CONFORMAL INVARIANTS AND HIGHER-ORDER SCHWARZ LEMMAS

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Abstract. We derive a generalization of the Grunsky inequalities using the Dirichlet principle. As a corollary, sharp distortion theorems for bounded univalent functions are proven for invariant differential expressions which are higher-order versions of the Schwarzian derivative. These distortion theorems can be written entirely in terms of conformal invariants depending on the derivatives of the hyperbolic metric, and can be interpreted as 'Schwarz lemmas'. In particular, sharp estimates on distortion of the derivatives of geodesic curvature of a curve under bounded univalent maps are given.

1. Introduction and statement of results

Let $\mathcal{B}$ denote the set of holomorphic univalent maps from the unit disc into itself. This paper is mainly concerned with finding necessary conditions for a map to be in $\mathcal{B}$. These conditions can be expressed in two ways: as estimates of the derivatives of the function, or as comparisons between kernel functions related to Green's function.

The estimates readily admit a geometric interpretation; they can be used to get a comparison between the geometry of the image domain to the geometry of the unit disc in which it is embedded. In a sense the theorems are like Schwarz lemmas, but for higher-order geometric quantities. The first order case is the Schwarz lemma itself. The second order case is a bound on the distortion of geodesic curvature under a univalent map, and was derived by Flinn and Osgood from the Schiffer-Tammi inequality [7]. This paper derives higher-order cases, for example bounds on the derivative of geodesic curvature.

One of the main themes is the relation between higher-order Schwarzians [18], [4] and the distortion of the derivatives of hyperbolic geodesic curature. Higher-order Versions of the Schwarzian derivatives were first considered by Aharanov [1] and Harmelin [8]. Their Schwarzians have a different invariance property than those in [4] and [18].

It is natural to try to relate univalence to the distortion of curves under a mapping. The relation of the Schwarzian to the distortion of the derivative of geodesic curvature appears in Osgood and Stowe [14] in a more general setting. Epstein [5],[6] relates the Schwarzian derivative of a map to the principal curvature of a certain surface in hyperbolic three-space naturally associated to the map. Furthermore, he explicitly relates univalence of the map to the curvature of this surface. This paper is another variation on the theme of relating curvature to univalence - more precisely, to change in curvature. However, here we do not leave the plane.

We now state the main theorem, which is a generalization of the Grunsky inequalities. Let g_i be Green's function of D_i, $i = 1, 2$, and let

$$K_i = -\frac{2}{\pi} \frac{\partial^2 g_i}{\partial z \partial \bar{w}}$$

be the Bergman kernel. Consider also the related kernel function (see [3])

$$l_i(z, w) = \frac{1}{\pi} \frac{1}{(z-w)^2} + \frac{2}{\pi} \frac{\partial^2 g_i}{\partial z \partial w}.$$

Then we have

Theorem 1. *If D_1 and D_2 are simply-connected domains, bounded by piecewise smooth curves, and $D_1 \subset D_2$, then for any collection of points $\zeta_\mu \in D_1$ and scalars $\alpha_\mu \in \mathbf{C}$, $\mu = 1, \ldots n$, and $m \geq 0$,*

$$\Re \left(\sum_{\mu,\nu} \left[\alpha_\mu \overline{\alpha}_\nu \frac{\partial^{2m} K_1}{\partial z^m \partial \overline{w}^m}(\zeta_\mu, \zeta_\nu) - \alpha_\mu \alpha_\nu \frac{\partial^{2m} l_1}{\partial z^m \partial w^m}(\zeta_\mu, \zeta_\nu) \right] \right)$$
$$\geq \Re \left(\sum_{\mu,\nu} \left[\alpha_\mu \overline{\alpha}_\nu \frac{\partial^{2m} K_2}{\partial z^m \partial \overline{w}^m}(\zeta_\mu, \zeta_\nu) - \alpha_\mu \alpha_\nu \frac{\partial^{2m} l_2}{\partial z^m \partial w^m}(\zeta_\mu, \zeta_\nu) \right] \right).$$

The proof of this theorem uses a method of Nehari's which exploits the Dirichlet principle. The derivatives of the kernel functions l and K are in fact geometric quantities, representing how the domain embeds in the plane; they are closely related to some 'Schwarzian tensors' generalizing that of Osgood and Stowe [13]. This will be explained in Section 5.

Since the kernel functions can be written in terms of a map from the unit disc to the domain in question, Theorem 1 immediately leads to estimates on the derivatives of functions in $\mathcal{B}$. More precisely, when $z = w$, the kernels

$$\frac{\partial^{2m} l}{\partial z^m \partial w^m}(z, w)$$

have expressions in terms of higher-order Schwarzians [18] which we now define.

Definition 1.

$$\begin{aligned} \sigma_3(f)(z) &= \frac{f'''}{f'} - \frac{3}{2}\frac{f''^2}{f'^2} \\ \sigma_{n+1}(f)(z) &= \sigma_n(f)'(z) - (n-1)\frac{f''}{f'}\sigma_n(f). \end{aligned}$$

(These are also best thought of as n-differentials.) They satisfy the invariance

$$\sigma_n(S \circ f \circ T)(z) = \sigma_n(f)(T(z))T'(z)^{n-1} \tag{1}$$

for all Möbius transformations T and affine S. These compare hyperbolic geometry on the domain to Euclidean on the image, and are appropriate for maps into the plane. A variation of these are more appropriate for maps into the unit disc:

Definition 2. *Let*

$$\sigma_n^h(f)(a) = \sigma_n(T_{-f(a)} \circ f)(a)$$

where

$$T_w = \frac{z + w}{1 + \overline{w}z}.$$

Note that $\sigma_3(f) = \sigma_3^h(f)$. These are invariant under both pre- and post-composition by Möbius transformations. The invariants $\sigma_n(f)$ are holomorphic while $\sigma_n^h(f)$ are not.

Finally let

$$D_n^h f(a) = \left.\frac{\partial^n}{\partial z^n}\right|_{z=0} T_{-f(a)} \circ f \circ T_a \tag{2}$$

be the non-holomorphic hyperbolic derivatives of Minda [10]. Then we have that

Theorem 2. *If $f \in \mathcal{B}$, then*

(1)
$$\frac{|\sigma_3^h(f)(z)|}{\lambda(z)^2} \leq 6\left(1 - |D_1^h f(z)|^2\right)$$

(2)
$$\frac{|\sigma_5^h(f)(z) + \sigma_3^h(f)(z)^2|}{30\lambda(z)^4} \leq \frac{|D_2^h f(z)|^2}{|D_1^h f(z)|^2} + 2\left(1 - |D_1^h f(z)|^4\right)$$

(3)
$$\frac{|18\sigma_7^h(f)(z) + 36\sigma_5^h(f)(z)\sigma_3^h(f)(z) + 81\sigma_4^h(f)(z)^2 + 80\sigma_3^h(f)(z)^3|}{2520\lambda(z)^6}$$
$$\leq \left|\frac{\sigma_3^h(f)(z)}{\lambda(z)^2} - \frac{3}{2}\frac{D_2^h f(z)^2}{D_1^h f(z)^2}\right|^2 + 18\frac{|D_2^h f|^2}{|D_1^h f(z)|^2} + 12\left(1 - |D_1^h f(z)|^6\right)$$

This is sharp for $T_1 \circ k \circ T_2$ where k is the Koebe function and T_i are disc automorphisms.

The first estimate is due to Nehari (see [17], p 99).

Theorem 1 only produces inequalities for odd orders of differentiation. However, we were also able to derive

Theorem 3. *If $f \in \mathcal{B}$, then*

$$\frac{|\sigma_4^h(f)|}{\lambda^3} + 2\frac{|D_2^h f|}{|D_1^h f|}\frac{|\sigma_3^h(f)|}{\lambda^2} \leq 96 - 48|D_1^h f|^2 + 48|D_1^h f|^3.$$

This is probably not sharp.

In section 5 we show that Theorems 2 and 3 have a natural formulation in terms of conformal invariants associated to a pair of hyperbolic domains, one a subset of the other. When written this way, the distortion theorems can be seen as higher-order Schwarz lemmas comparing the hyperbolic geometry of the image of f to that of the disc containing the image. In particular, we will derive the following corollary.

Corollary 1. *Let λ denote the hyperbolic line element on the unit disc. If γ is a smooth curve, and $f \in \mathcal{B}$, then, denoting by $k(\gamma)$ and $k(f \circ \gamma)$ the hyperbolic geodesic curvatures of γ and $f \circ \gamma$, and by ds hyperbolic arc length, we have the following inequalities.*

$$\begin{aligned} |k(f\circ\gamma)ds_{f\circ\gamma} - k(\gamma)ds_\gamma| &\leq 4\left(ds_\gamma - ds_{f\circ\gamma}\right) \\ \left|\frac{dk}{ds}(f\circ\gamma)ds_{f\circ\gamma}^2 - \frac{dk}{ds}(\gamma)ds_\gamma^2\right| &\leq 6\left(ds_\gamma^2 - ds_{f\circ\gamma}^2\right) \end{aligned}$$

If we furthermore assume that γ is a hyperbolic geodesic, then

$$\left|\frac{d^2k}{ds^2}(f\circ\gamma)\right| \leq 48\left(1+\frac{ds_{f\circ\gamma}^3}{ds_\gamma^3}+\left(1-\frac{ds_{f\circ\gamma}^2}{ds_\gamma^2}\right)\right) \tag{3}$$

The first two inequalities are sharp.

The first inequality was given by Flinn and Osgood [7] in a slightly different form.

2. Nehari's method and the main theorem

Nehari invented a method of deriving inequalities for classes of holomorphic functions directly from the Dirichlet principle. The method is as follows. Let D_1, D_2, and R be Riemann surfaces, with D_1 and D_2 bounded by piecewise smooth curves, and $D_1 \subset D_2 \subset R$. Let S be a real, single-valued harmonic function on R, with the exception of finitely many points, at which S has specified singularities.

Let p_1 and p_2 be the solutions of the Dirichlet problem

$$\Delta(p_i + S) = 0 \quad \text{in} \quad D_i, \qquad\qquad p_i|_{\partial D_i} = 0.$$

Applying the Dirichlet principle to the function

$$u = \begin{cases} S & D_2 \backslash D_1 \\ p_1 + S & D_1 \end{cases}$$

we get

$$\iint_{D_2} \nabla u \cdot \nabla u \, dA \geq \iint_{D_2} \nabla(p_2+S)\cdot\nabla(p_2+S)\, dA.$$

After using Green's formula we get Nehari's theorem [12]

Theorem 4 (Nehari)**.** *Let n denote the outward unit normal, and let R, D_i, S and p_i be as above. Then*

$$\int_{\partial D_1} S(z)\frac{\partial p_1}{\partial n}(z)\, ds \geq \int_{\partial D_2} S(z)\frac{\partial p_2}{\partial n}(z)\, ds.$$

This is the work-horse inequality, from which many other inequalities follow. The functions p_i depend on the singularity function S and the domains D_i; and in many cases can be written in terms of canonical domain functions (such as Green's function and its derivatives). Thus by judicious choice of S, inequalities can be derived for domain functions or for mapping functions onto canonical domains.

Below we derive the Schwarz lemma (for the special case of univalent maps) using this method. This is included both to illustrate the method with a simple example, and also to exhibit the unified nature of the approach: we will see in the proof of Theorem 1 that by modifying the order of the pole in the singularity function one gets Schwarz lemmas of various orders.

Example 1. *The Schwarz lemma*

Choosing $S(z) = -\Re\log(z-\zeta)$, we get the Schwarz lemma (in the special case of univalent maps into the disc). Let $F : D_1 \to D_2$ be a one-to-one, onto map. We get that

$$p_1(z,\zeta) = -g_1(z,\zeta) = \Re\log\frac{F(z)-F(\zeta)}{1-\overline{F(\zeta)}F(z)}$$

where g_1 denotes Green's function of the domain D_1. Let

$$q_1(z) = \log \frac{F(z) - F(\zeta)}{1 - \overline{F(\zeta)}F(z)}$$

and

$$\sigma(z) = -\log(z - \zeta).$$

so that $\Re(q_1) = p_1$, $\Re(\sigma) = S$ (note that q_1 and σ are multi-valued). Now $p_1 = 0$ on ∂D_1, and consequentially $\frac{\partial p_1}{\partial n} ds = \frac{1}{i} {q_1}' dz$; using both of these facts we can compute that

$$\int_{\partial D_1} S \frac{\partial p_1}{\partial n} ds = \int_{\partial D_1} (S + p_1) \frac{\partial p_1}{\partial n} ds = \Re\left(\frac{1}{i}\int_{\partial D_1} (\sigma + q_1){q_1}' dz\right).$$

But $\sigma + q_1$ is analytic and single-valued, so we can replace q_1' with $-\sigma'$:

$$\begin{aligned}\int_{\partial D_1} S \frac{\partial p_1}{\partial n} ds &= \Re\left(\frac{1}{i}\int_{\partial D_1}\left(-\log(z-\zeta) + \log \frac{F(z) - F(\zeta)}{1 - \overline{F(\zeta)}F(z)}\right)\frac{1}{z - \zeta} dz\right) \\ &= 2\pi \log \frac{|F'(\zeta)|}{1 - |F(\zeta)|^2}.\end{aligned}$$

In particular, letting $F(z) = z$ and $D_2 = D$, we get that

$$\int_{\partial D_2} S \frac{\partial p_2}{\partial n} ds = 2\pi \log \frac{1}{1 - |\zeta|^2}.$$

Now for any conformal map $f : D \to D_1$ let $F = f^{-1}$, so that

$$\int_{\partial D_1} S \frac{\partial p_1}{\partial n} ds = \frac{|f'(z)|}{1 - |f(z)|^2}.$$

Applying Nehari's theorem gives the Schwarz lemma.

We now derive the main theorem,

Proof. (of Theorem 1). Set k=m+1 in order to simplify notation in the proof. Let

$$\sigma_k = -\frac{(k-1)!}{2} \sum_\nu \frac{\alpha_\nu}{(z - \zeta_\nu)^k}$$

and

$$p_{1k}(z) = \Re\left(\sum_\nu \alpha_\nu \frac{\partial^k g_1}{\partial \zeta_\nu^k}(z, \zeta_\nu)\right),$$

where g_1 is Green's function. Also let $S_k = \Re(\sigma_k)$.

Lemma 1. *For piecewise smoothly bounded domains D_1, and fixed ζ, $\frac{\partial^k g_1}{\partial \zeta^k}(z, \zeta)$ vanishes as $z \to \partial D_1$.*

Proof. Let $F : D_1 \to D$ be a conformal map onto the unit disc, so that $g_1(z, \zeta) = g(F(z), F(\zeta))$. Let $w = F(z)$, $\xi = F(\zeta)$. It is easily checked directly that $\frac{\partial^k g}{\partial \xi^k} \to 0$ as $w \to \partial D$ for all k. Now apply the chain rule. □

Thus we have that $p_{1k}(z, \zeta) \to 0$ as $z \to \partial D_1$. Also, for fixed ζ_ν, $p_{1k} + S_k$ is harmonic in z and single-valued. It is non-singular because

$$p_{1k} + S_k = \Re\left(\sum_\nu \alpha_\nu \frac{\partial^k}{\partial \zeta_\nu^k}\left(g_1(z, \zeta_\nu) + \log|z - \zeta_\nu|\right)\right).$$

Let

$$q_1(z) = -\log \frac{F(z)-F(\zeta)}{1-\overline{F(\zeta)}F(z)}.$$

q_1 is analytic in z for fixed ζ and $g_1 = \Re(q_1)$ (q_1 is of course multi-valued). Then

$$\begin{aligned} p_{1k}(z) &= \Re\left(\sum_\nu \alpha_\nu \frac{\partial^k g_1}{\partial \zeta_\nu^k}(z,\zeta_\nu)\right) = \frac{1}{2}\Re\left(\sum_\nu \alpha_\nu \frac{\partial^k}{\partial \zeta_\nu^k}(q_1+\overline{q_1})\right) \\ &= \frac{1}{2}\Re\left(\sum_\nu \alpha_\nu \frac{\partial^k q_1}{\partial \zeta_\nu^k} + \alpha_\nu \frac{\partial^k \overline{q_1}}{\partial \zeta_\nu^k}\right) \\ &= \frac{1}{2}\Re\left(\sum_\nu \alpha_\nu \frac{\partial^k q_1}{\partial \zeta_\nu^k} + \overline{\alpha_\nu} \frac{\partial^k q_1}{\partial \overline{\zeta}_\nu^k}\right). \end{aligned}$$

Now let

$$h_{1k} = \frac{1}{2}\sum_\nu \left(\alpha_\nu \frac{\partial^k q_1}{\partial \zeta_\nu^k} + \overline{\alpha}_\nu \frac{\partial^k q_1}{\partial \overline{\zeta}_\nu^k}\right).$$

Lemma 2. *h_{1k} is analytic in z for fixed ζ_ν, and $h_{1k}+\sigma_k$ is non-singular.*

Proof. Note that $\frac{\partial^k q_1}{\partial \zeta_\nu^k}$ is single valued for $k \geq 1$. It's easy to compute that $\frac{\partial^k q_1}{\partial \overline{\zeta}_\nu^k}$ is non-singular using the explicit formula

$$q_1(z) = -\log \frac{F(z)-F(\zeta)}{1-\overline{F(\zeta)}F(z)}.$$

It is clearly analytic in z, as is $\frac{\partial^k q_1}{\partial \zeta_\nu^k}$. Now

$$\sum_\nu \frac{\alpha_\nu}{2}\frac{\partial^k q_1}{\partial \zeta_\nu^k} + \sigma_k = \sum_\nu \frac{\alpha_\nu}{2}\frac{\partial^k}{\partial \zeta_\nu^k}(q_1 + \log(z-\zeta_\nu))$$

and $q_1 + \log(z-\zeta_\nu)$ is single-valued and non-singular. The lemma follows. □

Now using the fact that $\frac{\partial p_{1k}}{\partial n} ds = \frac{1}{i} h_{1k}' dz$ on ∂D_1, (since $p_{1k}=0$ on ∂D_1),

$$\begin{aligned} \int_{\partial D_1} S_k \frac{\partial p_{1k}}{\partial n_z} ds &= \int_{\partial D_1} (S_k + p_{1k}) \frac{\partial p_{1k}}{\partial n_z} ds \\ &= \Re\left(\frac{1}{i}\int_{\partial D_1} (\sigma_k + h_{1k})\, h_{1k}'\, dz\right) \\ &= -\Re\left(\frac{1}{i}\int_{\partial D_1} (\sigma_k + h_{1k})\, \sigma_k'\, dz\right) \end{aligned}$$

since by the previous lemma we can replace h'_{1k} with $-\sigma'_k$. So

$$\begin{aligned}\int_{\partial D_1} S_k \frac{\partial p_{1k}}{\partial n_z} ds &= -\Re\left[\frac{1}{2i}\int_{\partial D_1}\sum_\nu\left(\alpha_\nu\frac{\partial^k q_1}{\partial \zeta_\nu^k} - \frac{(k-1)!\alpha_\nu}{(z-\zeta_\nu)^k} + \overline{\alpha}_\nu\frac{\partial^k q_1}{\partial\overline{\zeta}_\nu^k}\right)\cdot\right.\\ &\qquad\left.\left(\sum_\mu \frac{\alpha_\mu}{2}\frac{k!}{(z-\zeta_\mu)^{k+1}}\right)dz\right]\\ &= -\frac{\pi}{2}\Re\left[\sum_{\mu,\nu}\alpha_\mu\alpha_\nu\left(\frac{\partial^{2k}q_1}{\partial\zeta_\mu^k\partial\zeta_\nu^k} + (-1)^{k+1}\frac{(2k-1)!}{(\zeta_\mu-\zeta_\nu)^{2k}}\right)+\right.\\ &\qquad\left.\sum_{\mu,\nu}\alpha_\mu\overline{\alpha}_\nu\frac{\partial^{2k}q_1}{\partial\zeta_\mu^k\partial\overline{\zeta}_\nu^k}\right].\end{aligned}$$

By the Cauchy-Riemann equations, $\frac{\partial q_1}{\partial z} = 2\frac{\partial g_1}{\partial z}$, so the right-hand side becomes

$$\pi\Re\left[\sum_{\mu,\nu}\left(\alpha_\mu\overline{\alpha}_\nu\frac{\partial^{2k-2}K_1}{\partial\zeta_\mu^{k-1}\partial\overline{\zeta}_\nu^{k-1}}(\zeta_\mu,\zeta_\nu) - \alpha_\mu\alpha_\nu\frac{\partial^{2k-2}l_1}{\partial\zeta_\mu^{k-1}\partial\zeta_\nu^{k-1}}(\zeta_\mu,\zeta_\nu)\right)\right].$$

By Theorem 4, the corresponding quantity for D_2 is smaller. Letting $m = k-1$ finishes the proof. □

Remark 1. Choosing $m = 0$ and $D_2 = D$, we recover the Grunsky inequalities, in a form due to Bergman and Schiffer (see [12]). They also gave a variational proof [3] of the monotonicity of this quantity for $m = 0$.

3. Invariant derivatives

The results in this paper involve hyperbolic geometry. It is convenient therefore to make explicit use of covariant differentiation in the hyperbolic metric in the computations. This has the advantage of both making the geometry more explicit and computations themselves simpler.

This section has two purposes. The first is to briefly review the derivatives of Minda [9, 10]; the second is to introduce some sesquilinear and antisymmetric derivatives. These latter vastly simplify the explicit computations of the derivatives of l and K necessary to exploit Theorem 1.

We now define the covariant derivatives of Minda. These first appeared in function theory in a paper of Peschl [15] for the hyperbolic metric, and were later generalized by Minda [11] to arbitrary conformal metrics. We define them here in this generality; this introduces no extra difficulty and later allows us to deal with several cases at once. Let $w = f(z)$ be a locally univalent map from a domain D_1 to a domain D_2, and let $\rho(z)^2|dz|^2$ and $\sigma(z)^2|dz|^2$ be conformal metrics defined in D_1 and D_2 respectively. Let

$$\Gamma_\rho = 2\frac{\partial}{\partial z}\log\rho \qquad\text{and}\qquad \Gamma_\sigma = 2\frac{\partial}{\partial w}\log\sigma$$

be their Christoffel symbols. One differentiates a tensor of the form

$$g(z)\frac{\partial}{\partial w}\otimes dz^n$$

according to the rule

$$\nabla^{\rho,\sigma}\left(g\left(\frac{\partial}{\partial w}\right)^m \otimes dz^n\right) = \left[g' + \left(m\Gamma^\sigma \circ f \cdot f' - n\Gamma^\rho\right) g\right]\left(\frac{\partial}{\partial w}\right)^m \otimes dz^{n+1}.$$

(This is just the usual Riemannian covariant derivative followed by a projection onto the $\left(\frac{\partial}{\partial w}\right)^m \otimes dz^n$ component). It is easily checked that $\nabla^{\rho,\sigma}$ satisfies a Leibniz rule with respect to the multiplication

$$g_1\left(\frac{\partial}{\partial w}\right)^{m_1} \otimes dz^{n_1} \times g_2\left(\frac{\partial}{\partial w}\right)^{m_2} \otimes dz^{n_2} \mapsto g_1 g_2\left(\frac{\partial}{\partial w}\right)^{m_1+m_2} \otimes dz^{n_1+n_2}.$$

The n-th order derivatives of f are defined inductively by

$$\nabla_1^{\rho,\sigma} f = f' \frac{\partial}{\partial w} \otimes dz$$

and

$$\nabla_{n+1}^{\rho,\sigma} f = \nabla^{\rho,\sigma}(\nabla_n^{\rho,\sigma} f).$$

Now let $\lambda(z) = (1-|z|^2)^{-1}$ be the hyperbolic line element on the unit disc. In keeping with Minda's notation [9],[10] we define $D_n f$ and $D_n^h f$ by

$$\nabla_n^{\lambda,1} f = \lambda^n D_n f \frac{\partial}{\partial w} \otimes dz^n \tag{4}$$

$$\nabla_n^{\lambda,\lambda} f = \frac{\lambda^n}{\lambda \circ f} D_n^h f \frac{\partial}{\partial w} \otimes dz^n. \tag{5}$$

These satisfy certain invariances. Let S be an affine map, and T_1 and T_2 be disc automorphisms. Then

$$|D_n(S \circ f \circ T_2)| = |D_n(f) \circ T_2| \tag{6}$$

and

$$|D_n^h(T_1 \circ f \circ T_2)| = |D_n^h(f) \circ T_2| \tag{7}$$

Since $\Gamma_\lambda(0) = 0$, this proves the equivalence of (2) with the definition above. Furthermore we have the useful identities

$$D_n(T_{-f(a)} \circ f)(a) = D_n^h(f)(a) \tag{8}$$

and

$$\frac{(T_{-f(a)} \circ f)''}{(T_{-f(a)} \circ f)'} = \frac{f''}{f'} + \Gamma_\lambda \circ f \cdot f' \tag{9}$$

Next we define the invariant derivatives which will be useful in differentiating l and K. These are defined analogously to covariant derivatives, in a way which must produce invariant quantities.

We first note the following identities for disc automorphisms T.

Proposition 1. (1) $(T(z) - T(w))^2 = (z-w)^2 T'(z)T'(w)$
(2) $(1 - T(z)\overline{T(w)})^2 = (1 - z\overline{w})^2 T'(z)\overline{T'(w)}$
(3) $\frac{T''(z)}{T'(z)} + \frac{2}{z-w} = \frac{2T'(z)}{T(z)-T(w)}$
(4) $\frac{T''(z)}{T'(z)} - \frac{2\overline{w}}{1-z\overline{w}} = \frac{2\overline{T(w)}T'(z)}{1-T(z)\overline{T(w)}}$

These identities suggest making the following definitions.

Definition 3.

(1) $\alpha(z,w) = \frac{1}{1-z\overline{w}}$ *and* $G(z,w) = 2\frac{\partial}{\partial z}\log\alpha = \frac{2\overline{w}}{1-z\overline{w}}$
(2) $\beta(z,w) = \frac{1}{z-w}$ *and* $H(z,w) = 2\frac{\partial}{\partial z}\log\alpha = -\frac{2}{z-w}$

The functions α and β are like an antisymmetric and a sequilinear metric, respectively. G and H function as their Christoffel symbols. Differentiating using these Christoffel symbols leads to quantities invariant under composition with disc automorphisms.

Definition 4.

(1) *Let* $A_1(f)(z,w) = f'(z)$ *and* $A_{n+1}(f)(z,w) = A_{nz} - nG(z,w)A_n(z,w)$
(2) *Let* $B_1(f)(z,w) = f'(z)$ *and* $B_{n+1}(f)(z,w) = B_{nz} - nH(z,w)B_n(z,w)$

For example,

$$\begin{aligned} A_2(f)(z,w) &= f''(z) - \frac{2\overline{w}}{1-z\overline{w}}f'(z) \\ B_2(f)(z,w) &= f''(z) + \frac{2}{z-w}f'(z). \end{aligned}$$

We also clearly have $\lambda^n(z)A_n(f)(z,z) = D_n f(z)$.

Proposition 2. *For a disc automorphism T,*

(1) $A_n(f\circ T)(z,w) = A_n(f)(T(z),T(w))\cdot T'(z)^n$
(2) $B_n(f\circ T)(z,w) = B_n(f)(T(z),T(w))\cdot T'(z)^n$

Proof. Let $u = T(z)$ and $v = T(w)$. We use induction to prove 1.

$$\begin{aligned} A_{n+1}(f\circ T)(z,w) &= (A_n(f)(u,v)\cdot T'(z)^n)_z \\ &\quad -nG(z,w)A_n(f)(u,v)\cdot T'(z)^n \\ &= A_n(f)_u(u,v)\cdot T'(z)^{n+1} \\ &\quad +n\left(\frac{T''(z)}{T'(z)} - G(z,w)\right)A_n(f)(u,v)\cdot T'(z)^n \end{aligned}$$

The quantity in brackets is $-G(T(z),T(w))\cdot T'(z)$ by Proposition 1 part 3. The proof of part 2 makes similar use of Proposition 1 part 4. □

The invariants are best interpreted as nth-order differentials satisying

$$T^*(A_n(f)dz^n) = A_n(f\circ T)dz^n,$$

where T^* denotes the operation of pulling-back under the map T. We thus have two operators taking n-differentials to $n+1$-differentials, namely

$$\begin{aligned} D_s(f(z,w)\,dz^n) &= (f_z(z,w) - nG(z,w)f(z,w))dz^{n+1} \\ D_a(f(z,w)\,dz^n) &= (f_z(z,w) - nH(z,w)f(z,w))dz^{n+1} \end{aligned}$$

It's not hard to show that these both satisfy a Leibniz rule; namely, if one defines the product of k- and l-differentials by $f\,dz^k\cdot g\,dz^l = fg\,dz^{k+l}$, then,

$$D(f\,dz^k\cdot g\,dz^l) = D(f\,dz^k)\cdot g\,dz^l + f\,dz^k\cdot D(g\,dz^l).$$

The following identities follow immediately from Definition 3.

$$G_{\overline{w}}(z,w) = 2\alpha(z,w)^2 \tag{10}$$

$$H_w(z,w) = -2\beta(z,w)^2 \tag{11}$$

These allow us to derive two more useful relations.

Proposition 3.

(1) $A_n(f)_{\overline{w}}(z,w) = -n(n-1)\alpha(z,w)^2 A_{n-1}(f)(z,w)$
(2) $B_n(f)_w(z,w) = n(n-1)\beta(z,w)^2 B_{n-1}(f)(z,w)$

Proof. The proof is by induction. We drop the dependence on f, z, and w for notational convenience.

$$\begin{aligned}
(A_{n+1})_{\overline{w}} &= ((A_n)_z - nGA_n)_{\overline{w}} \\
&= -n(n-1)\alpha^2 (A_{n-1})_z - nG_{\overline{w}} A_n + n^2(n-1)\alpha^2 G A_{n-1} \\
&= -n(n-1)\alpha^2 \left((A_{n-1})_z - (n-1)GA_{n-1}\right) - 2n\alpha^2 A_n \\
&= -n(n+1)\alpha^2 A_n
\end{aligned}$$

We have used equation 10 and $2\alpha_z = G\alpha$. The proof of 2 is similar. $\square$

Finally, we have the useful identities

$$\sigma_3(f)(z) = \frac{B_3(f)(z,w)}{B_1(f)(z,w)} - \frac{3}{2}\frac{B_2(f)(z,w)^2}{B_1(f)(z,w)^2} \tag{12}$$

$$\sigma_3(f)(z) = \frac{A_3(f)(z,w)}{A_1(f)(z,w)} - \frac{3}{2}\frac{A_2(f)(z,w)^2}{A_1(f)(z,w)^2} \tag{13}$$

4. Distortion theorems

With the help of the invariants of the last section, it is now possible to find the derivatives of l_1 and K_1, and so to derive distortion theorems for bounded univalent maps.

The Bergman kernel and l-kernel obey certain transformation laws with respect to conformal mapping. If $f : D_2 \to D_1$ is a conformal mapping between simply connected domains, then we have

$$K_1(f(z), f(w)) f'(z)\overline{f'(w)} = K_2(z,w) \tag{14}$$

and

$$l_1(f(z), f(w)) f'(z) f'(w) = l(z,w) + \frac{1}{\pi}\left[\frac{f'(z)f'(w)}{(f(z)-f(w))^2} - \frac{1}{(z-w)^2}\right]. \tag{15}$$

These follow directly from the definition and properties of Green's function. On the unit disc, $l(z,w) = 0$ and $K(z,w) = \frac{1}{\pi(1-z\overline{w})^2}$, from which it follows that for $f : D \to D_1$ conformal,

$$l_1(f(z), f(w)) = \frac{1}{\pi}\left[\frac{1}{(f(z)-f(w))^2} - \frac{1}{f'(z)f'(w)(z-w)^2}\right] \tag{16}$$

and

$$K_1(f(z), f(w)) = \frac{1}{\pi f'(z)\overline{f'(w)}(1-z\overline{w})^2}. \tag{17}$$

Remark 2. *These quantities satisfy an invariance with respect to disc automorphisms T in the following sense: if one denotes $\Phi(f)(z,w) = l_1(f(z), f(w))$, then $\Phi(f \circ T)(z,w) = \Phi(f)(T(z), T(w))$; similarly for $K_1(f(z), f(w))$. This can be shown directly using Proposition 1 parts 3 and 4; or, it can be seen to follow directly from* *the validity of equations (16) and (17) for any choice of conformal map from the disc to D_1. It follows immediately that*

$$\frac{\partial^{k+l} l_1}{\partial\zeta^k \partial\eta^l}(f(z), f(w)) \qquad \text{and} \qquad \frac{\partial^{k+l} K_1}{\partial\zeta^k \partial\eta^l}(f(z), f(w))$$

are invariant in the same sense.

We now compute the derivatives of l_1. This is much easier if one groups terms into products of $\frac{\beta(z,w)^2}{B_1(z,w)}$ and 0-differentials (on which D_a acts as $\frac{\partial}{\partial z}$). Then one applies the Leibniz rule and either Proposition 3, and the identity

$$\frac{\partial}{\partial z}\left(\frac{\beta(z,w)^2}{B_1(z,w)}\right) = -\frac{\beta(z,w)^2 B_2(z,w)}{B_1(z,w)^2}.$$

The results are listed here. The arrangement of terms and products shown below is geared towards exploiting the rules just mentioned, rather than giving the most symmetric expression. For convenience, we abbreviate $\beta \equiv \beta(z,w)$, $\beta^t \equiv \beta(w,z)$, $B \equiv B(z,w)$, $B^t \equiv B(w,z)$, and so on.

$$l_{1\zeta 1}(f(z), f(w)) = -\frac{2}{(f(z)-f(w))^3} + \frac{{\beta^t}^2}{B_1^t}\frac{B_2}{B_1^3} \tag{18}$$

$$\begin{aligned} l_{1\zeta\zeta}(f(z), f(w)) &= \frac{6}{(f(z)-f(w))^4} + \frac{{\beta^t}^2}{B_1^t}\left(\frac{B_3}{B_1^4} - \frac{3}{2}\frac{B_2^2}{B_1^5}\right) \\ &- \frac{3}{2}\frac{{\beta^t}^2}{B_1^t}\frac{B_2^2}{B_1^5} \end{aligned} \tag{19}$$

$$l_{1\zeta\eta}(f(z), f(w)) = -\frac{6}{(f(z)-f(w))^4} - \frac{\beta^2}{B_1}\frac{B_2^t}{{B_1^t}^3}\frac{B_2}{B_1^2} + 2\frac{\beta^4}{B_1^2}\frac{1}{{B_1^t}^2} \tag{20}$$

$$\begin{aligned} l_{1\zeta\zeta\eta}(f(z), f(w)) &= \frac{24}{(f(z)-f(w))^5} - \frac{{\beta^t}^2}{B_1^t}\frac{B_2^t}{{B_1^t}^2}\left(\frac{B_3}{B_1^4} - \frac{3}{2}\frac{B_2^2}{B_1^5}\right) \\ &+\frac{3}{2}\frac{{\beta^t}^2}{B_1^t}\frac{B_2^t}{{B_1^t}^2}\frac{B_2^2}{B_1^5} - 6\frac{{\beta^t}^4}{{B_1^t}^2}\frac{B_2}{B_1^4} \end{aligned} \tag{21}$$

$$\begin{aligned} l_{1\zeta\zeta\eta\eta}(f(z), f(w)) &= \frac{120}{(f(z)-f(w))^6} + 18{\beta^t}^4\frac{B_2}{B_1^4}\frac{B_2^t}{{B_1^t}^4} - 12\frac{{\beta^t}^6}{B_1^3 {B_1^t}^3} \\ &- {\beta^t}^2\left(\frac{B_3}{B_1^4} - 3\frac{B_2^2}{B_1^5}\right)\left(\frac{B_3^t}{{B_1^t}^4} - 3\frac{{B_2^t}^2}{{B_1^t}^5}\right) \end{aligned} \tag{22}$$

Similar tricks allow one to compute the derivatives of K_1.

$$K_{1\zeta}(f(z),f(w)) = -\frac{\overline{\alpha}^{t\,2}}{\overline{A_1}^{t}}\frac{A_2}{A_1^3} \tag{23}$$

$$K_{1\zeta\zeta}(f(z),f(w)) = -\frac{\alpha^{t\,2}}{\overline{A_1}^{t\,3}}\left(\frac{A_3}{A_1^4}-\frac{3}{2}\frac{A_2^2}{A_1^5}\right)+\frac{3}{2}\frac{\alpha^2}{\overline{A_1}^{t}}\frac{A_2^2}{A_1^5} \tag{24}$$

$$K_{1\zeta\eta}(f(z),f(w)) = \frac{\alpha^2}{A_1}\frac{A_2}{A_1^2}\frac{\overline{A_2}^{t}}{\overline{A_1}^{t\,3}}+2\frac{\alpha^4}{A_1^2}\frac{1}{\overline{A_1}^{t\,2}} \tag{25}$$

$$\begin{aligned} K_{1\zeta\zeta\eta}(f(z),f(w)) &= \frac{\overline{\alpha}^{t\,2}}{\overline{A_1}^{t}}\frac{\overline{A_2}^{t}}{\overline{A_1}^{t\,2}}\left(\frac{A_3}{A_1^4}-\frac{3}{2}\frac{A_2^2}{A_1^5}\right) \\ &\quad -\frac{3}{2}\frac{\overline{\alpha}^{t\,2}}{\overline{A_1}^{t}}\frac{\overline{A_2}^{t}}{\overline{A_1}^{t\,2}}\frac{A_2^2}{A_1^5}-6\frac{\overline{\alpha}^{t\,4}}{\overline{A_1}^{t\,2}}\frac{A_2}{A_1^4} \end{aligned} \tag{26}$$

$$\begin{aligned} K_{1\zeta\zeta\eta\eta}(f(z),f(w)) &= \alpha^2\left(\frac{A_3}{A_1^4}-3\frac{A_2^2}{A_1^5}\right)\left(\frac{\overline{A_3}^{t}}{\overline{A_1}^{t\,4}}-3\frac{\overline{A_2}^{t\,2}}{\overline{A_1}^{t\,5}}\right) \\ &\quad +18\alpha^2\frac{\overline{A_2}^{t}}{\overline{A_1}^{t\,4}}\frac{A_2}{A_1^4}+12\frac{\alpha^6}{A_1^3\overline{A_1}^{t\,3}} \end{aligned} \tag{27}$$

An attempt to compute these directly will quickly convince one of the utility of the invariant derivatives. Setting $z=w$, tedious but straightforward computations making use of the identities (12), yield

$$l_1(f(z),f(z)) = -\frac{1}{6f'(z)^2}\sigma_3(f)(z) \tag{28}$$

$$l_{1\zeta}(f(z),f(z)) = -\frac{1}{12f'(z)^3}\sigma_4(f)(z) \tag{29}$$

$$l_{1\zeta\zeta}(f(z),f(z)) = -\frac{1}{120f'(z)^4}\left(6\sigma_5(f)(z)+4\sigma_3(f)(z)^2\right) \tag{30}$$

$$l_{1\zeta\eta}(f(z),f(z)) = \frac{1}{30f'(z)^4}\left(\sigma_5(f)(z)+\sigma_3(f)(z)^2\right) \tag{31}$$

$$l_{1\zeta\zeta\eta}(f(z),f(z)) = \frac{1}{60f'(z)^5}\left(\sigma_6(f)(z)+2\sigma_4(f)(z)\sigma_3(f)(z)\right) \tag{32}$$

$$\begin{aligned} l_{1\zeta\zeta\eta\eta}(f(z),f(z)) &= \frac{1}{2520f'(z)^6}\big(18\sigma_7(f)(z)+36\sigma_5(f)(z)\sigma_3(f)(z) \\ &\quad +81\sigma_4(f)(z)^2+80\sigma_3(f)(z)^3\big) \end{aligned} \tag{33}$$

and, with considerably less work,

$$K_{1\zeta}(f(z),f(z)) = -\frac{D_2 f(z)}{D_1 f(z)^2\overline{D_1 f(z)}} \tag{34}$$

$$K_{1\zeta\overline{\eta}}(f(z),f(z)) = \frac{|D_3 f(z)|^2}{|D_1 f(z)|^6} + \frac{2}{|D_1 f(z)|^4} \tag{35}$$

$$K_{1\zeta\zeta}(f(z),f(z)) = -\frac{\sigma_3 f(z)}{\lambda(z)^2 D_1 f(z)^3\overline{D_1 f(z)}} + \frac{3}{2}\frac{D_2 f(z)^2}{D_1 f(z)^5\overline{D_1 f(z)}} \tag{36}$$

$$\begin{aligned} K_{1\zeta\zeta\overline{\eta}}(f(z),f(z)) &= \frac{\overline{D_2 f(z)}\sigma_3(f)(z)}{\lambda(z)^2|D_1 f(z)|^3} - \frac{3}{2}\frac{D_2 f(z)^2\overline{D_2 f(z)}}{D_1 f(z)^5\overline{D_1 f(z)}^3} \\ &- 6\frac{D_2 f(z)}{D_1 f(z)^4\overline{D_1 f(z)}^2} \end{aligned} \tag{37}$$

$$\begin{aligned} K_{1\zeta\zeta\overline{\eta\eta}}(f(z),f(z)) &= \left|\frac{\sigma_3(f)}{\lambda(z)^2} - \frac{3}{2}\frac{D_2 f(z)^2}{D_1 f(z)^2}\right|^2 \frac{1}{|D_1 f(z)|^6} \\ &+ 18\frac{|D_2 f(z)|^2}{|D_1 f(z)|^8} + \frac{12}{|D_1 f(z)|^6} \end{aligned} \tag{38}$$

These identities were also verified with the use of Mathematica.

It is now possible to prove Theorem 2.

Proof. (of Theorem 2) On the unit disc, we have that $l \equiv 0$ and

$$K(z,w) = \frac{1}{\pi(1-\bar{w}z)^2}. \tag{39}$$

Also,

$$K_{z\bar{w}}(z,z) = \frac{2+4|z|^2}{\pi(1-|z|^2)^4} \tag{40}$$

$$K_{zz\bar{w}\bar{w}}(z,z) = \frac{12+72|z|^2+36|z|^4}{\pi(1-|z|^2)^6}. \tag{41}$$

To derive the first inequality, apply Theorem 1 for the case $m = 0$; choose only one point $\zeta = f(z)$ and choose $\alpha = e^{i\theta}$. By (28), (14), and (39), we get

$$\Re\left(\frac{1}{|D_1 f(z)|^2} - \frac{e^{2i\theta}\sigma_3(f)(z)}{6f'(z)^2}\right) \geq \frac{1}{(1-|f(z)|^2)^2}.$$

Letting $2\theta = -\arg(f'(z)^{-2}\sigma_3(f)(z))$ and rearranging results in the first inequality.

The second and third inequalities are derived in exactly the same way: for the second, use the case $m = 1$ of Theorem 1, (35), (31), and (40); for the third, use the case $m = 2$, (38), (33), and (41).

These inequalities are sharp for the functions $\phi_t = k_t^{-1}\circ k$ where k is the Koebe function and $k_t = e^t k$. A (lengthy) computation shows that

$$\begin{aligned} \sigma_3(\phi_t)(0) &= -6+6e^{-t} \\ \sigma_4(\phi_t)(0) &= 48-48e^{-t} \\ 30^{-1}(\sigma_5(\phi_t)(0)+\sigma_3(\phi_t)(0)^2) &= -18+8e^{-4t}-16e^{-2t}+32e^{-t} \end{aligned}$$

and

$$\begin{aligned}&2520^{-1}(18\sigma_7(\phi_t)(0)+36\sigma_5(\phi_t)(0)\sigma_3(\phi_t)(0)+81\sigma_4(\phi_t)(0)^2+80\sigma_3(\phi_t)(0)^3)\\&\qquad=-12(-e^{-6t}+27e^{-4t}-144e^{-3t}+306e^{-2t}-288e^{-t}+100).\end{aligned}$$

We then use

$$\begin{aligned}|D_1^h\phi_t(0)| &= e^{-t}\\ \left|\frac{D_2^h\phi_t(0)}{D_1^h\phi_t(0)}\right| &= 4-4e^{-t}\end{aligned}$$

to compare with the upper bounds. □

This method does not produce estimates for even orders of differentiation. We include two such even-order estimates derived from other methods. The first is simply the Schiffer-Tammi inequality [7]:

Theorem 5. *If $f\in\mathcal{B}$, then*

$$\frac{|D_2^hf|}{|D_1^hf|}\leq 4(1-|D_1^hf|).$$

Proof. Apply the estimate

$$\left|\frac{a_2}{a_1}\right|\leq 2(1-|a_1|)$$

to the transformed function $T_{-f(a)}\circ f\circ T_a$. □

The second is Theorem 3. As previously mentioned, it is probably not sharp. On the other hand, letting f be ϕ_t, the proof of sharpness in Theorem 2 shows that the right-hand side in Theorem 3 approaches the sharp bound as $t\to\infty$.

Proof. Let k be the Koebe function; we have the sharp estimate [18]

$$\frac{|\sigma_4(k\circ f)|}{\lambda^3}\leq 48.$$

It follows from the composition law for the Schwarzian

$$\sigma_3(k\circ f)=\sigma_3(k)\circ f f'^2+\sigma_3(f)$$

that

$$\sigma_4(k\circ f)=\sigma_4(k)\circ f f'^3-2\frac{k''\circ f}{k'\circ f}f'\sigma_3(f)+\sigma_4(f).$$

Assume for the moment that f is normalized so that $f(0)=0$. Using the facts that $\sigma_4(k)(0)=48$ and $k''(0)/k'(0)=4$,

$$|\sigma_4(f)(0)-8\sigma_3(f)f'(0)|\leq 48(1+|D_1^hf|^3).$$

If f is an arbitrary element of $\mathcal{B}$, we can apply the previous inequality to the normalized function $\tilde{f}=T_{-f(a)}\circ f\circ T_a$, yielding

$$\left|\frac{\sigma_4^h(f)(a)}{\lambda^3(a)}-8D_1^hf(a)\frac{\sigma_3^h(f)(a)}{\lambda^2(a)}\right|\leq 48(1+|D_1^hf(a)|^3). \tag{42}$$

Now applying the Schiffer-Tammi inequality,

$$\frac{|\sigma_4^h(f)|}{\lambda^3}+2\frac{|D_2^hf|}{|D_1^hf|}\frac{|\sigma_3^h(f)|}{\lambda^2}\leq\frac{|\sigma_4^h(f)|}{\lambda^3}-8|D_1^hf|\frac{|\sigma_3^h(f)|}{\lambda^2}+8\frac{|\sigma_3^h(f)|}{\lambda^2}.$$

The desired inequality now follows from Theorem 2 part 1, and the triangle inequality. □

5. Higher-order Schwarz lemmas

In this section it will be shown how the distortion theorems are much like the Schwarz lemma, but rather than bounding the distortion of distance, they bound the distortion of higher-order geometric quantities such as geodesic curvature. (Of course, unlike the Schwarz lemma, in this case the hypothesis of univalence is necessary).

In the first subsection, the distortion theorems are written in terms of conformal invariants. These conformal invariants depend on derivatives of the hyperbolic metric. In the second subsection, we derive estimates on the distortion of curves under a bounded univalent map (Corollary 1). More precisely, bounds are given on the change in geodesic curvature and its derivatives under the map.

5.1. **Conformal Invariants.** In this subsection, we show how Theorem 2 can be written entirely in terms of conformal invariants. These conformal invariants are natural geometric quantities which compare the hyperbolic geometry of $f(D)$ and D.

In order to do this, we show how the higher-order Schwarzians are special cases of a series of 'Schwarzian tensors', generalizing that of Osgood and Stowe [13]. This requires only slightly more development, and has two advantages. First, it will allow us to clearly illustrate how the Schwarzian derivatives relate to change of curvature under a conformal change of metric. This is the subject of the next subsection. Secondly, it allows one to deal with more than one case simulateously.

First, we illustrate the conformal invariance of the simplest inequality, namely the Schiffer-Tammi inequality. Denote by λ_E the hyperbolic line element on a domain E. Denote the pullback of a line element ρ by $f^*(\rho) = \rho \circ f|f'|$. We have that $f^*(\lambda_{f(D)}) = \lambda_D$. We can then write the Schwarz-Pick inequality as

$$1 - \frac{\lambda_D}{\lambda_{f(D)}} \geq 0. \tag{43}$$

Using similar notation, if g is any conformal map then

$$\Gamma_{g(E)} \circ g \cdot g' = \Gamma_E - \frac{g''}{g'} \tag{44}$$

It follows directly from the definition of $D_n^h f$ (5) and (44) that

$$\lambda_D \frac{D_2^h f}{D_1^h f} = -\Gamma_{f(D)} \circ f \cdot f' + \Gamma_D \circ f \cdot f'. \tag{45}$$

Thus the Schiffer-Tammi inequality (Theorem 5) can be immediately rewritten as

Theorem 6.

$$\lambda_{f(D)}^{-1} |\Gamma_{f(D)} - \Gamma_D| \leq 4 \left(1 - \frac{\lambda_D}{\lambda_{f(D)}}\right). \tag{46}$$

The quantities on both sides of the equation are conformal invariants:

$$\lambda_{g(f(D))}^{-1} \circ g \, |\Gamma_{g(f(D))} \circ g - \Gamma_{g(D)} \circ g| = \lambda_{f(D)}^{-1} |\Gamma_{f(D)} - \Gamma_D|. \tag{47}$$

More precisely, let E and F be hyperbolic domains, with $F \subset E$. The triple (E, F, z) is said to be conformally equivalent to (E', F', z') if there is a conformal map g of E onto E' such that $g(F) = F'$ and $g(z) = z'$. Then defining

$$I_0(E, F, z) = \frac{\lambda_E(z)}{\lambda_F(z)}$$

and

$$I_1(E, F, z) = \lambda_F^{-1}(\Gamma_E(z) - \Gamma_F(z)),$$

we have that whenever (E, F, z) is conformally equivalent to (E', F', z')

$$I_j(E, F, z) = I_j(E', F', z') \qquad j = 1, 2.$$

These quantities are conformally invariant in much the same way that curvature is ([2] p 12); however here, a pair of metrics is involved, rather than just one.

Theorems 2 and 3 can also be written completely in terms of conformal invariants. We do this now.

First it is necessary to define the higher-order Schwarzian tensors. Let ρ be a conformal line element, and $e^{\psi}\rho$ another line element conformal to ρ. Then

Definition 5. *The Schwarzian tensors are defined inductively by*

$$\begin{aligned} B_2^{\rho}(\psi) &= (2\psi_{zz} - 2\psi_z^2 - 2\Gamma_{\rho}\psi_z)dz^2 \\ B_{n+1}^{\rho}(\psi) &= \nabla_{\rho e^{\psi}} B_n^{\rho}(\psi) \end{aligned}$$

Note that $\Re(B_2^{\rho}(\psi))$ is the Schwarzian tensor of Osgood and Stowe.

These behave nicely under pull-back.

Proposition 4. *If g is locally univalent, then*

$$g^*(B_n^{\rho}(\psi)) = B_n^{g^*(\rho)}(\psi \circ g).$$

Proof. Let $\rho \mapsto g^*(\rho)$ and $\psi \mapsto \psi \circ g$. Then it can be directly computed that

$$\begin{aligned} &2(\psi \circ g)_{ww} - 2(\psi \circ g)_w \Gamma_{g^*(\rho)} - 2(\psi \circ g)_w^2 \\ &\qquad = [2\psi_{zz} \circ g - 2\psi_z \circ g \, \Gamma_{\rho} \circ g - 2\psi_z \circ g^2] {g'}^2, \end{aligned}$$

proving the claim for $n = 2$. Now apply induction. We have that

$$\Gamma_{g^*(e^{\psi}\rho)} = \Gamma_{e^{\psi}\rho} \circ g \, g' + \frac{g''}{g'},$$

so, abusing notation somewhat and letting B_n denote the coefficient of the differential,

$$\begin{aligned} \nabla_{g^*(e^{\psi}\rho)} B_n^{g^*(\rho)}(\psi \circ g) &= (B_n^{g^*(\rho)}(\psi \circ g))_w - n\Gamma_{g^*(e^{\psi}\rho)} B_n^{g^*(\rho)}(\psi \circ g) \\ &= (B_n^{\rho}(\psi))_z \circ g \, {g'}^{n+1} - n\Gamma_{\rho e^{\psi}} \circ g \, B_n^{\rho}(\psi) \circ g \, {g'}^{n+1} \end{aligned}$$

□

This invariance under pull-back means that the Schwarzian tensors are conformally invariant in precisely the same sense as above. Letting $\rho = \lambda_F$ and $e^{\psi}\rho = \lambda_E$, Proposition 4 implies that

$$\lambda_{g(F)}^{-n} |B_n^{\lambda_{g(F)}}(\log \frac{\lambda_{g(E)}}{\lambda_{g(F)}})| = \lambda_F^{-n} |B_n^{\lambda_F}(\log \frac{\lambda_E}{\lambda_F})|.$$

We now show the relation between the Schwarzian tensors and higher-order Schwarzian derivatives. If, for $f \in \mathcal{B}$, one chooses $\rho = \lambda_{f(D)}$ and $e^{\psi}\lambda_{f(D)} = 1$, then the Schwarzian tensors compare the hyperbolic geometry of $f(D)$ to Euclidean geometry. Then we have

$$B_n^{\lambda_{f(D)}}(-\log \lambda_{f(D)}) = \frac{\sigma_{n+1}(f) \circ f^{-1}}{f'^n \circ f^{-1}}. \tag{48}$$

To see this, we have by 4 that

$$f^*(B_n^{\lambda_{f(D)}}(-\log \lambda_{f(D)})) = B_n^{\lambda_D}\left(\log \frac{|f'|}{\lambda_D}\right).$$

It can be computed directly that $B_2^{\lambda_D}\left(\log \frac{|f'|}{\lambda_D}\right) = \sigma_3(f)dz^2$. Equation (48) follows using induction and the fact that

$$2\left(\log \frac{|f'|}{\lambda_D}\right)_z = \frac{f''}{f'} - \Gamma_D.$$

If on the other hand one chooses $\rho = \lambda_{f(D)}$ and ψ such that $e^\psi \lambda_{f(D)} = \lambda_D$, then

$$B_n^{\lambda_{f(D)}}\left(\log \frac{\lambda_D}{\lambda_{f(D)}}\right) = \frac{\sigma^h_{n+1}(f) \circ f^{-1}}{f'^n \circ f^{-1}} \tag{49}$$

This is proved in a similar way to (48). We show that $B_2^{\lambda_D}\left(\log \frac{|f'|\lambda_D \circ f}{\lambda_D}\right) = \sigma_3^h(f)dz^2$. By (9),

$$2\left(\log \frac{|f'|\lambda_D \circ f}{\lambda_D}\right)_z(a) + \Gamma_D(a) = \frac{(T_{-f(a)} \circ f)''(a)}{(T_{-f(a)} \circ f)'(a)};$$

using this it follows by induction that

$$B_n^{\lambda_D}\left(\log \frac{|f'|\lambda_D \circ f}{\lambda_D}\right) = \sigma^h_{n+1}(f)dz^n.$$

Equation 49 now follows from the fact that

$$f^*\left(B_n^{\lambda_{f(D)}}\left(\log \frac{\lambda_D}{\lambda_{f(D)}}\right)\right) = B_n^{\lambda_D}\left(\log \frac{|f'|\lambda_D \circ f}{\lambda_D}\right).$$

It is now a simple matter to write Theorems 2 and 3 in terms of the conformal invariants. (They then become valid on any domain). For simplicity we denote

$$\begin{aligned} I_0 &= \frac{\lambda_E(z)}{\lambda_F(z)} \\ I_1 &= \lambda_F^{-1}(z)(\Gamma_F(z) - \Gamma_E(z)) \\ I_n &= \lambda_F^{-n}(z) B_n^{\lambda_F}\left(-\log \frac{\lambda_E}{\lambda_F}\right)(z) \end{aligned}$$

for E, F, and z as above.

Corollary 2. *Let $F \subset E$ be two hyperbolic simply connected domains, and let $z \in F$. Let I_n be the conformal invariants associated to this configuration. Then*

$$|I_2| \leq 6\left(1 - I_0^2\right),$$

$$|I_3| + 2|I_1 I_2| \leq 48\left(1 + I_0^3\right) + 48\left(1 - I_0^2\right),$$

$$|I_4 + I_2^2| \leq I_1^2 + 2\left(1 - I_0^4\right),$$

$$|18I_6 + 36I_4I_2 + 81I_3^2 + 80I_2^3| \leq |I_2^2 - \frac{3}{2}I_1^2| + 18I_1^2 + 12\left(1 - I_0^6\right).$$

All but the second inequality are sharp.

5.2. **Distortion of curves.** It is natural to try to relate distortion of geodesic curvature to univalence; demanding that the image of a geodesic curve not bend too much might guarantee univalence, and vice versa. Before proving the necessary conditions, we give a simple illustration of this idea in the form of a geometric proof of a distortion theorem of Pommerenke.

Let $\rho|dz|$ and $\rho e^{\psi}|dz|$ be line elements conformal to the Euclidean. Let $\gamma(t)$ be a curve in the plane. Denote arc length in the ρ and ρe^{ψ} metrics by ds and $d\bar{s}$ respectively, and in the euclidean metric by ds_e. Then

$$d\bar{s} = e^{\psi} ds = \rho e^{\psi} ds_e = \rho e^{\psi} |\dot{\gamma}| dt.$$

We denote the geodesic curvatures of γ in the Euclidean, ρ, and ρe^{ψ} metrics as k_e, k, and $\bar{k}$. The Euclidean geodesic curvature is defined by

$$ik_e \frac{\dot{\gamma}}{|\dot{\gamma}|} = \frac{d}{ds_e} \frac{\dot{\gamma}}{|\dot{\gamma}|}$$

(note that this implies that k_e is real, since the derivative of the unit tangent must be perpendicular to the curve). We can then compute that

$$k_e = \Im \left(\frac{\ddot{\gamma}}{|\dot{\gamma}|\dot{\gamma}} \right), \tag{50}$$

$$\rho k = k_e + \Im \left(\Gamma_\rho \frac{\dot{\gamma}}{|\dot{\gamma}|} \right), \tag{51}$$

and

$$\rho e^{\psi} \bar{k} = \rho k + \Im \left(2\psi_z \frac{\dot{\gamma}}{|\dot{\gamma}|} \right) \tag{52}$$

Given a simple smooth curve γ enclosing a domain G, the rotation index $r(\gamma)$, which is the winding number of $\frac{\dot{\gamma}}{|\dot{\gamma}|}$ must be 1. By the local Gauss-Bonnet theorem, for any conformal line element ρ,

$$1 = \frac{1}{2\pi} \int_\gamma k_\rho \, ds_\rho + \iint_G K_\rho dA_\rho,$$

where K_ρ is the Gaussian curvature. If f is a map from the disc into the complex plane, we can compare the rotation index of γ and $f \circ \gamma$ by

$$r(f \circ \gamma) - r(\gamma) = \frac{1}{2\pi} \int_\gamma [k_{euc}(f \circ \gamma) ds_{euc} - k_\lambda(\gamma) ds_\lambda] - \frac{1}{2\pi} \iint_G K_\lambda dA_\lambda. \tag{53}$$

Thus the change in rotation index under the map f is related to the distortion of geodesic and Gaussian curvature. Since $k_{euc}(f \circ \gamma) ds_{euc} = k_{|f'|}(\gamma) ds_{|f'|}$, applying (52) with $\rho = \lambda$ and $e^{\psi} = |f'|\lambda^{-1}$, and the fact that $K_\lambda = -4$,

$$\begin{aligned} r(f \circ \gamma) - r(\gamma) &= \frac{1}{2\pi} \int \Im \left(\frac{D_2 f}{D_1 f} \lambda dz \right) + \frac{2}{\pi} \iint dA_\lambda \\ &= \frac{1}{2\pi i} \int \frac{D_2 f}{D_1 f} \lambda dz + \frac{2}{\pi} \iint_G dA_\lambda \end{aligned} \tag{54}$$

The second equality follows from the fact that

$$\Re \left(\frac{D_2 f}{D_1 f} \lambda dz \right) = d \log |D_1 f|$$

which has zero periods when $D_1 f \neq 0$. Equation (54) could also be derived directly from the argument principle.

If f is locally univalent, it's easy to see geometrically that $r(f \circ \gamma) = r(\gamma)$. This leads to a simple geometric proof of

Theorem 7. *(Pommerenke) If f is a locally univalent map from the disc into the complex plane, then*

$$\sup_{z \in D} \left| \frac{D_2 f}{D_1 f} \right| \geq 2.$$

Proof. Let γ be the circle of radius s about the origin; then the hyperbolic area of G is $\pi s^2 (1-s^2)^{-1}$, and the hyperbolic length of γ is $2\pi s(1-s^2)^{-1}$. If there were a $c < 2$ such that

$$\left| \frac{D_2 f}{D_1 f} \right| \leq c$$

in D, then by (54),

$$\begin{aligned} 0 &= \left| \frac{1}{2\pi i} \int \frac{D_2 f}{D_1 f} \lambda dz + \frac{2}{\pi} \iint_G dA_\lambda \right| \\ &\geq \frac{1}{2\pi} \left(\frac{4\pi s^2}{1-s^2} - \frac{2\pi c\, s}{1-s^2} \right). \end{aligned}$$

For $1 > s > c/2$, the right-hand side is positive, a contradiction. $\square$

To prove Corollary 1 we require formulas for the change of derivatives of geodesic curvature under conformal change of metric.

The following lemma, which is a kind of product rule, will be useful in the computations.

Lemma 3. *With notation as above, we have that*

$$\begin{aligned} \frac{1}{\rho e^\psi} \frac{d}{ds_e} \left[\rho^{-(n+m)} e^{-(n+m)\psi} g \frac{\dot\gamma^n \bar{\dot\gamma}^m}{|\dot\gamma|^{n+m}} \right] &= \rho^{-(n+m+1)} e^{-(n+m+1)\psi} \times \\ \left[\nabla_{\rho e^\psi} g \frac{\dot\gamma^{n+1} \bar{\dot\gamma}^m}{|\dot\gamma|^{n+m+1}} + \bar\nabla_{\rho e^\psi} g \frac{\dot\gamma^n \bar{\dot\gamma}^{m+1}}{|\dot\gamma|^{n+m+1}} \right. &\left. + i(n-m) \rho e^\psi g\, \bar{k} \frac{\dot\gamma^n \bar{\dot\gamma}^m}{|\dot\gamma|^{n+m}} \right] \end{aligned}$$

Proof.

$$\begin{aligned} &\frac{1}{\rho e^\psi} \frac{d}{ds_e} \left[\frac{g}{\rho^{n+m} e^{(n+m)\psi}} \frac{\dot\gamma^n \bar{\dot\gamma}^m}{|\dot\gamma|^{n+m}} \right] \\ &= \frac{1}{\rho^{n+m+1} e^{(n+m+1)\psi}} \left[\left(g_z \frac{\dot\gamma}{|\dot\gamma|} - n \Re \left((\Gamma_\rho + 2\psi_z) \frac{\dot\gamma}{|\dot\gamma|} \right) g \right) \frac{\dot\gamma^{n+m}}{|\dot\gamma|^{n+m}} \right. \\ &+ \left. \left(g_{\bar z} \frac{\bar{\dot\gamma}}{|\dot\gamma|} - n \Re \left((\bar\Gamma_\rho + 2\psi_{\bar z}) \frac{\bar{\dot\gamma}}{|\dot\gamma|} \right) g \right) \frac{\dot\gamma^{n+m}}{|\dot\gamma|^{n+m}} + i(n-m) g k_e \frac{\dot\gamma^n \bar{\dot\gamma}^m}{|\dot\gamma|^{n+m}} \right] \end{aligned}$$

where in the last term we have used (50). Applying

$$\rho e^\psi \bar{k} = \rho k + \Im \left[(\Gamma_\rho + 2\psi_z) \frac{\dot\gamma}{|\dot\gamma|} \right]$$

completes the proof. $\square$

Proposition 5. *With notation as above,*

$$\rho^2 e^{2\psi} \frac{d\bar{k}}{d\bar{s}} = \rho^2 \frac{dk}{ds} + \Im \left(B_2^\rho(\psi) \frac{\dot\gamma^2}{|\dot\gamma|^2} \right)$$

Proof. Differentiating (52), and applying Lemma 3,

$$\begin{aligned}\frac{d\bar{k}}{d\bar{s}} &= \frac{1}{e^{2\psi}}\frac{dk}{ds} - \Re\left[\frac{2\psi_z}{\rho e^{2\psi}}\frac{\dot{\gamma}}{|\dot{\gamma}|}\right]k + \frac{1}{\rho^2 e^{2\psi}}\Im\left[\left(\nabla_{\rho e^{\psi}}(2\psi_z) - 2\psi_z^2\right)\frac{\dot{\gamma}^2}{|\dot{\gamma}|^2}\right] \\ &\quad + \frac{1}{\rho^2 e^{2\psi}}\Im\left[i\,2\psi_z\,\rho e^{\psi}\bar{k}\frac{\dot{\gamma}}{|\dot{\gamma}|}\right] + \frac{1}{\rho^2 e^{2\psi}}\Im\left[(2\psi_{z\bar{z}} - 4\psi_z\psi_{\bar{z}})\frac{\bar{\dot{\gamma}}\dot{\gamma}}{|\dot{\gamma}|^2}\right]\end{aligned}$$

The last term is zero because the argument is real. Applying (52) results in the desired expression. □

We continue differentiating.

Proposition 6.

$$\begin{aligned}\rho^3 e^{3\psi}\frac{d^2\bar{k}}{d\bar{s}^2} &= \rho^3\frac{d^2k}{ds^2} + \Im\left[B_3^{\rho}(\psi)\frac{\dot{\gamma}^3}{|\dot{\gamma}|^3}\right] + \Im\left[B_2^{\rho}(\psi)_{\bar{z}}\frac{\bar{\dot{\gamma}}\dot{\gamma}^2}{|\dot{\gamma}|^3}\right] \\ &\quad -2\rho^2\Re\left[2\psi_z\frac{\dot{\gamma}}{|\dot{\gamma}|}\right]\frac{dk}{ds} + 2\rho\Re\left[B_2^{\rho}(\psi)\frac{\dot{\gamma}}{|\dot{\gamma}|^2}\right]k \\ &\quad +2\Re\left[B_2^{\rho}(\psi)\frac{\dot{\gamma}^2}{|\dot{\gamma}|^2}\right]\Im\left[2\psi_z\frac{\dot{\gamma}}{|\dot{\gamma}|}\right]\end{aligned}$$

Proof. Differentiating the expression from Proposition 5, we get

$$\frac{d^2\bar{k}}{d\bar{s}^2} = -4\Re\left[\frac{\psi_z}{\rho e^{3\psi}}\frac{\dot{\gamma}}{|\dot{\gamma}|}\right]\frac{dk}{ds} + \frac{1}{e^{3\phi}}\frac{d^2k}{ds^2} + \Im\left[\frac{d}{d\bar{s}}\left(\frac{B_2^{\rho}(\psi)}{\rho^2 e^{2\psi}}\frac{\dot{\gamma}}{|\dot{\gamma}|}\right)\right]$$

Applying Lemma 3,

$$\begin{aligned}\frac{d^2\bar{k}}{d\bar{s}^2} &= \frac{1}{e^{3\psi}}\frac{d^2k}{ds^2} - 2\Re\left[\frac{2\psi_z}{\rho e^{3\psi}}\frac{\dot{\gamma}}{|\dot{\gamma}|}\right]\frac{dk}{ds} + \frac{1}{\rho^3 e^{3\psi}}\Im\left[B_3^{\rho}(\psi)\frac{\dot{\gamma}^3}{|\dot{\gamma}|^3}\right] \\ &\quad + \frac{1}{\rho^3 e^{3\psi}}\Im\left[B_2^{\rho}(\psi)_{\bar{z}}\frac{\bar{\dot{\gamma}}\dot{\gamma}^2}{|\dot{\gamma}|^3}\right] + \frac{1}{\rho^2 e^{2\psi}}\Im\left[2iB_2^{\rho}(\psi)\bar{k}\frac{\dot{\gamma}^2}{|\dot{\gamma}|^2}\right]\end{aligned}$$

Applying (52) finishes the proof. □

We can now prove Corollary 1.

Proof. (of Corollary 1) First note that choosing $\rho = \lambda$ and $e^{\psi} = |f'|\lambda \circ f\lambda^{-1}$, we have that

$$\psi_z = \lambda\frac{D_2^h f}{D_1^h f} \tag{55}$$

and

$$B_n^{\lambda}(\psi) = \sigma_{n+1}^h(f). \tag{56}$$

The first inequality now follows from Theorem 5, (52), and (55): we have that

$$\begin{aligned}|k(f\circ\gamma)ds_{f\circ\gamma} - k(\gamma)ds_{\gamma}| &= \left||f'\circ\gamma|\,k(f\circ\gamma)\lambda(f\circ\gamma) - \lambda(\gamma)k(\gamma)\right| \\ &\leq 4\lambda(\gamma)\left(1 - \frac{\lambda(f\circ\gamma)|f'\circ\gamma|}{\lambda(\gamma)}\right) \\ &= 4|ds_{\gamma} - ds_{f\circ\gamma}|.\end{aligned}$$

One proves the second inequality similarly, using Theorem 2, Proposition 5, and (56). To prove the last one, we apply Proposition 6, with ρ and ψ as above, noting that $B_2^{\lambda}(\psi)_{\bar{z}} = 0$, $k = 0$, and $\frac{dk}{ds} = 0$. □

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