2)
\end{aligned}$$

where we have applied the mean value theorem in the second equality. Since $(\partial H_{t_0}/\partial n)(z,\zeta)$ is C^∞ in z and ζ away from the diagonal $z - \zeta = 0$, this estimate is uniform in ζ for compact sets in $\mathcal{G}_t$. Finally, by an application of Lemma 2.6 we have that

$$\begin{aligned}
\frac{1}{2\pi m}\left(H_{t_0}^{\zeta}, H_{t_0}^{\eta}\right)_{\mathcal{G}_{t_0}\backslash\mathcal{G}_t} &= -(t-t_0)\frac{2}{\pi m}\int_{\partial\mathcal{G}_{t_0}} \mathrm{Re}\left[\frac{\partial H_{t_0}}{\partial z}(z,\zeta)\frac{\partial H_{t_0}}{\partial z}(z,\eta)\dot{z}^2\right]\nu_{t_0}(z) ds_z \\
&\quad - (t-t_0)\frac{1}{\pi m}\int_{\partial\mathcal{G}_{t_0}} \kappa_{t_0}(z)^2\,\nu_{t_0}(z) ds_z + O(|t-t_0|^2)
\end{aligned} \tag{3.7}$$

where again the estimate is uniform in ζ on compact subsets of $\mathcal{G}_t$.

Next we estimate the third term. We will apply Lemma 2.7 with $t_1 = t_0$ and $t_2 = t$, and $z = z_t = z_{t_0} + \Delta n_{t_0} n_{t_0}$. We have that

$$H_{t_0}(z,\zeta) = H_{t_0}(z_{t_0} + \Delta n_{t_0} n_{t_0},\zeta) = H_{t_0}(z_{t_0},\zeta) + (t-t_0)\nu_{t_0}\frac{\partial H_{t_0}}{\partial n}(z_{t_0},\zeta) + O(|t-t_0|^2). \tag{3.8}$$

Let $u(t) = \Delta n_{t_0} n_{t_0}$ as in the proof of Lemma 2.7. Since

$$ds_t = |\dot{z}_{t_0} + \dot{u}(t)| ds_{t_0}$$

and by equation (2.2)

$$\left.\frac{d}{dt}\right|_{t=t_0} |\dot{z}_{t_0} + \dot{u}(t)| = \mathrm{Re}\left(\frac{\dot{u}'(t_0)}{\dot{z}_{t_0}}\right) = -\kappa_{t_0}\nu_{t_0}$$

it follows that

$$ds_t = \left(1 - (t-t_0)\kappa_{t_0}\nu_{t_0} + O(|t-t_0|^2)\right) ds_{t_0}. \tag{3.9}$$

Combining this with Lemma 2.7 we have

$$\kappa_t ds_t = \left(\kappa_{t_0} + (t-t_0)\ddot{\nu}_{t_0} + O(|t-t_0|^2)\right) ds_{t_0}.$$

Thus by (3.8)

$$\int_{\partial\mathcal{G}_t} H_{t_0}(z,\zeta)\kappa(z)ds = \int_{\partial\mathcal{G}_{t_0}} H_{t_0}(z_{t_0},\zeta)\kappa_{t_0}ds_{t_0}$$
$$+ (t-t_0)\int_{\partial\mathcal{G}_{t_0}} \left(H_{t_0}(z_{t_0},\zeta)\ddot{\nu}_{t_0} - (\kappa_{t_0})^2\nu_{t_0}\right) ds_{t_0} + O(|t-t_0|^2)$$

so integrating by parts twice and using (1.9) we have

$$\int_{\partial\mathcal{G}_t} H_{t_0}(z,\zeta)\kappa(s)ds = (t-t_0)\int_{\partial\mathcal{G}_{t_0}} \left(\frac{\partial^2 H_{t_0}}{\partial {s_{t_0}}^2}(z_{t_0},\zeta) - {\kappa_{t_0}}^2\right)\nu_{t_0}\, ds_{t_0} + O(|t-t_0|^2).$$

Applying the identity (3.3) to the above we have that

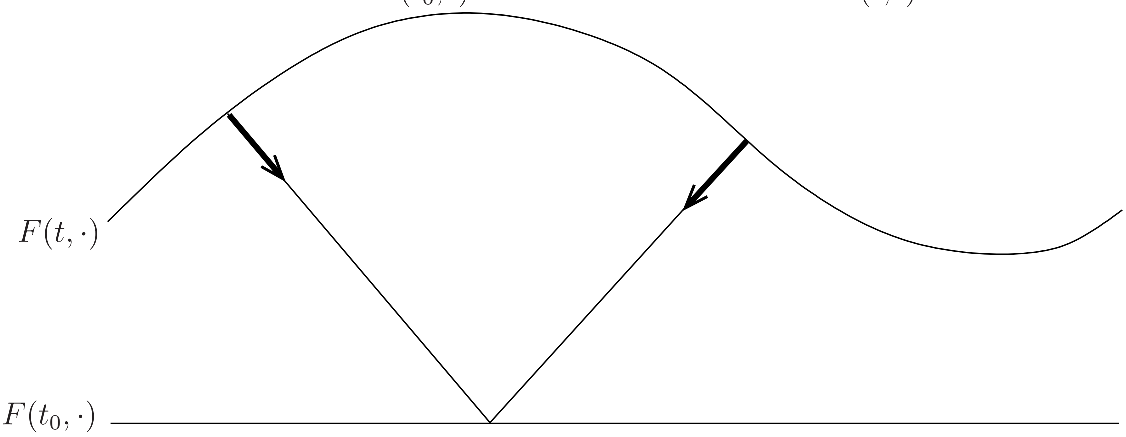
$$\int_{\partial\mathcal{G}} H_{t_0}(z,\zeta)\kappa(s)ds = (t-t_0)\int_{\partial\mathcal{G}_{t_0}} \left(2\mathrm{Re}\left(\frac{\partial^2 H_{t_0}}{\partial z^2}\dot{z}_{t_0}^2\right) - 2\kappa_{t_0}^2\right)\nu_{t_0}\, ds_{t_0} + O(|t-t_0|^2). \tag{3.10}$$

The estimation of the fourth term gives the identical result. Combining equations (3.6), (3.7) and (3.10), and using the fact that the second term is $O(|t-t_0|^2)$, proves the Theorem in the case that $t < t_0$.

Finally, we consider the case that $t > t_0$. To do this, we enclose both domains $\mathcal{G}_t$ and $\mathcal{G}_{t_0}$ inside the larger domain $\mathcal{G}_T$ for $T > t$, and apply the first part of the proof. The main difficulty is that although it may be possible to reach the curve $F(t,\cdot)$ from $F(t_0,\cdot)$ by a normal variation, it is not obvious that you can enclose the two curves in a larger one $F(t,\tau)$ from which it is possible to reach them by a normal variation). (For example, it is even possible that $F(t,\cdot)$ can be reached from $F(t_0,\cdot)$ by a normal variation, but $F(t_0,\cdot)$ cannot be reached from $F(t,\cdot)$ as in Figure 3.1.)

Lemma 2.5 solves the problem. Fixing t_0; let $[c,d]$ be a compact interval such that $t_0 \in (c,d)$, and let ε_1 be as in Lemma 2.5. By possibly shrinking ε_1 we can ensure

FIGURE 2. $F(t_0,\cdot)$ not a normal variation of $F(t,\cdot)$



that $(t_0-\varepsilon_1, t_0+\varepsilon_1) \subset [c,d]$. Choose $\varepsilon = \varepsilon_1/2$. For any $t \in [t_0+\varepsilon)$, set T so that $T - t_0 = 2(t-t_0)$. By the first part of the proof we have that

$$H_t(\zeta,\eta) - H_T(\zeta,\eta) = \frac{t-T}{2\pi m}\int_{\mathcal{G}_T}\left(\mathrm{Re}\left(R_T(z,\zeta,\eta)\dot{z}^2\right) + 2\kappa_T^2(z)\right)\nu_T(z)ds_z + \mathcal{R}_1(T,\eta;t,\zeta)$$

and

$$H_{t_0}(\zeta,\eta) - H_T(\zeta,\eta) = \frac{t_0-T}{2\pi m}\int_{\mathcal{G}_T}\left(\mathrm{Re}\left(R_T(z,\zeta,\eta)\dot{z}^2\right) + 2\kappa_T^2(z)\right)\nu_T(z)ds_z + \mathcal{R}_1(T,\eta;t_0,\zeta).$$

Subtracting the second equation from the first leads to

$$\begin{aligned} H_t(\zeta,\eta) - H_{t_0}(\zeta,\eta) &= \frac{t-t_0}{2\pi m}\int_{\mathcal{G}_T}\left(\mathrm{Re}\left(R_T(z,\zeta,\eta)\dot{z}^2\right) + 2\kappa_T^2(z)\right)\nu_T(z)ds_z \\ &\quad + \mathcal{R}_1(T,\eta;t,\zeta) - \mathcal{R}_1(T,\eta;t_0,\zeta). \end{aligned}$$

Since $T - t = t - t_0$ and $T - t_0 = 2(t-t_0)$ it follows that both remainder terms are $O(|t-t_0|^2)$. Furthermore since H_T and its derivatives converge uniformly on compact sets to H_{t_0}, R_T converges uniformly on compact sets to R_{t_0}. Also since F is C^3, $\kappa_T \to \kappa_{t_0}$ and $\nu_T \to \nu_{t_0}$, both uniformly.

Therefore we obtain

$$H_t(\zeta,\eta) - H_{t_0}(\zeta,\eta) = \frac{t-t_0}{2\pi m}\int_{\partial\mathcal{G}_t}\left(\mathrm{Re}\left(R_{t_0}(z,\zeta,\eta)\dot{z}^2\right) + 2\kappa_{t_0}^2(z)\right)\nu_{t_0}(z)\,ds_z + \mathcal{R}_1(t_0,\eta;t,\zeta),$$

and this proves the theorem.

□

Remark 3.4. Figure 3.1 provides an example where a normal variation based at the curve $F(t_0,\cdot)$ traces out the curve $F(t,\cdot)$ in a one-to-one way, whereas the normal variation based at $F(t,\cdot)$ tracing out $F(t_0,\cdot)$ is not one-to-one. In order to complete the proof, it was necessary to establish that given t_0, there is an interval containing t_0 on which this does not happen. Thus Lemma 2.5 is indispensible.

In fact, this issue already appears in the proof of the weaker Hadamard variational formula. Many sources do not treat this point carefully.

3.2. **Variational formula for the modified Green's function.** Since G is constant on each boundary,

$$\frac{\partial G}{\partial s} = 0$$

on all boundaries, so if $z(s)$ parametrizes the boundary by arc length,

$$2\frac{\partial G}{\partial z}\dot{z} = i\frac{\partial G}{\partial n}. \tag{3.11}$$

Therefore

$$4\frac{\partial G}{\partial z}(z,\zeta)\frac{\partial G}{\partial z}(z,\eta)\dot{z}^2 = -\frac{\partial G}{\partial n}(z,\zeta)\frac{\partial G}{\partial n}(z,\eta). \tag{3.12}$$

It is easy to see that

$$\nabla G(z,\zeta)\cdot\nabla G(z,\eta) = \frac{\partial G}{\partial s}(z,\zeta)\frac{\partial G}{\partial s}(z,\eta) + \frac{\partial G}{\partial n}(z,\zeta)\frac{\partial G}{\partial n}(z,\eta)$$

Hence since $\partial G/\partial s = 0$

$$\nabla G(z,\zeta)\cdot\nabla G(z,\eta) = \frac{\partial G}{\partial n}(z,\zeta)\frac{\partial G}{\partial n}(z,\eta) = -4\frac{\partial G}{\partial z}(z,\zeta)\frac{\partial G}{\partial z}(z,\eta)\dot{z}^2 \tag{3.13}$$

by equation (3.12). Note that the last expression is real.

Theorem 3.5. *Let $\mathcal{G}_t$, $t \in (a,b)$ be a collection of domains bounded by m simple closed C^2 curves and F an admissible C^2 homotopy for $\mathcal{G}_t$. Then*

$$G_t(\zeta,\eta) - G_{t_0}(\zeta,\eta) = -(t-t_0)\frac{1}{2\pi}\int_{\partial\mathcal{G}_t} 4\frac{\partial G}{\partial z}(z,\zeta)\frac{\partial G}{\partial z}(z,\eta)\dot{z}^2\nu_{t_0}(z)\,ds_z + \mathcal{R}_2(t_0,\eta;t,\zeta)$$

where the remainder term $\mathcal{R}_2$ is harmonic in ζ and satisfies $\mathcal{R}_2(t_0,\eta;t,\zeta) = O(|t-t_0|^2)$ uniformly in ζ in the sense of Theorem 3.1 and Remark 3.2.

Remark 3.6. Again, since G is symmetric, $\mathcal{R}_2$ is also harmonic in η and the estimate is uniform in compact sets in the same sense.

Proof. Using (1.13), (1.14) and (1.15), we have for any t

$$\begin{aligned}\frac{1}{2\pi}\int_{\partial\mathcal{G}_t} G_t(z,\zeta)\frac{\partial G_t}{\partial n}(z,\eta)ds_z &= \frac{1}{2\pi}\sum_{\mu=1}^{m} k_\mu(t,\zeta)\int_{C_\mu}\frac{\partial G_t}{\partial n}(z,\eta)ds_z \\ &= \frac{-1}{2\pi m}\sum_{\mu=1}^{m} k_\mu(t,\zeta) = 0,\end{aligned} \tag{3.14}$$

where $k_\mu(t,\zeta)$ denotes the boundary value of $G_t(z,\zeta)$.

Since G_t has a logarithmic pole (1.12) it satisfies Green's third identity

$$u(\zeta) = \frac{1}{2\pi} \int_{\partial \mathcal{G}_t} \left(G_t(z,\zeta) \frac{\partial u}{\partial n}(z) - u(z) \frac{\partial G_t}{\partial n}(z,\zeta) \right) ds_z \tag{3.15}$$

for any u harmonic on $\mathcal{G}$ and for any t. Setting $u(\cdot) = G_t(\cdot,\eta) - G_{t_0}(\cdot,\eta)$ we have using equation (3.14)

(3.16)

$$G_t(\zeta,\eta) - G_{t_0}(\zeta,\eta) = \frac{1}{2\pi} \int_{\partial \mathcal{G}_t} G_{t_0}(z,\eta) \frac{\partial G_t}{\partial n}(z,\zeta) ds_z - \frac{1}{2\pi} \int_{\partial \mathcal{G}_t} G_t(z,\zeta) \frac{\partial G_{t_0}}{\partial n}(z,\eta) ds_z.$$

Using Green's identity and equation (3.14) we have

$$\frac{1}{2\pi} \left(G_{t_0}^\zeta, G_{t_0}^\eta \right)_{\mathcal{G}_{t_0} \setminus \mathcal{G}_t} = -\frac{1}{2\pi} \int_{\partial \mathcal{G}_t} G_{t_0}(z,\zeta) \frac{\partial G_{t_0}}{\partial n}(z,\eta) ds_z$$

and

$$\begin{aligned} \frac{1}{2\pi} \left(G_t^\zeta - G_{t_0}^\zeta, G_t^\eta - G_{t_0}^\eta \right)_{\mathcal{G}_t} = &- \int_{\partial \mathcal{G}_t} G_{t_0}(z,\zeta) \frac{\partial G_t}{\partial n}(z,\eta) ds_z - \int_{\partial \mathcal{G}_t} G_t(z,\zeta) \frac{\partial G_{t_0}}{\partial n}(z,\eta) ds_z. \\ &+ \int_{\partial \mathcal{G}_t} G_{t_0}(z,\zeta) \frac{\partial G_{t_0}}{\partial n}(z,\eta) ds_z \end{aligned}$$

Putting the last three equations together we have that

(3.17)

$$\begin{aligned} G_t(\zeta,\eta) - G_{t_0}(\zeta,\eta) = &\frac{1}{2\pi} \left(G_{t_0}^\zeta, G_{t_0}^\eta \right)_{\mathcal{G}_{t_0} \setminus \mathcal{G}_t} + \frac{1}{2\pi} \left(G_t^\zeta - G_{t_0}^\zeta, G_t^\eta - G_{t_0}^\eta \right)_{\mathcal{G}_t} \\ &+ \frac{1}{2\pi} \int_{\partial \mathcal{G}_t} G_{t_0}(z,\zeta) \frac{\partial G_t}{\partial n}(z,\eta) ds_z + \frac{1}{2\pi} \int_{\partial \mathcal{G}_t} G_{t_0}(z,\eta) \frac{\partial G_t}{\partial n}(z,\zeta) ds_z. \end{aligned}$$

We first show that

$$G_t(\zeta,\eta) - G_{t_0}(\zeta,\eta) = O(|t - t_0|)$$

uniformly on $\mathcal{G}_{t_0}$.

To do this we use the representation (3.16). Denote the right hand side by $I + II$.

To estimate the first term I, as in the proof of Theorem 3.1 set $z = z_{t_0} + \Delta n_{t_0}(t,\tau(s)) n_{t_0}(s)$. Equations (1.13), (1.14) and (1.15) yield

$$\begin{aligned} I &= \frac{1}{2\pi} \int_{\partial \mathcal{G}_t} G_{t_0}(z,\eta) \frac{\partial G_t(z,\zeta)}{\partial n} ds = \frac{1}{2\pi} \int_{\partial \mathcal{G}_t} G_{t_0}(z_{t_0},\eta) \frac{\partial G_{t_0}(z_{t_0},\zeta)}{\partial n} ds + O(|t - t_0|) \\ &= \frac{1}{2\pi} \int_{\partial \mathcal{G}_{t_0}} G_{t_0}(z_{t_0},\eta) \frac{\partial G_{t_0}(z_{t_0},\zeta)}{\partial n} ds_{t_0} + O(|t - t_0|) \\ &= \frac{1}{2\pi} \sum_\mu k_\mu(t_0,\eta) \int_{C_\mu^{t_0}} \frac{\partial G_{t_0}(z_{t_0},\zeta)}{\partial n} ds_{t_0} + O(|t - t_0|) = O(|t - t_0|) \end{aligned}$$

where $C_\mu^{t_0}$ is the $\mu-$th boundary curve of $\mathcal{G}_{t_0}$.

This yields $I = O(|t-t_0|)$. To show that $II = O(|t-t_0|)$, we observe that

$$\begin{aligned}
II &= -\frac{1}{2\pi}\int_{\partial\mathcal{G}_t} G_t(z,\zeta)\frac{\partial G_{t_0}}{\partial n}(z,\eta)ds_z \\
&= -\sum_{\mu} k_\mu(t,\zeta)\int_{C_\mu^t}\frac{\partial G_{t_0}}{\partial n}(z,\eta)ds \\
&= -\sum_{\mu} k_\mu(t,\zeta)\int_{C_\mu^{t_0}}\frac{\partial G_{t_0}}{\partial n}(z,\eta)ds + O(|t-t_0|) \\
&= \frac{1}{m}\sum_{\mu} k_\mu(t,\zeta) + O(|t-t_0|) = O(|t-t_0|).
\end{aligned}$$

Therefore $I + II = O(|t-t_0|)$ and thereby $G_t(\zeta,\eta) - G_{t_0}(\zeta,\eta) = O(|t-t_0|)$.

Since the difference is harmonic, the ζ-derivative is holomorphic and we have that $\frac{\partial}{\partial\zeta}(G_t - G_{t_0})(\zeta,\eta)$ converges to zero uniformly on compact sets, and similarly for $\frac{\partial}{\partial\bar{\zeta}}(G_t - G_{t_0})(\zeta,\eta)$. Thus we also have that

$$\frac{1}{2\pi}\left(G_t^\zeta - G_{t_0}^\zeta, G_t^\eta - G_{t_0}^\eta\right)_{\mathcal{G}_t} = O(|t-t_0|^2).$$

Next we estimate the first term in equation (3.17). Proceeding as in the proof of Theorem 3.1

(3.18)

$$\begin{aligned}
\frac{1}{2\pi}\left(G_{t_0}^\zeta, G_{t_0}^\eta\right)_{\mathcal{G}_{t_0}\setminus\mathcal{G}_t} &= -\frac{2}{\pi}\int_{\partial\mathcal{G}_{t_0}}\int_{\Delta n_{t_0}(t,\tau(s))}^0 \frac{\partial G_{t_0}}{\partial z}(z,\zeta)\frac{\partial G_{t_0}}{\partial z}(z,\eta)\dot{z}^2\left(1 + O(|t-t_0|)\right)dr ds_z \\
&= \frac{2}{\pi}\int_{\partial\mathcal{G}_{t_0}}\frac{\partial G_{t_0}}{\partial z}(z,\zeta)\frac{\partial G_{t_0}}{\partial z}(z,\eta)\dot{z}^2\Delta n_{t_0}(t,\tau(s))ds_z + O(|t-t_0|^2) \\
&= (t-t_0)\frac{2}{\pi}\int_{\partial\mathcal{G}_{t_0}}\frac{\partial G_{t_0}}{\partial z}(z,\zeta)\frac{\partial G_{t_0}}{\partial z}(z,\eta)\dot{z}^2\nu_{t_0}(z)ds_z + O(|t-t_0|^2)
\end{aligned}$$

where the last equality follows from Lemma 2.6. This estimate is uniform in ζ for compact sets in $\mathcal{G}_t$.

Next we estimate the third term of equation (3.17). Letting $z = z_{t_0} + \Delta n_{t_0}(t,\tau(s))n_{t_0}(\tau(s))$ as above, since

$$G_{t_0}(z,\zeta) = G_{t_0}(z_{t_0},\zeta) + \Delta n_{t_0}(t,\tau(s))\frac{\partial G_{t_0}}{\partial n}(z_{t_0},\zeta) + O(|t-t_0|^2)$$

we have that

$$\frac{1}{2\pi}\int_{\partial\mathcal{G}_t} G_{t_0}(z,\zeta)\frac{\partial G_t}{\partial n}(z,\eta)ds_z = \frac{1}{2\pi}\int_{\partial\mathcal{G}_t} G_{t_0}(z_{t_0},\zeta)\frac{\partial G_t}{\partial n}(z,\eta)\Delta n_{t_0}(t,\tau(s))$$

$$+\frac{1}{2\pi}\int_{\partial\mathcal{G}_t}\frac{\partial G_{t_0}}{\partial n}(z_{t_0},\zeta)\frac{\partial G_{t_0}}{\partial n}(z,\eta)\Delta n_{t_0}(t,\tau(s))ds_z + O(|t-t_0|^2).$$

We have used the fact that $\Delta n_{t_0} = O(|t-t_0|)$.

The first term of the above equation is

$$\frac{1}{2\pi}\int_{\partial\mathcal{G}_t} G_{t_0}(z_{t_0},\zeta)\frac{\partial G_t}{\partial n}(z,\eta)\Delta n_{t_0}(t,\tau(s))ds_z = \frac{1}{2\pi}\sum_{\mu=1}^{m} k_\mu(t_0,\zeta)\int_{C_\mu}\frac{\partial G_t}{\partial n}(z,\eta)ds_z = 0$$

by equation (1.15).

Now since the arc lengths $ds_{t_0}(z_{t_0})$ on $\mathcal{G}_{t_0}$ and $ds_t(z)$ on $\mathcal{G}_t$ differ by $O(|t-t_0|)$ uniformly on $\partial\mathcal{G}_t$, and

$$\frac{\partial G_{t_0}}{\partial n}(z,\eta) = \frac{\partial G_{t_0}}{\partial n}(z_{t_0},\eta) + O(|t-t_0|),$$

uniformly on ∂G_t, we have that

$$\frac{1}{2\pi}\int_{\partial\mathcal{G}_t} G_{t_0}(z,\zeta)\frac{\partial G_t}{\partial n}(z,\eta)ds_z = \frac{1}{2\pi}\int_{\partial G_{t_0}}\frac{\partial G_{t_0}}{\partial n}(z,\zeta)\frac{\partial G_{t_0}}{\partial n}(z,\eta)\Delta n_{t_0}(t,\tau(s))ds_z + O(|t-t_0|^2)$$

$$= -(t-t_0)\frac{2}{\pi}\int_{\partial\mathcal{G}_{t_0}}\frac{\partial G_{t_0}}{\partial z}(z,\zeta)\frac{\partial G_{t_0}}{\partial z}(z,\eta)\dot{z}^2\nu_{t_0}(z)ds_z + O(|t-t_0|^2) \tag{3.19}$$

by Lemma 2.6 and equation (3.13). The same expression approximates the fourth term. Adding equations (3.18) to (3.19) and using the fact that the second term is $O(|t-t_0|^2)$ the theorem is proven in the case that $t < t_0$. The case that $t > t_0$ is dealt with in the same manner as in the proof of Theorem 3.1. □

3.3. **Variation of the Neumann function.** We now establish the variational formula for N under an arbitrary homotopy. Surprisingly, we were unable to find even the Hadamard variational formula for N in the literature. Bergman [2] gives a formula for a difference of Neumann functions and Bergman and Schiffer [4] give a formula for a Neumann function satisfying the normalization $\partial N/\partial n = 0$ (which is correspondingly different than the one given here).

Theorem 3.7. *Let $\mathcal{G}_t$, $t \in (a,b)$ be a collection of domains bounded by m simple closed C^3 curves and F an admissible C^3 homotopy for $\mathcal{G}_t$. Then*

$$N_t(\zeta,\eta) - N_{t_0}(\zeta,\eta) = -(t-t_0)\frac{1}{2\pi}\int_{\partial\mathcal{G}_t} Re\left(4\frac{\partial N}{\partial z}(z,\zeta)\frac{\partial N}{\partial z}(z,\eta)\dot{z}^2\right)\nu_{t_0}(z)\,ds_z$$

$$-(t-t_0)\frac{1}{L}\int_{\partial\mathcal{G}_{t_0}}\left(N_{t_0}(z,\zeta)+N_{t_0}(z,\eta)\right)\kappa_{t_0}(z)\nu_{t_0}ds + \mathcal{R}_3(t_0,\eta;t,\zeta)$$

where the remainder term $\mathcal{R}_3$ is harmonic in ζ and satisfies $\mathcal{R}_3(t_0, \eta; t, \zeta) = O(|t - t_0|^2)$ uniformly in ζ in the sense of Theorem 3.1 and Remark 3.2.

Remark 3.8. Since N is symmetric $\mathcal{R}_3$ is harmonic in η and the estimate is uniform in η in the same sense.

Proof. The proof is very similar to the proofs of Theorems 3.1 and 3.5, so we will be brief.

Assume that $t < t_0$. Setting $u(\zeta) = N_t(\zeta, \eta) - N_{t_0}(\zeta, \eta)$ in equation (1.6) and using equations (1.4) and (1.5) we have

$$N_t(\zeta,\eta) - N_{t_0}(\zeta,\eta) = \frac{1}{2\pi}\left(N_t^\zeta - N_{t_0}^\zeta, N_t^\eta - N_{t_0}^\eta\right)_{\mathcal{G}_t} + \frac{1}{2\pi}\left(N_{t_0}^\zeta, N_{t_0}^\eta\right)_{\mathcal{G}_{t_0}\setminus\mathcal{G}_t} + \frac{1}{L}\int_{\partial\mathcal{G}_t}\left(N_{t_0}(z,\zeta) + N_{t_0}(z,\eta)\right) ds. \tag{3.20}$$

It can easily be shown as before that the first term of equation (3.20) is $O(|t - t_0|^2)$. Using the fact that

$$\nabla N_{t_0}(z,\zeta)\cdot\nabla N_{t_0}(z,\eta) = 4\mathrm{Re}\left(\frac{\partial N}{\partial z}(z,\zeta)\frac{\partial N}{\partial z}(z,\eta)\dot{z}^2\right) + 2\frac{\partial N}{\partial n}(z,\zeta)\frac{\partial N}{\partial n}(z,\eta)$$

and proceeding as in the proofs of Theorems 3.1 and 3.5 and using equation (1.4) we have that

$$\frac{1}{2\pi}\left(N_{t_0}(\cdot,\zeta), N_{t_0}(\cdot,\eta)\right)_{\mathcal{G}_{t_0}\setminus\mathcal{G}_t} = -(t-t_0)\frac{2}{\pi}\int_{\partial\mathcal{G}_{t_0}}\mathrm{Re}\left(\frac{\partial N}{\partial z}(z,\zeta)\frac{\partial N}{\partial z}(z,\eta)\dot{z}^2\right)\nu_{t_0}(z)ds_z - (t-t_0)\frac{4\pi}{L^2}\int_{\partial\mathcal{G}_{t_0}}\nu_{t_0}ds_z + O(|t-t_0|^2). \tag{3.21}$$

Finally, the third term can be estimated in the same manner as before. Using equations (3.9) and (1.4)

$$\begin{aligned}\frac{1}{L}\int_{\partial\mathcal{G}_t}N_{t_0}(z,\zeta)ds_z &= \frac{1}{L}\int_{\partial\mathcal{G}_{t_0}}\left(N_{t_0}(z,\zeta) + \frac{\partial N_{t_0}}{\partial n}(z,\zeta)\Delta n_{t_0}(t,\tau(s)) + O(|t-t_0|^2)\right)\\ &\quad\cdot\left(1 - (t-t_0)\kappa_{t_0}(z)\nu_{t_0}(z) + O(|t-t_0|^2)\right)ds_z\\ &= (t-t_0)\frac{2\pi}{L^2}\int_{\partial\mathcal{G}_{t_0}}\nu_{t_0}(z)ds_z\\ &\quad - (t-t_0)\frac{1}{L}\int_{\partial\mathcal{G}_{t_0}}N_{t_0}(z,\zeta)\kappa_{t_0}(z)\nu_{t_0}(z)ds_z + O(|t-t_0|^2).\end{aligned}$$

Combining this with equation (3.21) proves the theorem in the case that $t < t_0$. The case that $t > t_0$ is dealt with as in the proof of Theorem 3.1. □

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