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Behaviour of Kernel Functions under Homotopic Variations of Planar Domains

Eric Schippers

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Abstract. A variational formula is derived for Green's function of multiply connected planar domains under homotopy of the boundary. The formula shows that up to first order, a homotopy behaves like the Hadamard variation. This is applied to show that certain expressions in the derivatives of Green's function are monotonic with respect to set inclusion.

Keywords. Green's function, Hadamard variation, Bergman kernel.

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1. Introduction

In this paper we derive inequalities between derivatives of Green's function, the Bergman kernel K, and L-kernel for multiply-connected domains. This is accomplished with a variational formula for Green's function under homotopy of the boundary of the domain. It is similar to the Hadamard variational formula; however, the method of Hadamard is local in nature, whereas the technique presented here applies to pairs of domains which are not nearby each other. This makes it possible to use the method to show that expressions in the derivatives of Green's function are monotonic under set theoretic inclusion.

Similar inequalities were derived by the author in [7] using the Dirichlet principle. However, the methods used there seem to give inequalities only for even orders of differentiation of K and L. One of the main objectives of this paper is to derive inequalities for odd orders of differentiation; this is accomplished in Theorems 12 and 14.

Briefly, the Hadamard variation consists of varying the boundary by flowing along the normal vector; i.e. if $\gamma(\tau)$ parametrizes the boundary, $\nu(\tau)$ is a smooth

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function, and $n(\tau)$ is the unit outward normal, then a family of nearby domains are obtained with boundary $\gamma(\tau) + \epsilon\nu(\tau)n(\tau)$. A drawback of this method is that it requires some order of smoothness (although this requirement can be weakened [8]). Since an extremal domain is generically not smooth, this is a serious limitation on the method. This limitation was one of the reasons for the invention of boundary variation by Schiffer. However, in the situation here, it is not necessary to anchor the variation at the extremal domain, so the smoothness requirement does not pose a problem. Also, once the inequalities are established for pairs of smooth domains, the smoothness requirement can be removed with an exhaustion argument (see Remark 2).

A more serious drawback of Hadamard variation for the applications in this paper is that one can only prove that inequalities hold up to first order. Here, for example, we show that certain quantities increase as the domain increases, up to first order. Immediately, one is led to try some sort of compactness argument in order to patch up the small pieces on which the quantity is actually increasing.

This leads naturally to the following approach. One shows that up to first order a general smooth injective homotopy between the boundaries of two domains behaves like a Hadamard variation. This observation was applied in specific cases by Barnard and Lewis [1]. When this approach is taken, it is possible to provide rigorous proofs for the inequalities derived here with elementary arguments. However, a general treatment does not seem to exist in the literature, so the details are given in Section 2 for the sake of completeness and the conscience of the author.

In Section 3, we exhibit the monotonicity theorems which can be derived from this method. Corollaries for univalent functions are given in Section 4.

Two results should be mentioned in this context. Rodin [6] showed that if the boundary of a simply connected domain depends holomorphically on a complex parameter, then for small values of the parameter the domain must stay simply connected. He also showed that the Riemann mapping function depends holomorphically on this parameter. Warschawski [9] generalized Hadamard's first order variational formula for Green's function to higher orders of differentiation.

2. The variational formula

We first recall the Hadamard variational formula. Let D be a smoothly bounded simply connected domain, with the boundary parametrized by $\gamma(\tau)$. We construct a variation of the domain. Let $n(\tau)$ be the unit outward normal. Given a smooth function $\nu(\tau)$, we vary the curve $\gamma(\tau)$ according to

$$\gamma(\tau, \epsilon) = \gamma(\tau) + \epsilon\nu(\tau)n(\tau).$$

If ϵ is sufficiently small, then $\gamma(\cdot, \epsilon)$ is also a smooth simple closed curve, bounding a domain D_ϵ.

Under this variation, Green's function of the domain D_ϵ varies according to the formula

$$g_\epsilon(z,w) = g(z,w) + \epsilon\delta g(z,w) + \epsilon^2\gamma_\epsilon(z,w) \tag{1}$$

where the function δg is given by

$$\delta g(z,w) = \frac{1}{2\pi}\int_{\partial D} \frac{\partial g}{\partial n_u}(z,u)\frac{\partial g}{\partial n_u}(u,w)\nu(u)\,ds_u. \tag{2}$$

Here, s is arc length (in the variable u), n is understood to be the unit outward normal as above, ν is a smooth function (not necessarily strictly positive or negative), and γ_ϵ is bounded and harmonic in each compact subdomain of $D \cup D_\epsilon$.

We will consider here a more general variation of a domain, namely an injective homotopy of the boundary of a fixed domain. Locally, this is a normal variation, but it is not a Hadamard variation, since the distance from the fixed boundary to the boundary of the new domain is not linear in the homotopy variable. However, we can isolate the first order part of the variation. Green's function varies according to the Hadamard formula determined by this first order part.

More precisely, let $F\colon (a,b)\times[0,2\pi]\to\mathbb{C}$ be a C^2-injective homotopy of curves $\Gamma_t(\tau) = F(t,\tau)$ with non-vanishing Jacobian determinant. For some $t_0\in(a,b)$, let $n_{t_0}(\tau)$ be the outward unit normal vector to the curve Γ_{t_0} at τ. Let $\Delta n_{t_0}(t,\tau)$ be the distance from $\Gamma_{t_0}(\tau)$ to Γ_t in the direction of the normal $n_{t_0}(\tau)$. Define

$$\nu_{t_0}(\tau) = \frac{d}{dt}\bigg|_{t=t_0} \Delta n_{t_0}(t,\tau).$$

Thus the distance to Γ_t along $n_{t_0}(\tau)$ is $(t-t_0)\nu_{t_0}(\tau)+O(|t-t_0|^2)$. The remainder is uniform in τ (as will be shown in Lemma 4).

It is believable that at least for $|t-t_0|$ small, the normal line at $F(t_0,\tau)$ will intersect Γ_t once and only once. However, uniform control on the t for which this holds is necessary. This will be established in Lemmas 2 and 3. Assuming this control, we have the following theorem.

Theorem 1. *Let D_t, $t\in(a,b)$ be a family of domains, each bounded by n C^2-simple closed curves, satisfying the following conditions.*

1) *There exists a collection of injective C^2-homotopies $F_i\colon (a,b)\times[0,2\pi]\to\mathbb{C}$, $i=1,\dots,n$ with non-vanishing Jacobian determinant between the curves $\Gamma_t^i(\tau) = F_i(t,\tau)$, such that $\partial D_t = \bigcup_i \Gamma_t^i$.*
2) *$D_{t'}\subset D_t$ whenever $t'<t$.*

Let $n_{t_0}(\tau)$, $\Delta n_{t_0}(t,\tau)$ and $\nu_{t_0}(\tau)$ be as above. Then,

$$g_t(z,\zeta) - g_{t_0}(z,\zeta) = \frac{t-t_0}{2\pi}\int_{\partial D_{t_0}} \frac{\partial g_{t_0}}{\partial n_u}(u,z)\frac{\partial g_{t_0}}{\partial n_u}(u,\zeta)\nu_{t_0}(u)\,ds_u + \mathcal{O}(|t-t_0|^2).$$

Here, $\mathcal{O}(|t-t_0|^2)$ is uniform for ζ in any compact set in $D_t\cap D_{t_0}$ and z in a compact set in $D_t\cap D_{t_0}$. Furthermore, the remainder term is harmonic.

The proof depends on some technical lemmas. Their proofs will follow the proof of the above theorem.

Lemma 2. *Let γ be a C^2-simple closed curve in the plane, parametrized by arc length s. Assume that the signed curvature $k(s)$ is uniformly bounded by K. Then, on any interval $[\alpha, \beta]$ of arc length less than $\pi/(4K)$, the mapping*

$$(s, r) \mapsto \gamma(s) + rn(s)$$

is one-to-one for $|r| < 1/(\sqrt{2}K)$.

Lemma 3. *Let F be a homotopy as in Theorem 1, and $[c, d]$ be a compact subinterval of (a, b). There is a fixed $\epsilon > 0$, such that for every $t_0 \in [c, d]$ and $t \in (t_0 - \epsilon, t_0 + \epsilon)$, the curve $\Gamma_t(\tau) = F(t, \tau)$ intersects the normal line $r \mapsto F(t_0, \tau_0) + rn(t_0, \tau_0)$ once and only once for every τ_0.*

Lemma 4. *The quantity ϵ of Lemma 3 can be chosen so that*

$$\Delta n_{t_0}(t, \tau) = (t - t_0)\nu_{t_0}(\tau) + \mathcal{O}(|t - t_0|^2)$$

for $|t - t_0| < \epsilon$, where the remainder is uniform for $(t_0, \tau) \in [c, d] \times [0, 2\pi]$.

Proof of Theorem 1. This is a modification of a proof of the Hadamard variational formula found in [3, pp. 293–294]. We assume first that the variation is inward directed; i.e. that $t < t_0$.

We will assume, for notational convenience, that the domains D_t are bounded by only one smooth curve Γ_t with a single homotopy $F(t, \tau)$ between them. This assumption does not affect in the proof in any way; it only simplifies the presentation.

Consider the function

$$\sigma(z, \zeta) = g_{t_0}(z, \zeta) + \frac{t - t_0}{2\pi} \int_{\partial D_{t_0}} \frac{\partial g_{t_0}}{\partial n_u}(u, z) \frac{\partial g_{t_0}}{\partial n_u}(u, \zeta) \nu_{t_0}(u)\, ds_u.$$

Fix a $\tau \in [0, 2\pi]$, and denote $z(t) = F(t_0, \tau) + \Delta n_{t_0}(t, \tau) n_{t_0}(\tau)$ (this point lies on the boundary of D_t). Also let $z = z(t_0) = F(t_0, \tau)$. We claim that

$$\sigma(z(t), \zeta) = \mathcal{O}(|t - t_0|^2). \tag{3}$$

The estimate is uniform for ζ restricted to compact sets in D_t.

To prove this, first note that by Lemma 4 and the fact that $g_{t_0}(z, \zeta) = 0$,

$$\begin{aligned} g_{t_0}(z(t), \zeta) &= g_{t_0}(z(t), \zeta) - g_{t_0}(z, \zeta) \\ &= -\frac{\partial g_{t_0}}{\partial n_z}(z, \zeta)|z(t) - z| + \mathcal{O}(|z(t) - z|^2) \\ &= \frac{\partial g_{t_0}}{\partial n_z}(z, \zeta)(t - t_0)\nu_{t_0}(\tau) + \mathcal{O}(|t - t_0|^2). \end{aligned}$$

So

$$\sigma(z(t),\zeta) = \frac{\partial g_{t_0}}{\partial n_z}(z,\zeta)(t-t_0)\nu_{t_0}(\tau) + \frac{t-t_0}{2\pi}\int_{\partial D_{t_0}} \frac{\partial g_{t_0}}{\partial n_u}(u,z)\frac{\partial g_{t_0}}{\partial n_u}(u,\zeta)\nu_{t_0}(u)\,ds_u + \mathcal{O}(|t-t_0|^2).$$

By the reproducing property of $\partial g/\partial n$

$$\frac{\partial g_{t_0}}{\partial n_z}(z,\zeta)\nu_{t_0}(\tau) = -\frac{1}{2\pi}\int_{\partial D_{t_0}} \frac{\partial g_{t_0}}{\partial n_u}(u,z)\frac{\partial g_{t_0}}{\partial n_u}(u,\zeta)\nu_{t_0}(u)\,ds_u,$$

which proves (3).

The variational formula now follows easily. Since g_t vanishes on ∂D_t, we have that

$$g_t(z,\zeta) - \sigma(z,\zeta) = \mathcal{O}(|t-t_0|^2)$$

for $z \in \partial D_t$. Further, the left hand side is harmonic in z, so this estimate extends to all $z \in D_t$. This proves the theorem in the special case that the variation is inward directed.

In the case that the variation is outwardly directed, i.e. that $t > t_0$, we enclose both domains in a larger domain D_T. Some care is needed in choosing this domain. Enclose t_0 in a compact interval $[c,d]$ and let ϵ be as in Lemmas 3 and 4. Choose T so that $|T-t_0| < \epsilon$, and consider $t \in (t_0,T)$. We then have that

$$g_t(z,\zeta) - g_T(z,\zeta) = \frac{t-T}{2\pi}\int_{\partial D_T} \frac{\partial g_T}{\partial n_u}(u,z)\frac{\partial g_T}{\partial n_u}(u,\zeta)\nu_T(u)\,ds_u + \mathcal{O}(|t-T|^2),$$

and also a similar formula for $g_{t_0} - g_T$. Subtracting these two, and noting that $|t_0 - T| < C|t-t_0|$ and $|t-T| < C|t-t_0|$,

$$g_{t_0}(z,\zeta) - g_T(z,\zeta) = \frac{t-t_0}{2\pi}\int_{\partial D_T} \frac{\partial g_T}{\partial n_u}(u,z)\frac{\partial g_T}{\partial n_u}(u,\zeta)\nu_T(u)\,ds_u + \mathcal{O}(|t-t_0|^2).$$

By the assumptions on F, $\nu_T(\tau)$ must be C^1 in both T and τ. Also, $\partial g_T/\partial n_u$ is C^∞ away from ζ, so

$$\int_{\partial D_T} \frac{\partial g_T}{\partial n_u}(u,z)\frac{\partial g_T}{\partial n_u}(u,\zeta)\nu_T(u)\,ds_u = \int_{\partial D_{t_0}} \frac{\partial g_T}{\partial n_u}(u,z)\frac{\partial g_T}{\partial n_u}(u,\zeta)\nu_T(u)\,ds_u + \mathcal{O}(|t-t_0|).$$

Also

$$\frac{\partial g_T}{\partial n_u}(u,z) = \frac{\partial g_{t_0}}{\partial n_u}(u,z) + \mathcal{O}(|T-t_0|)$$

uniformly on compact sets in u away from z, so we get that

$$\int_{\partial D_{t_0}} \frac{\partial g_T}{\partial n_u}(u,z)\frac{\partial g_T}{\partial n_u}(u,\zeta)\nu_T(u)\,ds_u$$
$$= \int_{\partial D_{t_0}} \frac{\partial g_{t_0}}{\partial n_u}(u,z)\frac{\partial g_{t_0}}{\partial n_u}(u,\zeta)\nu_{t_0}(u)\,ds_u + \mathcal{O}(|t-t_0|),$$

completing the proof. ∎

Proof of Lemma 2. Let s_1 and s_2 be such that $\alpha \leq s_1 < s_2 \leq \beta$. We will show that the normal lines $r \mapsto \gamma(s_i) + rn(s_i)$ do not intersect for $r < 1/(\sqrt{2}K)$.

By reparametrizing, we can assume that $s_1 = 0$, and by applying a Euclidean motion we can further place γ so that $\gamma(0) = (0,0)$ and $\gamma'(0)$ is parallel to the y-axis. Denote $\gamma(s) = (x(s), y(s))$, and the angle between $\gamma'(s)$ and the vertical by $\theta(s)$. For small s this will be close to zero.

Since $d\theta/ds = k(s)$, we have $|\theta(s_2)| \leq Ks_2$. Thus $|\theta(s)| \leq \pi/4$, which implies $\cos\theta(s) \geq 1/\sqrt{2}$ and $y(s_2) \geq s_2/\sqrt{2}$. We also have $|\sin\theta(s_2)| \leq |\theta(s_2)| \leq Ks_2$. The r for which the line $r \mapsto \gamma(s_2) + rn(s_2)$ intersects the real axis (that is, the normal line at s_1) must be at least $y(s_2)/|\sin\theta(s_2)| \geq 1/(\sqrt{2}K)$. Since this argument is independent of the direction of the parametrization, the line $r \mapsto \gamma(s_1) + rn(s_1)$ cannot intersect the normal line at s_2 for $r < 1/(\sqrt{2}K)$ either. ∎

Proof of Lemma 3. To reduce notation in the proof, we assume that there is only one curve in the homotopy F. The proof carries over into the case of several curves without any difficulties, and we will not comment on it further.

First we fix some notation. Let $F = (F_1, F_2)$, and $G = F^{-1} = (G_1, G_2)$; that is, $t = G_1(x,y)$ and $\tau = G_2(x,y)$. It is understood that G is defined only on the image of F.

Since F is C^2, the curvature of each curve $\Gamma_t(\tau)$ is bounded above by a uniform constant K on $[c,d]\times[0,2\pi]$. Let ϵ_0 be small enough so that $[c-\epsilon_0, d+\epsilon_0] \subset (a,b)$. Then $|\partial F/\partial t|$ is bounded above by some constant C_1 on the interval $[c-\epsilon_0, d+\epsilon_0]$. Set $\epsilon_1 = \min\{\epsilon_0, (\sqrt{2}KC_1)^{-1}\}$. We then have that for any $t_0 \in [c,d]$ and t such that $|t-t_0| < \epsilon_1$, the curve Γ_t is contained in the region of injectivity of the normal lines given by Lemma 2 (we will refer to this as the 'band of injectivity at t_0'). To see this, just note that for any τ, the disc of radius $1/(\sqrt{2}K)$ centered at $F(t_0,\tau)$ is in the band of injectivity, and $|F(t,\tau) - F(t_0,\tau)| \leq C_1|t-t_0| < 1/(\sqrt{2}K)$.

In order to show that a normal line to a curve intersects nearby curves once and only once, we will use the assumptions on F in order to control the angle of the tangent vector to Γ_t, thus preventing the curve Γ_t from 'looping back'.

Construct the normal line at $F(t_0,\tau_0)$. This normal line intersects the curve Γ_t at a closest point $F(t,\tau(t))$. We claim that on some uniform interval in t,

$|F(t,\tau(t)) - F(t,\tau_0)| \leq C|t-t_0|$ for a uniform constant C. To prove this, consider the function $t(r,t_0,\tau_0) = G_1(F(t_0,\tau_0) + rn(t_0,\tau_0))$. We then have that

$$\frac{dt}{dr} = \nabla G_1 \cdot n \tag{4}$$

where ∇G_1 depends on r. Now $|\nabla G_1| \neq 0$ since F has non-vanishing Jacobian determinant, and in fact on $F([c-\epsilon_1, d+\epsilon_1]\times[0,2\pi])$ we have that $|\nabla G_1| \geq m_1$ for some uniform constant m_1. Now $\nabla G_1(F(t_0,\tau_0))$ is parallel to $n(t_0,\tau_0)$. Since F is C_2 and has non-vanishing Jacobian determinant, ∇G_1 is C^1, so in particular the argument of $\nabla G_1/n$ (treating them as complex numbers) is bounded in $(-\pi/4,\pi/4)$ for some interval $-\epsilon_2 < r < \epsilon_2$. (Since the argument must stay small for r close to zero, there are no worries about choosing a branch of the argument.) We can choose this constant uniformly for $(t_0,\tau_0) \in [c,d]\times[0,2\pi]$ (ϵ_2 must be chosen at least smaller than $1/(\sqrt{2}K)$). In particular, we have that

$$\left|\frac{dt}{dr}\right| = |\nabla G_1| \cos\left(\arg\frac{\nabla G_1}{n}\right) \geq \frac{m_1}{\sqrt{2}}. \tag{5}$$

Thus for any fixed point (t_0,τ_0) the function $t(r)$ is invertible on the interval $(t_0 - m_1\epsilon_2/\sqrt{2}, t_0 + m_1\epsilon_2/\sqrt{2})$, and the inverse r is the first point at which the normal intersects the curve Γ_t. On this interval,

$$\left|\frac{dr}{dt}\right| \leq \frac{\sqrt{2}}{m_1}$$

so

$$|r| = |F(t,\tau(t)) - F(t_0,\tau_0)| \leq \frac{\sqrt{2}}{m_1}|t-t_0|, \tag{6}$$

thus proving the claim.

Next we need an estimate on $|\tau(t)-\tau_0|$ (recall that $\tau(t)$ is the value of τ at which the normal line at $F(t_0,\tau_0)$ intersects the curve Γ_t). Choose a uniform lower bound m_2 such that $m_2 \leq |\nabla G_2|$ on $F([c-\epsilon_1, d+\epsilon_1]\times[0,2\pi])$. Let $\epsilon_3 = \min\{\epsilon_1,\epsilon_2\}$. For $t \in (t_0-\epsilon_3, t_0+\epsilon_3)$, using (6), we have that

$$\begin{aligned}|F(t_0,\tau(t)) - F(t_0,\tau_0)| &\leq |F(t,\tau(t)) - F(t_0,\tau_0)| + |F(t_0,\tau(t)) - F(t,\tau(t))| \\ &\leq \left(\frac{\sqrt{2}}{m_1} + C_1\right)|t-t_0|.\end{aligned}$$

Thus

$$|\tau(t)-\tau_0| \leq m_2|F(t_0,\tau_0) - F(t_0,\tau(t))| \leq C|t-t_0| \tag{7}$$

on $(t_0-\epsilon_3, t_0+\epsilon_3)$ where $C = \sqrt{2}m_1 + C_1$ is independent of (t_0,τ_0).

We can now put a uniform bound on the change in the argument of $\partial F/\partial\tau$ to complete the proof. Let K' be a uniform lower bound for $\partial F/\partial\tau$ on the rectangle

$[c-\epsilon_3, d+\epsilon_3] \times [0, 2\pi]$, and M_1 and M_2 be the bounds

$$\left|\frac{\partial^2 F}{\partial\tau\partial t}\right| \leq M_1, \qquad \left|\frac{\partial^2 F}{\partial\tau^2}\right| \leq M_2$$

on the same region. We then have

$$\begin{aligned}
&\left|\arg\left(\frac{\partial F}{\partial\tau}(t,\tau(t))\right) - \arg\left(\frac{\partial F}{\partial\tau}(t_0,\tau_0)\right)\right| \\
&\quad\leq \left|\log\left(\frac{\partial F}{\partial\tau}(t,\tau(t))\right) - \log\left(\frac{\partial F}{\partial\tau}(t_0,\tau(t))\right)\right| \\
&\qquad+ \left|\log\left(\frac{\partial F}{\partial\tau}(t_0,\tau(t))\right) - \log\left(\frac{\partial F}{\partial\tau}(t_0,\tau_0)\right)\right| \\
&\quad\leq \frac{M_1}{K'}|t-t_0| + \frac{M_2}{K'}|\tau(t)-\tau_0| \\
&\quad\leq \left(\frac{M_1}{K'} + \frac{M_2 C}{K'}\right)|t-t_0|
\end{aligned}$$

by equation (7). Thus, by choosing a small enough positive $\epsilon < \epsilon_3$, we can guarantee that the argument of the tangent to Γ_t does not deviate more than, say, $\pi/4$ from the argument of the tangent to Γ_{t_0}. This implies that the normal lines to Γ_{t_0} cannot intersect Γ_t more than once for any $t_0 \in [c,d]$ and $t \in [t_0-\epsilon, t_0+\epsilon]$, where ϵ is independent of t_0. ■

Proof of Lemma 4. Let $t(r)$ be as in the proof of Lemma 3. From (4) and the assumptions on F it follows that d^2t/dr^2 exists and is bounded uniformly in τ and t, if t is chosen so that $|t-t_0| < \epsilon$. Because of the uniform lower bound (5), we then have that $|d^2r/dt^2|$ is bounded above uniformly for $(t_0,\tau) \in [c,d] \times [0,2\pi]$ as long as $|t-t_0| < \epsilon$. The function $r(t)$ is just the function $\Delta n_{t_0}(t,\tau)$ so the lemma follows. ■

3. Main theorems

We will now prove inequalities for the derivatives of Green's function, the Bergman kernel, and L-kernel. The Bergman kernel and L-kernel are defined by

$$K(\zeta,\eta) = -\frac{2}{\pi}\frac{\partial^2 g}{\partial\zeta\partial\bar{\eta}}(\zeta,\eta)$$

and

$$L(\zeta,\eta) = -\frac{2}{\pi}\frac{\partial^2 g}{\partial\zeta\partial\eta}(\zeta,\eta).$$

Theorem 5. *Let D_1 and D_2 be multiply connected domains, each bounded by n C^2-simple closed curves $\gamma_j^i(s)$, $j=1,\ldots,n$, $i=1,2$, respectively. Assume that there exists a collection $F_j(t,\tau)$ of C^2-injective homotopies between the curves*

γ_j^1 and γ_j^2 with non-vanishing Jacobian, such that if D_t denotes the n-connected domain bounded by the curves $F_j(t,\cdot)$, then $D_t \subset D_{t'}$ whenever $t < t'$.

Then, for every value of λ, and all complex parameters α_μ, β_μ and points $\zeta_\mu \in D_1$, $\mu = 1,\dots,n$, the following quantity is an increasing function of t:

$$\sum_{\mu,\nu} \alpha_\mu \overline{\alpha}_\nu \frac{1}{2\pi} g_t(\zeta_\mu,\zeta_\nu) - 2\lambda \operatorname{Re}\left(\sum_{\mu,\nu} \alpha_\nu \beta_\mu \frac{1}{\pi}\frac{\partial g_t}{\partial \zeta}(\zeta_\mu,\zeta_\nu)\right) - \lambda^2 \sum_{\mu,\nu} \beta_\mu \overline{\beta}_\nu K_t(\zeta_\mu,\zeta_\nu).$$

Here g_t and K_t denote Green's function and the Bergman kernel respectively of D_t. In particular the corresponding quantity for D_2 is larger than that for D_1.

Remark 1. The condition that the two domains be homotopic in the sense of Theorem 5 is not very restrictive. Consider two simply connected domains $D_1 \subset D_2$, each bounded by a C^2-curve, with positive minimum distance between ∂D_1 and ∂D_2. There exists a biholomorphic map g taking $D_2 \backslash \overline{D_1}$ to an annulus. On the annulus, one easily constructs a radial homotopy $\tilde{F}$. The map $g^{-1} \circ \tilde{F}$ then provides a homotopy satisfying the conditions of Theorem 5. One can extend this procedure to multiply connected domains.

Remark 2. Once inequalities such as the above are established for C^2-bounded domains with positive distance between the boundary curves, they can be extended to a wide class of domains. The smoothness hypothesis can be weakened considerably. We need only assume that D_1 can be exhausted by a sequence of smooth domains U_i, i.e. $U_i \subset U_{i+1}$ and $\bigcup_{i\in\mathbb{N}} U = D_1$, such that each U_i is bounded by n curves which are homotopic to the boundary curves of D_2 in the sense above. In order to see this, just note that the sequence of Green's functions will converge uniformly on compact sets to Green's function of D_1 ([5, p. 108]). Hence, its derivatives will also converge uniformly on compact sets, and the expression above for D_1 will be approximated by the corresponding expression for the domain U_i. One can then perform a similar process to weaken the conditions on the boundary of D_2. The same process can be used to remove the hypothesis that the boundaries are a finite distance apart.

A sufficient condition can be expressed in terms of the level curves of Green's functions g_1 and g_2 of D_1 and D_2. It is enough to assume that there is some ϵ_0 such that for all $0 < \epsilon < \epsilon_0$ and fixed $\zeta \in D_1$, the level sets $g_1(z,\zeta) = \epsilon$ and $g_2(z,\zeta) = \epsilon$ each consist of n simple smooth closed curves $\gamma_j^i(\epsilon,\cdot)$, $j = 1,\dots n$ and $i = 1,2$, which are homotopic in the sense above.

To prove Theorem 5 we will need to determine the first order variation of the first few derivatives of Green's function.

Lemma 6. *Let g be Green's function of D and K the Bergman kernel. Then*

$$\delta g(\zeta,\eta) = \frac{2}{\pi}\int_{\partial D} \frac{\partial g}{\partial u}(\zeta,u)\frac{\partial g}{\partial \bar{u}}(u,\eta)\nu(u)\,ds_u = \frac{2}{\pi}\int_{\partial D} \frac{\partial g}{\partial \bar{u}}(\zeta,u)\frac{\partial g}{\partial u}(u,\eta)\nu(u)\,ds_u,$$

$$\delta\frac{\partial g}{\partial \zeta}(\zeta,\eta) = -\int_{\partial D} L(\zeta,u)\frac{\partial g}{\partial \bar{u}}(u,\eta)\nu(u)\,ds_u = -\int_{\partial D} K(\zeta,u)\frac{\partial g}{\partial u}(u,\eta)\nu(u)\,ds_u,$$

$$\delta K(\zeta,\eta) = -\int_{\partial D} L(\zeta,u)\overline{L(u,\eta)}\nu(u)\,ds_u = -\int_{\partial D} K(\zeta,u)K(u,\eta)\nu(u)\,ds_u.$$

Proof of Lemma 6. We will make repeated use of the following identities, which hold along the boundary of D:

$$\frac{\partial g}{\partial n_u}(\zeta,u)\,ds_u = \frac{2}{i}\frac{\partial g}{\partial u}(\zeta,u)\,du = -\frac{2}{i}\frac{\partial g}{\partial \bar{u}}(\zeta,u)\,d\bar{u}. \tag{8}$$

The symmetry of Green's function gives the same identities in the first variable. Applying this to the integrand of equation (2), and using the fact that $ds^2 = |du|^2$, we get that

$$\delta g(\zeta,\eta) = \frac{2}{\pi}\int_{\partial D} \frac{\partial g}{\partial u}(\zeta,u)\frac{\partial g}{\partial \bar{u}}(u,\eta)\nu(u)\,ds_u. \tag{9}$$

The other two formulas follow by differentiating under the integral sign. (Strictly speaking, one differentiates the variational formula in Theorem 1, and uses the harmonicity of the remainder to conclude that the remainder term of the derivative is also harmonic and $\mathcal{O}(|t-t_0|^2)$. This can be done as many times as desired.) ■

Proof of Theorem 5. It is only necessary to show that the first variation is positive, using the variational formula of Theorem 1. Indeed, if the first variation is positive, then there is an open interval around every t in the homotopy, on which the quantity is actually increasing. Since t varies over a compact interval the theorem follows.

We now turn to the problem of showing that the first variation is positive. By Lemma 6 we have that

$$\begin{aligned}\delta\left(\sum_{\mu,\nu}\alpha_\mu\overline{\alpha}_\nu g(\zeta_\mu,\zeta_\nu)\right) &= \frac{2}{\pi}\int_{\partial D}\sum_{\mu,\nu}\alpha_\mu\overline{\alpha}_\nu\frac{\partial g}{\partial u}(\zeta_\mu,u)\frac{\partial g}{\partial \bar{u}}(\zeta_\nu,u)\nu(u)\,ds_u \\ &= \frac{2}{\pi}\int_{\partial D}\left|\sum_{\mu}\alpha_\mu\frac{\partial g}{\partial u}(\zeta_\mu,u)\right|^2\nu(u)\,ds_u,\end{aligned} \tag{10}$$

$$\delta\left(\sum_{\mu,\nu}\alpha_\nu\beta_\mu\frac{\partial g}{\partial z}(\zeta_\mu,\zeta_\nu)\right) = -\int_{\partial D}\sum_{\mu,\nu}\alpha_\nu\beta_\mu K(\zeta_\mu,u)\frac{\partial g}{\partial u}(u,\zeta_\nu)\nu(u)\,ds_u, \tag{11}$$

$$\delta\left(\sum_{\mu,\nu}\beta_\mu\overline{\beta_\nu}K(\zeta_\mu,\zeta_\nu)\right) = -\int_{\partial D}\left|\sum_{\mu}\beta_\mu K(\zeta_\mu,u)\right|^2\nu(u)\,ds_u. \tag{12}$$

So, combining equations (10), (11), and (12) we get

$$\delta\Bigg(\sum_{\mu,\nu}\alpha_\mu\overline{\alpha}_\nu\frac{1}{2\pi}g(\zeta_\mu,\zeta_\nu)-2\lambda\operatorname{Re}\Bigg(\sum_{\mu,\nu}\alpha_\nu\beta_\mu\frac{1}{\pi}\frac{\partial g}{\partial\zeta}(\zeta_\mu,\zeta_\nu)\Bigg)$$
$$-\lambda^2\sum_{\mu,\nu}\beta_\mu\overline{\beta}_\nu K(\zeta_\mu,\zeta_\nu)\Bigg)$$
$$=\int_{\partial D}\left|\sum_{\mu}\alpha_\mu\frac{1}{\pi}\frac{\partial g}{\partial u}(\zeta_\mu,u)+\lambda\overline{\beta}_\mu\overline{K(\zeta_\mu,u)}\right|^2\nu(u)\,ds_u.$$

■

Since this inequality holds for every choice of λ, we must have that the discriminant is less than or equal to zero. So we have the following theorem.

Theorem 7. *Let D_1 and D_2 be domains satisfying the conditions of Theorem 5. Let g_i denote their Green's functions and K_i denote their Bergman kernels, for $i=1,2$. Let $\zeta_\mu\in D_1$, $\mu=1,\dots n$, and let $\alpha_\mu,\beta_\mu\in\mathbb{C}$. Then the following inequality holds:*

$$\left(\operatorname{Re}\sum_{\mu,\nu}\alpha_\nu\beta_\mu\left(\frac{\partial g_1}{\partial\zeta}(\zeta_\mu,\zeta_\nu)-\frac{\partial g_2}{\partial\zeta}(\zeta_\mu,\zeta_\nu)\right)\right)^2$$
$$\leq\frac{\pi}{2}\left(\sum_{\mu,\nu}\beta_\mu\overline{\beta_\nu}\left(K_1(\zeta_\mu,\zeta_\nu)-K_2(\zeta_\mu,\zeta_\nu)\right)\right)$$
$$\times\left(\sum_{\mu,\nu}\alpha_\mu\overline{\alpha_\nu}\left(g_2(\zeta_\mu,\zeta_\nu)-g_1(\zeta_\mu,\zeta_\nu)\right)\right).$$

It is easy to generalize the variational formulas for derivatives of Green's function to all orders of differentiation, and with these, derive many inequalities.

Lemma 8. *For a domain D bounded by n smooth curves, the first order variation in the kernel functions K and L are given by*

$$\delta\frac{\partial^{m+n}K}{\partial\zeta^m\partial\overline{\eta}^n}(\zeta,\eta)=-\int_{\partial D}\frac{\partial^m K}{\partial\zeta^m}(\zeta,u)\frac{\partial^n K}{\partial\overline{\eta}^n}(u,\eta)\nu(u)\,ds_u$$
$$=-\int_{\partial D}\frac{\partial^m L}{\partial\zeta^m}(\zeta,u)\overline{\frac{\partial^n L}{\partial\eta^n}(u,\eta)}\nu(u)\,ds_u$$

and

$$\delta\frac{\partial^{m+n}L}{\partial\zeta^m\partial\eta^n}(\zeta,\eta)=-\int_{\partial D}\frac{\partial^m L}{\partial\zeta^m}(\zeta,u)\overline{\frac{\partial^n K}{\partial\overline{\eta}^n}(u,\eta)}\nu(u)\,ds_u$$
$$=-\int_{\partial D}\frac{\partial^m K}{\partial\zeta^m}(\zeta,u)\frac{\partial^n L}{\partial\eta^n}(u,\eta)\nu(u)\,ds_u.$$

Proof. As in Lemma 6, one simply differentiates the variational formula of Theorem 1. ■

With the help of Lemma 6 and Lemma 8, it is easy to derive many inequalities for derivatives of K and L.

Theorem 9. *Let D_1 and D_2 be two domains satisfying the conditions of Theorem 5, with Bergman kernels K_1 and K_2 respectively. Then*

$$\sum_{\mu,\nu} \beta_\mu \overline{\beta}_\nu \left(\frac{\partial^{2m} K_1}{\partial \zeta^m \partial \overline{\eta}^m}(\zeta_\mu, \zeta_\nu) - \frac{\partial^{2m} K_2}{\partial \zeta^m \partial \overline{\eta}^m}(\zeta_\mu, \zeta_\nu) \right) \geq 0.$$

Remark 3. It is possible to derive this from [7, Theorem 1], but only in the case that the domains are simply connected. The proof given here is also much more direct.

Proof. By Lemma 8,

$$\delta \left(\sum_{\mu,\nu} \beta_\mu \overline{\beta}_\nu \frac{\partial^{2m} K}{\partial \zeta^m \partial \overline{\eta}^m}(\zeta_\mu, \zeta_\nu) \right) = - \int_{\partial D} \left| \sum_{\mu} \beta_\mu \frac{\partial^m K}{\partial \zeta^m}(\zeta_\mu, u) \right|^2 \nu(u)\, ds_u.$$

■

Corollary 10. *Let D_1 be a simply-connected planar domain. Then,*

$$\sum_{\mu,\nu} \beta_\mu \overline{\beta}_\nu \frac{\partial^{2m} K_1}{\partial \zeta^m \partial \overline{\eta}^m}(\zeta_\mu, \zeta_\nu) \geq 0.$$

Proof. First assume that D_1 is bounded by a C^∞-simple closed curve. By letting D_2 be a disc of radius r, and letting $r \to \infty$, this follows from Theorem 9. One then applies the argument of Remark 2. ■

Theorem 11. *Let D_1 and D_2 satisfy the conditions of Theorem 5. Then, the following quantity is a decreasing function of t:*

$$\sum_{\mu,\nu} \bar{\alpha}_\mu \alpha_\nu \frac{\partial^{2m} K_t}{\partial \zeta^m \partial \overline{\eta}^m}(\zeta_\mu, \zeta_\nu) + 2\lambda \,\mathrm{Re} \left(\sum_{\mu,\nu} \alpha_\nu \beta_\mu \frac{\partial^{2m+1} K_t}{\partial \zeta^{m+1} \partial \overline{\eta}^m}(\zeta_\mu, \zeta_\nu) \right)$$
$$+ \lambda^2 \sum_{\mu,\nu} \beta_\mu \overline{\beta}_\nu \frac{\partial^{2m+2} K_t}{\partial \zeta^{m+1} \partial \overline{\eta}^{m+1}}(\zeta_\mu, \zeta_\nu).$$

In particular, it is larger for D_1 than for D_2.

Proof. By Lemma 8,

$$\delta\left(\sum_{\mu,\nu}\overline{\alpha}_\mu\alpha_\nu\frac{\partial^{2m}K}{\partial\zeta^m\partial\overline{\eta}^m}(\zeta_\mu,\zeta_\nu)+2\lambda\,\mathrm{Re}\left(\sum_{\mu,\nu}\alpha_\nu\beta_\mu\frac{\partial^{2m+1}K}{\partial\zeta^{m+1}\partial\overline{\eta}^m}(\zeta_\mu,\zeta_\nu)\right)\right.$$
$$\left.+\lambda^2\sum_{\mu,\nu}\beta_\mu\overline{\beta}_\nu\frac{\partial^{2m+2}K}{\partial\zeta^{m+1}\partial\overline{\eta}^{m+1}}(\zeta_\mu,\zeta_\nu)\right)$$
$$=-\int_{\partial D}\left|\sum_\mu\overline{\alpha}_\mu\frac{\partial^m K}{\partial\zeta^m}(\zeta_\mu,u)+\lambda\beta_\mu\frac{\partial^{m+1}K}{\partial\zeta^{m+1}}(\zeta_\mu,u)\right|^2\nu(u)\,ds_u$$

■

By taking the discriminant, we have the following corollary.

Corollary 12. *Denoting $\Delta K(\zeta,\eta)=K_1(\zeta,\eta)-K_2(\zeta,\eta)$, we have*

$$\left(\mathrm{Re}\sum_{\mu,\nu}\alpha_\nu\beta_\mu\Delta\frac{\partial^{2m+1}K}{\partial\zeta^{m+1}\partial\eta^m}(\zeta_\mu,\zeta_\nu)\right)^2$$
$$\le\left(\sum_{\mu,\nu}\overline{\alpha}_\mu\alpha_\nu\Delta\frac{\partial^{2m}K}{\partial\zeta^m\partial\overline{\eta}^m}(\zeta_\mu,\zeta_\nu)\right)\left(\sum_{\mu,\nu}\beta_\mu\overline{\beta}_\nu\Delta\frac{\partial^{2m+2}K}{\partial\zeta^{m+1}\partial\overline{\eta}^{m+1}}(\zeta_\mu,\zeta_\nu)\right).$$

Theorem 13. *Under the conditions of Theorem 5, the following quantity is larger for D_1 than for D_2:*

$$\sum_{\mu,\nu}\alpha_\mu\overline{\alpha}_\nu\frac{\partial^{2m}K}{\partial\zeta^m\partial\overline{\eta}^m}(\zeta_\mu,\zeta_\nu)+2\lambda\,\mathrm{Re}\left(\sum_{\mu,\nu}\beta_\mu\alpha_\nu\frac{\partial^{2m+1}L}{\partial\zeta^{m+1}\partial\eta^m}(\zeta_\mu,\zeta_\nu)\right)$$
$$+\lambda^2\sum_{\mu,\nu}\beta_\mu\overline{\beta}_\nu\frac{\partial^{2m+2}K}{\partial\zeta^{m+1}\partial\overline{\eta}^{m+1}}(\zeta_\mu,\zeta_\nu).$$

Proof. By Lemma 8,

$$\delta\left(\sum_{\mu,\nu}\alpha_\mu\overline{\alpha}_\nu\frac{\partial^{2m}K}{\partial\zeta^m\partial\overline{\eta}^m}(\zeta_\mu,\zeta_\nu)+2\lambda\,\mathrm{Re}\left(\sum_{\mu,\nu}\beta_\mu\alpha_\nu\frac{\partial^{2m+1}L}{\partial\zeta^{m+1}\partial\overline{\eta}^m}(\zeta_\mu,\zeta_\nu)\right)\right.$$
$$\left.+\lambda^2\sum_{\mu,\nu}\beta_\mu\overline{\beta}_\nu\frac{\partial^{2m+2}K}{\partial\zeta^{m+1}\partial\overline{\eta}^{m+1}}(\zeta_\mu,\zeta_\nu)\right)$$
$$=-\int_{\partial D}\left|\sum_\mu\alpha_\mu\frac{\partial^m K}{\partial\zeta^m}(\zeta_\mu,u)+\lambda\overline{\beta}_\mu\overline{\frac{\partial^{m+1}L}{\partial\zeta^{m+1}}(\zeta_\mu,u)}\right|^2\nu(u)\,ds_u.$$

■

Again taking the discriminant, we get the following corollary.

Corollary 14. *With D_1 and D_2 as in Theorem 5,*

$$\left(\operatorname{Re}\sum_{\mu,\nu}\alpha_\nu\beta_\mu\Delta\frac{\partial^{2m+1}L}{\partial\zeta^{m+1}\partial\eta^m}(\zeta_\mu,\zeta_\nu)\right)^2 \leq \left(\sum_{\mu,\nu}\alpha_\mu\overline{\alpha}_\nu\Delta\frac{\partial^{2m}K}{\partial\zeta^m\partial\overline{\eta}^m}(\zeta_\mu,\zeta_\nu)\right)\left(\sum_{\mu,\nu}\beta_\mu\overline{\beta}_\nu\Delta\frac{\partial^{2m+2}K}{\partial\zeta^{m+1}\partial\overline{\eta}^{m+1}}(\zeta_\mu,\zeta_\nu)\right).$$

Finally, we would like to observe that Theorem 1 in [7] can be extended to the multiply connected case: i.e. for D_1, D_2, satisfying the conditions of Theorem 5, and notation as above, we have

$$\operatorname{Re}\Delta\left(\sum_{\mu,\nu}\alpha_\mu\alpha_\nu\frac{\partial^{2m}L}{\partial\zeta^m\partial\eta^n}(\zeta_\mu,\zeta_\nu)\right)-\Delta\left(\sum_{\mu,\nu}\alpha_\mu\overline{\alpha}_\nu\frac{\partial^{2m}K}{\partial\zeta^m\partial\overline{\eta}^m}(\zeta_\mu,\zeta_\nu)\right)\leq 0. \tag{13}$$

To prove this, just note that

$$\delta\left(\sum_{\mu,\nu}\alpha_\mu\overline{\alpha}_\nu\frac{\partial^{2m}K}{\partial\zeta^m\partial\overline{\eta}^m}(\zeta_\mu,\zeta_\nu)\right)=-\int_{\partial D}\left|\sum_{\nu}\overline{\alpha}_\nu\frac{\partial^m K}{\partial\overline{\eta}^m}(u,\zeta_\nu)\right|^2\nu(u)\,ds_u$$

and

$$\delta\left(\sum_{\mu,\nu}\alpha_\mu\alpha_\nu\frac{\partial^{2m}L}{\partial\zeta^m\partial\eta^m}(\zeta_\mu,\zeta_\nu)\right)=-\int_{\partial D}\left(\sum_{\mu,\nu}\alpha_\mu\frac{\partial^m K}{\partial\zeta^m}(\zeta_\mu,u)\right)^2\left(\frac{du}{ds_u}\right)^2\nu(u)\,ds_u,$$

using the identity $L(\zeta,\eta)d\zeta=-\overline{K(\zeta,\eta)d\overline{\zeta}}$ (see [2, p. 208] or (8)), and the fact that when integrating with respect to arc length, $|du/ds_u|=1$. The case $m=0$ of this theorem was proved by Bergman and Schiffer [2] using other techniques. They also establish that this quantity increases when the domain decreases under a Hadamard variation, in the same way that we do here in the general case.

4. Estimates for univalent functions

Let $f\colon\mathbb{D}\to\mathbb{D}$ be a bounded univalent function. By letting the outside domain be the unit disc, and expressing the kernel functions of the inside domain in terms of the map f, we can derive estimates for bounded univalent functions. Although there is much room for experimentation, we limit ourselves here to two such inequalities.

We define the hyperbolic derivatives invented by Minda [4]:

$$D_i^h(f)(a)=\left.\frac{\partial^i}{\partial z^i}\right|_{z=0}(S\circ f\circ T)(z)$$

where

$$S(z)=\frac{z-f(a)}{1-\overline{f(a)}z},\qquad T(z)=\frac{z+a}{1+\bar{a}z}.$$

Also, let $\sigma_3(f)$ denote the Schwarzian derivative, and

$$\sigma_4(f) = \sigma_3(f)_z - 2\frac{f''}{f'}\sigma_3(f),$$

and define

$$\sigma_4^h(f)(a) = \sigma_4(S \circ f)(a).$$

The hyperbolic line element on the disc is given by

$$\lambda(z) = \frac{1}{1-|z|^2}.$$

Then we have the following distortion theorem.

Corollary 15. *Let f be a univalent map from the unit disc into itself. Then,*

$$\left|\frac{\sigma_4^h(f)(z)}{12\lambda(z)^3}\right|^2 \le \left(1 - |D_1^h f(z)|^2\right)\left(\left|\frac{D_2^h f(z)}{D_1^h f(z)}\right|^2 + 2\right). \tag{14}$$

Proof. This follows from Corollary 14, by choosing $n = 1$ and $\alpha = \beta = 1$. The computation of the diagonal terms appears in [7]. ■

Also, we have the following estimate for a univalent function f from the unit disc into itself.

Corollary 16.

$$\frac{1}{2}\left(\mathrm{Re}\sum_{\mu,\nu}\alpha_\mu\beta_\nu\frac{1-|z_\nu|^2}{f'(z_\mu)(z_\mu - z_\nu)(1-\bar{z}_\nu z_\mu)} - \frac{1-|f(z_\nu)|^2}{(f(z_\mu)-f(z_\nu))(1-\overline{f(z_\nu)}f(z_\mu))}\right)^2$$
$$\le \left(\sum_{\mu,\nu}\alpha_\mu\overline{\alpha}_\nu \log\left|\frac{f(z_\mu)-f(z_\nu)}{z_\mu - z_\nu}\frac{1-\bar{z}_\nu z_\mu}{1-\overline{f(z_\nu)}f(z_\mu)}\right|\right)$$
$$\times\left(\sum_{\mu,\nu}\beta_\mu\overline{\beta}_\nu\frac{1}{(1-\overline{f(z_\nu)}f(z_\mu))^2} - \frac{1}{f'(z_\mu)\overline{f'(z_\nu)}(1-\overline{z_\nu}z_\mu)^2}\right).$$

Proof. Apply Theorem 5. ■

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Eric Schippers E-MAIL: eric_schippers@umanitoba.ca
ADDRESS: *Department of Mathematics, University of Manitoba, Winnipeg, Manitoba R3T 2N2, Canada.*