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Estimates on Kernel Functions of Elliptically Schlicht Domains

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Abstract. Inequalities for derivatives of the Bergman and L-kernel are derived for elliptically schlicht domains, using the Dirichlet Principle. By introducing a generalization of elliptically schlicht domains involving metrics of constant positive curvature, it is seen that the inequalities smoothly interpolate the bounded, unbounded and elliptically schlicht cases.

Keywords. Elliptically schlicht, univalent, Bergman kernel, Green's function, inequalities, conformal metrics, elliptic capacity, Dirichlet Principle.

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1. Introduction

The purpose of this paper is to demonstrate that a method of proving function theoretic inequalities due to Nehari [11] can be extended to the Riemann sphere. We derive inequalities for 'elliptically schlicht' mappings; these are mappings whose image does not contain any pairs of antipodal points on the sphere. These maps were invented by Grunsky [5], and studied, for instance, by Kühnau [8], [9] and Jenkins [7].

The method of Nehari uses the fact that the solution to the Dirichlet Problem has minimum energy to generate inequalities for domain functions. Varying the boundary conditions gives rise to different inequalities. Of course, there is no sensible Dirichlet Problem for the Riemann sphere. This obstacle is overcome by observing that the Dirichlet Principle is equivalent to the positivity of the energy of the difference between the harmonic solution and a competing function with the same boundary values. Nehari's method then easily carries over to the sphere.

The main theorem is an extension of the following theorem, proven in [13].

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Theorem 1. *If D and D' are simply connected hyperbolic domains, bounded by piecewise smooth curves, and $D \subset D'$, then for any collection of points $\xi_\mu \in D$ and scalars $\alpha_\mu \in \mathbb{C}$, $\mu = 1, \dots n$, and $m \geq 0$,*

$$\operatorname{Re}\left(\sum_{\mu,\nu}\left[\alpha_\mu\overline{\alpha}_\nu\frac{\partial^{2m}K}{\partial\zeta^m\partial\overline{\eta}^m}(\xi_\mu,\xi_\nu) - \alpha_\mu\overline{\alpha}_\nu\frac{\partial^{2m}K'}{\partial\zeta^m\partial\overline{\eta}^m}(\xi_\mu,\xi_\nu)\right]\right)$$
$$\geq \operatorname{Re}\left(\sum_{\mu,\nu}\left[\alpha_\mu\alpha_\nu\frac{\partial^{2m}L}{\partial\zeta^m\partial\eta^m}(\xi_\mu,\xi_\nu) - \alpha_\mu\alpha_\nu\frac{\partial^{2m}L'}{\partial\zeta^m\partial\eta^m}(\xi_\mu,\xi_\nu)\right]\right).$$

Here K and K' are the Bergman kernels of D and D' respectively, and

$$L(\zeta,\eta) = -\frac{2}{\pi}\frac{\partial^2 g}{\partial\zeta\partial\eta}$$

where g is Green's function of D; L' is defined in the same way using Green's function g' of D'.

We show that this theorem extends to the case where the outer domain D' is the Riemann sphere. This extension requires a natural definition of K and L on the sphere. In the following section we give definitions and some motivation. In Section 3 the main inequalities (Theorem 2) are stated and proved.

In order to provide some context we discuss, in Section 4, a Theorem of Kühnau [8], completing a 'rectangle' of theorems. Kühnau's Theorem is a generalization to the elliptically schlicht case of a Theorem of Nehari estimating Green's functions of two domains. Theorem 1 is a higher-order version of Nehari's inequality, while Theorem 2 is a higher-order version of Kühnau's Theorem. By 'higher-order', we mean that the inequalities are estimates on higher derivatives of the mapping function or Green's function. We also give a sharpening of Kühnau's Theorem provided there is some additional information about the complement of the domain and its reflection.

Also built in is a slight generalization of the concepts of elliptically schlicht domains and functions, based on the following idea. The notion of 'antipodal point' is dependent on the choice of metric. Here, the sphere is endowed with a metric of constant positive curvature; unlike the negative curvature case, the metric is not uniquely determined by the curvature. A one-parameter family of metrics with curvature 4 is given, each matching up different pairs of points as antipodal. In this way we have a different family of 'elliptically schlicht' domains for each choice of parameter. This is discussed in Section 2.1.

The advantage of this generalization is that the parameter appears in the main inequalities here (Theorem 2), and in this way the bounded, unbounded, and elliptically schlicht case are continuously interpolated. Setting the parameter to 1, 0, and -1, results in the elliptically schlicht, Euclidean, and hyperbolic cases respectively.

Finally, in Section 4 we give an elementary inequality relating the elliptic capacity of Duren and Kühnau [3] to a natural notion of 'elliptic reduced module'.

2. Preliminaries

2.1. Constant curvature metrics and elliptically k-schlicht domains. Let D_k be the disc of radius $1/\sqrt{-k}$ centred at the origin for $k \leq 0$. In the case that $k = 0$ it is understood that we mean $\mathbb{C}$, and if $k > 0$, D_k is taken to be the Riemann sphere $\overline{\mathbb{C}}$. Endow D_k with the conformal metric

$$\lambda_k(z) = \frac{\sqrt{|k|}}{1 + k|z|^2}$$

in the case that $k \neq 0$, and

$$\lambda_0(z) = 1$$

in the case that $k = 0$. This metric has curvature 4, 0, and -4, in the cases that k is positive, zero, or negative respectively. The set of isometries of this metric are

$$\mathrm{Isom}(\lambda_k) = \left\{ T(z) = e^{i\theta} \frac{z + w}{1 - k\bar{w}z} : w \in D_k \right\}. \tag{1}$$

The fact that this is the set of one-to-one and onto isometries, follows from the identity

$$\frac{|T'(z)|}{1 + k|T(z)|^2} = \frac{1}{1 + k|z|^2}.$$

The mappings (1) are the only Möbius transformations satisfying the identity above. Note that for $k = 0$ these are the Euclidean isometries.

In fact, it can be shown that for each k, a mapping which is an isometry on any open set must indeed be one of the global isometries (1). For the case $k < 0$, this is just the equality statement of the Schwarz Lemma. The case $k > 0$ is somewhat trickier [10], but is not necessary in the following.

In [8], Kühnau considers four distinct kinds of schlicht mappings, of which two will be considered here. In the first case, he assumes that the image of f cannot contain both the points w and $1/\bar{w}$. It is easily seen that this condition is equivalent to the assumption that f is bounded either outside or inside the unit disc. In the second case, he assumes that the image of f cannot contain both w and $-1/\bar{w}$; mappings of this sort are called 'elliptically schlicht'.

We extend these classes as follows. We will assume that f always contains 0 in its image. For each k, we have a class of mappings $\mathcal{B}_k$.

- If $k < 0$, then we assume that the image of f cannot contain both the points w and $1/|k|\bar{w}$, for any w. Equivalently, the image of f is contained in the domain D_k.
- If $k = 0$, then we assume only that the image of f does not contain ∞.

- If $k > 0$, we assume that the image of f cannot contain both the points w and $-1/k\bar{w}$, for any w.

Simply connected domains that never contain both of the points w and $-1/k\bar{w}$ for $k > 0$ will be called 'elliptically k-schlicht'. Univalent mappings onto such domains will also be referred to as 'elliptically k-schlicht'.

The notion of an elliptically k-schlicht domain has a simple geometric interpretation: the points w and $-1/(k\bar{w})$ are antipodal points in the metric λ_k. (By 'antipodal' we mean that the two points are a maximal distance apart.) To see this, just note that 0 and ∞ are antipodal points in λ_k by the radial symmetry of the metric. Isometries take pairs of antipodal points to pairs of antipodal points, so applying a transformation of the form (1) we see that w and $-1/(k\bar{w})$ are indeed antipodal.

Remark 1. It is well known that there is a unique complete constant curvature metric on the disc (up to scale). However, this is not the case on the sphere: every pull-back of λ_1 under a Möbius transformation is a complete constant curvature metric. The metrics λ_k are all such metrics for which 0 and ∞ are antipodal (up to scale); however as seen above each choice of k gives different associations of antipodal points. This arbitrariness in the choice of antipodal points is another reason for introducing the parameter k. It is remarkable that this arbitrariness directly leads to a natural positive curvature counterpart to the radius of the disc in the bounded case. Finally, note that if we want to interpolate the bounded, unbounded, and elliptic cases as mentioned in the introduction, it is necessary to demand that 0 and ∞ are antipodal. This restricts our attention to the metrics λ_k.

2.2. Definition of the kernel functions on the plane and Riemann sphere. In this section we give natural generalizations of the Bergman kernel K and the kernel L to the case of the Riemann sphere and the plane.

First, we define 'Green's function' of the domain D_k to be

$$g_k(\zeta, \eta) = -\log \left| \frac{\sqrt{|k|}(\zeta - \eta)}{(1 + k\bar{\eta}\zeta)} \right|.$$

If $k < 0$, this is of course Green's function of D_k. The function above is a reasonable extension when $k > 0$ because then g_k is the unique (up to a constant) harmonic function with logarithmic singularities of opposite sign at the antipodal points η and $-1/(k\bar{\eta})$. On the other hand, g_k is not defined for $k = 0$.

The kernel functions K and L can now be defined for the domains D_k for $k \geq 0$. We define

$$K_k(\zeta, \eta) = -\frac{2}{\pi} \frac{\partial^2 g_k}{\partial \zeta \partial \bar{\eta}}(\zeta, \eta)$$

and

$$L_k(\zeta,\eta) = -\frac{2}{\pi}\frac{\partial^2 g_k}{\partial\zeta\partial\eta}(\zeta,\eta).$$

When $k < 0$, these agree with the standard Bergman and L kernel (see for instance [2]). Of course these are defined on any domain with Green's function, but we restrict our attention to the discs D_k.

In the case that a domain is a disc, we always have that

$$L(\zeta,\eta) = \frac{1}{\pi(\zeta-\eta)^2}.$$

This extends to the case where $k > 0$, so we use this as the L kernel of the Riemann sphere, for all choices of k. Thus we define

$$L_k(\zeta,\eta) = \frac{1}{\pi(\zeta-\eta)^2}. \tag{2}$$

Although L_k does not depend on k, we retain the subscript 'k' in order to emphasize the context in which the kernel appears.

On the other hand we can compute that

$$K_k(\zeta,\eta) = -\frac{1}{\pi}\frac{k}{(1+k\bar{\zeta}\eta)^2}.$$

Note that although g_k is not defined for $k=0$, we can easily extend the definitions of K_k and L_k to the case where $k=0$; take $K_0 \equiv 0$ and L_0 as in (2). We now have natural analogues of the L and K kernels in the case of the Riemann sphere and the plane.

3. Inequalities for elliptically schlicht domains

Let E be an elliptically schlicht, simply connected domain, with piecewise smooth boundary, and E^* be the set of antipodal points in the metric of curvature k. Let

$$g_k(\zeta,\eta) = -\log\left|\frac{\sqrt{k}(\zeta-\eta)}{1+k\bar{\eta}\zeta}\right|$$

denote Green's function on D_k, and let g and g^* be Green's functions of E and E^* respectively. If $F\colon E \to D$ is a conformal mapping, then $\zeta \mapsto \overline{F(-1/(k\bar{\zeta}))}$ is a conformal mapping from E^* to D, and we have that Green's functions of E and E^* are

$$\begin{aligned} g(\zeta,\eta) &= -\log\left|\frac{F(\zeta)-F(\eta)}{1-\overline{F(\eta)}F(\zeta)}\right|, \\ g^*(\zeta,\eta) &= -\log\left|\frac{\overline{F(-1/(k\bar{\zeta}))}-\overline{F(-1/(k\bar{\eta}))}}{1-F(-1/(k\bar{\eta}))\overline{F(-1/(k\bar{\zeta}))}}\right|. \end{aligned}$$

Finally let L_k and K_k be as in Section 2.2, and L and K be the corresponding kernel functions for E.

Theorem 2. *Let E be a simply connected, elliptically k-schlicht planar domain, and let E^* be the domain antipodal to E in the metric λ_k. Further, let L, K, L_k, and K_k be the associated kernel functions as above. Then for any collection of points $\xi_i \in E$ and complex parameters α_i, $i = 1, \dots n$, and $m \geq 0$, we have*

$$\operatorname{Re}\left(\sum_{\mu,\nu}\left[\alpha_\mu\overline{\alpha}_\nu\frac{\partial^{2m}K}{\partial\zeta^m\partial\overline{\eta}^m}(\xi_\mu,\xi_\nu) - \alpha_\mu\overline{\alpha}_\nu\frac{\partial^{2m}K_k}{\partial\zeta^m\partial\overline{\eta}^m}(\xi_\mu,\xi_\nu)\right]\right)$$
$$\geq \operatorname{Re}\left(\sum_{\mu,\nu}\left[\alpha_\mu\alpha_\nu\frac{\partial^{2m}L}{\partial\zeta^m\partial\eta^m}(\xi_\mu,\xi_\nu) - \alpha_\mu\alpha_\nu\frac{\partial^{2m}L_k}{\partial\zeta^m\partial\eta^m}(\xi_\mu,\xi_\nu)\right]\right).$$

Remark 2. The following proof also works in the case $k < 0$, and this is in fact a special case of Theorem 1, by letting the outside domain be a disc of radius $1/\sqrt{|k|}$. In the case $m = 0$, letting $k \to 0$, one recovers the well-known inequality of Bergman and Schiffer [2] for domains in the plane. The l-kernel of Bergman and Schiffer can be interpreted as $L_0 - L$, where

$$L_0(\zeta,\eta) = \frac{1}{\pi(\zeta-\eta)^2}$$

is the L-kernel of the complex plane.

Proof. We first note that it is sufficient to prove the theorem in the case that the boundary of E is smooth. To see this, first observe that one can write Green's function of E in terms of a mapping function f from the unit disc $\mathbb{D}$ onto the domain E. Consider the sequence of maps of the unit disc $f_r(z) = rf(z/r)$ for $r > 1$. It can easily be checked that the domains $f_r(\mathbb{D})$ are elliptically (k/r)-schlicht and smoothly bounded, and that the sequence f_r approximates f uniformly on compact subsets of $\mathbb{D}$. The expressions in the theorem can all be written in terms of derivatives of the mapping function, and the expressions for $f_r(\mathbb{D})$ tend to the expressions for E.

Let $l = m + 1$ (this simplifies notation in the proof). Define the harmonic functions

$$p(\zeta) = \operatorname{Re}\left(\sum_{\nu=1}^{n}\alpha_\nu\frac{\partial^l}{\partial\eta^l}\bigg|_{\xi_\nu} g(\zeta,\eta)\right)$$

and

$$p_k(\zeta) = \operatorname{Re}\left(\sum_{\nu=1}^{n}\alpha_\nu\frac{\partial^l}{\partial\eta^l}\bigg|_{\xi_\nu} g_k(\zeta,\eta)\right)$$

on E and D_k respectively. Also, let

$$p^*(\zeta) = -p(-1/(k\bar{\zeta})).$$

These functions are all clearly harmonic, and have the following properties.

(i) $p(-1/(k\bar{\zeta})) = -p^*(\zeta)$

(ii) $p_k(-1/(k\bar{\zeta})) = -p_k(\zeta)$

(iii) $p(\zeta) = 0$ for $\zeta \in \partial E$, and $p^*(\zeta) = 0$ for $\zeta \in \partial E^*$.

(iv) $p - p_k$ is non-singular on E, and $p^* - p_k$ is non-singular on E^*.

The first property is of course just a repetition of the definition of p^*. The second property follows from the easily-verified fact that

$$g_k(-1/(k\bar{\zeta}), \eta) = -g_k(\zeta, \eta).$$

Property (iii) follows from the fact that $g(\zeta, \eta) = 0$ for $\zeta \in E$, along with the fact that g continues smoothly up to the boundary. Property (iv) follows from Properties (i) and (iii), and the fact that $g - g_k$ is non-singular.

Thus the function

$$\epsilon(\zeta) = \begin{cases} p(\zeta) - p_k(\zeta) & \text{if } \zeta \in E \\ -p_k(\zeta) & \text{if } \zeta \in \overline{\mathbb{C}} \backslash (E \cup E^*) \\ p^*(\zeta) - p_k(\zeta) & \text{if } \zeta \in E^* \end{cases}$$

is continuous, non-singular, and piecewise harmonic. The fact that this has non-negative Dirichlet energy will be the source of the desired inequality.

We now compute the Dirichlet energy $D(\epsilon)$, using Green's Theorem. Let D_r be the disc of radius r centred at 0.

$$\begin{aligned} D(\epsilon) &= \iint_{\overline{\mathbb{C}}} \nabla\epsilon \cdot \nabla\epsilon \, dA \\ &= \iint_E \nabla(p - p_k) \cdot \nabla(p - p_k)\, dA + \iint_{\overline{\mathbb{C}} \backslash (E \cup E^*)} \nabla p_k \cdot \nabla p_k \, dA \\ &\quad + \lim_{r\to\infty} \iint_{E^* \cap D_r} \nabla(p^* - p_k) \cdot \nabla(p^* - p_k)\, dA \\ &= \lim_{r\to\infty} \int_{\partial D_r} \frac{2}{i}(p^* - p_k) \frac{\partial}{\partial\zeta}(p^* - p_k)\, d\zeta - \int_{\partial E^*} \frac{2}{i}(p^* - p_k)\frac{\partial}{\partial\zeta}(p^* - p_k)\, d\zeta \\ &\quad + \int_{\partial E^*} \frac{2}{i} p_k \frac{\partial p_k}{\partial\zeta}\, d\zeta - \int_{\partial E} \frac{2}{i} p_k \frac{\partial p_k}{\partial\zeta}\, d\zeta + \int_{\partial E} \frac{2}{i}(p - p_k)\frac{\partial}{\partial\zeta}(p - p_k)\, d\zeta. \end{aligned}$$

Using the properties of p, p^* and p_k described above, we see that for $\xi = -1/(k\bar{\zeta})$,

$$\frac{2}{i}(p^* - p_k)(\xi)\,\frac{\partial}{\partial\xi}(p^* - p_k)(\xi)\, d\xi = -\overline{\frac{2}{i}(p - p_k)(\zeta)\,\frac{\partial}{\partial\bar{\zeta}}(p - p_k)(\zeta)\, d\bar{\zeta}}$$

This implies in particular that

$$\lim_{r\to\infty} \int_{\partial D_r} \frac{2}{i}(p^* - p_k)(\xi)\,\frac{\partial}{\partial\zeta}(p^* - p_k)\, d\zeta = 0.$$

Now using the fact that $p = 0$ on ∂E and $p^* = 0$ on ∂E^* we simplify this expression:

$$\begin{aligned} D(\epsilon) &= -\int_{\partial E^*} \frac{2}{i}(p^* - p_k)\frac{\partial p^*}{\partial \zeta}\, d\zeta + \int_{\partial E} \frac{2}{i}(p - p_k)\frac{\partial p}{\partial \zeta}\, d\zeta. \\ &= 2\,\mathrm{Re}\left(\int_{\partial E} \frac{2}{i}(p - p_k)\frac{\partial p}{\partial \zeta}\, d\zeta\right). \end{aligned}$$

To evaluate this, let

$$q(\zeta, \eta) = -\log \frac{F(\zeta) - F(\eta)}{1 - \overline{F(\eta)}F(\zeta)}$$

and

$$h(\zeta) = \frac{1}{2}\left(\sum_{\nu} \alpha_\nu \frac{\partial^l}{\partial \eta^l}\bigg|_{\eta=\xi_\nu} q(\zeta, \eta) + \overline{\alpha}_\nu \frac{\partial^l}{\partial \overline{\eta}^l}\bigg|_{\eta=\xi_\nu} q(\zeta, \eta)\right).$$

The function q is multi-valued on E; however $\partial q/\partial \eta$ and $\partial q/\partial \bar{\eta}$ are single-valued. Also, letting

$$q_k(\zeta, \eta) = -\log \frac{\sqrt{k}(\zeta - \eta)}{1 + k\bar{\eta}\zeta},$$

we get that

$$\frac{\partial^l q}{\partial \eta^l} - \frac{\partial^l q_k}{\partial \eta^l}$$

is non-singular and holomorphic in ζ on E. Writing $2(g - g_k) = (q - q_k) + (\bar{q} - \bar{q}_k)$, we have

$$\begin{aligned} p - p_k &= \mathrm{Re}\left(\sum_{\nu=1}^{n} \alpha_\nu \frac{\partial^l}{\partial \eta^l}\bigg|_{\eta=\xi_\nu} (g - g_k)\right) \\ &= \frac{1}{2}\,\mathrm{Re}\left(\sum_{\nu=1}^{n} \alpha_\nu \frac{\partial^l}{\partial \eta^l}\bigg|_{\eta=\xi_\nu} (q - q_k) + \bar{\alpha}_\nu \frac{\partial^l}{\partial \bar{\eta}^l}\bigg|_{\eta=\xi_\nu} (q - q_k)\right). \end{aligned}$$

The Cauchy-Riemann equations imply that

$$2\frac{\partial p}{\partial \zeta}\, d\zeta = \frac{\partial h}{\partial \zeta}\, d\zeta.$$

Using this and the fact that

$$\frac{1}{i}\frac{\partial p}{\partial \zeta}\, d\zeta$$

is real when restricted to ∂E, we get

$$\begin{aligned} D(\epsilon) &= 2\,\mathrm{Re}\left(\int_{\partial E}\sum_{\nu=1}^{n}\left[\alpha_\nu\frac{\partial^l}{\partial\eta^l}\bigg|_{\eta=\xi_\nu}(q-q_k)+\bar{\alpha}_\nu\frac{\partial^l}{\partial\bar{\eta}^l}\bigg|_{\eta=\xi_\nu}(q-q_k)\right]\frac{1}{i}\frac{\partial p}{\partial\zeta}\,d\zeta\right) \\ &= \mathrm{Re}\left(\int_{\partial E}\sum_{\nu=1}^{n}\left[\alpha_\nu\frac{\partial^l}{\partial\eta^l}\bigg|_{\eta=\xi_\nu}(q-q_k)+\bar{\alpha}_\nu\frac{\partial^l}{\partial\bar{\eta}^l}\bigg|_{\eta=\xi_\nu}(q-q_k)\right]\frac{1}{i}\frac{\partial h}{\partial\zeta}\,d\zeta\right). \end{aligned}$$

Furthermore,

$$\begin{aligned} h(\zeta)-\frac{1}{2}\sum\alpha_\nu\frac{(l-1)!}{(\zeta-\xi_\nu)^l} &= \frac{1}{2}\sum\alpha_\nu\frac{\partial^l}{\partial\eta^l}\bigg|_{\eta=\xi_\nu}[q(\zeta,\eta)+\log(\zeta-\eta)] \\ &\quad +\frac{1}{2}\overline{\alpha}_\nu\frac{\partial^l}{\partial\overline{\eta}^l}\bigg|_{\eta=\xi_\nu}q(\zeta,\eta) \end{aligned}$$

is holomorphic, so we can replace $h'(\zeta)$ with $1/2\sum\alpha_\nu l!\,(\zeta-\xi_\nu)^{-(l+1)}$ so that

$$\begin{aligned} D(\epsilon) &= \frac{1}{2}\,\mathrm{Re}\Bigg(\int_{\partial E}\sum\left[\alpha_\nu\frac{\partial^l}{\partial\eta^l}\bigg|_{\eta=\xi_\nu}(q-q_k)+\bar{\alpha}_\nu\frac{\partial^l}{\partial\bar{\eta}^l}\bigg|_{\eta=\xi_\nu}(q-q_k)\right] \\ &\qquad \cdot\frac{l!}{i}\sum\frac{\alpha_\nu}{(\zeta-\xi_\nu)^{l+1}}\,d\zeta\Bigg) \\ &= \pi\,\mathrm{Re}\Bigg(\sum_{\mu,\nu}\alpha_\mu\alpha_\nu\left[\frac{\partial^{2l}q}{\partial\zeta^l\partial\eta^l}(\xi_\mu,\xi_\nu)-\frac{\partial^{2l}q_k}{\partial\zeta^l\partial\eta^l}(\xi_\mu,\xi_\nu)\right] \\ &\qquad +\sum_{\mu,\nu}\alpha_\mu\overline{\alpha}_\nu\left[\frac{\partial^{2l}q}{\partial\zeta^l\partial\overline{\eta}^l}(\xi_\mu,\xi_\nu)-\frac{\partial^{2l}q_k}{\partial\zeta^l\partial\overline{\eta}^l}(\xi_\mu,\xi_\nu)\right]\Bigg). \end{aligned}$$

The theorem follows immediately upon observing that

$$2\frac{\partial g}{\partial\zeta}=\frac{\partial q}{\partial\zeta}.$$

■

Remark 3 (The equality case)**.** The inequality is equivalent to the fact that the Dirichlet energy of the function ϵ is non-negative. Thus, equality occurs if and only if the Dirichlet energy is zero. This happens exactly when $\mathbb{C}\backslash(E\cup E^*)$ has zero area, and $p=p_k$ on E. The second condition implies that $p_k=0$ on ∂E, so that the boundary must consist of arcs along which

$$p_k(\zeta)=\mathrm{Re}\left(\sum_{\nu=1}^{n}\alpha_\nu\frac{\partial^l}{\partial\eta^l}\bigg|_{\xi_\nu}g_k(\zeta,\eta)\right)=0.$$

4. A theorem of Kühnau, elliptic capacity, and elliptic reduced module

In this section we will show that a Theorem of Kühnau (namely, Satz IV in [8]) can be sharpened using the Dirichlet energy method of Nehari. This sharpening involves information about the complement of the region. The sharpened theorem leads to an elementary inequality between 'elliptic capacity' [3] and 'elliptic reduced module'. In the hyperbolic case this is a known sharpening of the Schwarz Lemma.

Kühnau's Theorem has two cases, one for univalent functions bounded in the unit disc, and the other for elliptically schlicht functions. Since it is simple to add the parameter k, we do this here. However, we would like to emphasize that in contrast to Theorem 2, there is no Euclidean version of this theorem.

Let E be a simply connected planar domain. There are two distinct cases here: $k > 0$ and $k < 0$. If $k < 0$, we assume that E is bounded in the disc D_k.

Theorem 3. *Let $f \in \mathcal{B}_k$, $k \neq 0$, and let α_μ, $\mu = 1, \dots n$ be complex parameters satisfying $\operatorname{Im}(\alpha_1 + \cdots + \alpha_n) = 0$, and $z_\mu \in \mathbb{D}$. Also, let $d(E, E^*)$ denote the extremal distance between the domains E and E^*. Then, we have that*

$$\operatorname{Re}\left(\sum_{\mu,\nu} \overline{\alpha_\mu}\alpha_\nu \log \frac{1 + k\overline{f(z_\mu)}f(z_\nu)}{1 - \overline{z_\mu}z_\nu} - \sum_{\mu,\nu} \alpha_\mu\alpha_\nu \log \frac{\sqrt{k}(f(z_\mu) - f(z_\nu))}{z_\mu - z_\nu}\right)$$
$$\geq d(E, E^*) \left(\sum_\nu \alpha_\nu\right)^2 .$$

Proof. We will prove this only in the case that $k > 0$. The case $k < 0$ is similar, and appears in Nehari [11] (for $k = -1$). Assume then that k is strictly positive. Let f be a mapping from the unit disc onto E. Let $F = f^{-1}$, and $\zeta_\nu = f(z_\nu)$. We define the functions

$$\begin{aligned}
p(\zeta) &= \operatorname{Re}\left(\sum_\nu \alpha_\nu \log\left(F(\zeta) - F(\zeta_\nu)\right) - \overline{\alpha_\nu} \log\left(1 - \overline{F(\zeta_\nu)}F(\zeta)\right)\right), && \zeta \in E, \\
p^*(\zeta) &= -\operatorname{Re}\left(\sum_\nu \overline{\alpha_\nu} \log\left(\overline{F(\zeta^*)} - \overline{F(\zeta_\nu)}\right) - \alpha_\nu \log\left(1 - F(\zeta_\nu)\overline{F(\zeta^*)}\right)\right), && \zeta \in E^*, \\
p_k(\zeta) &= \operatorname{Re}\left(\sum_\nu \alpha_\nu \log \sqrt{k}(\zeta - \zeta_\nu) - \overline{\alpha_\nu} \log\left(1 + k\overline{\zeta_\nu}\zeta\right)\right), && \zeta \in \overline{\mathbb{C}},
\end{aligned}$$

where $\zeta^* = -1/(k\overline{\zeta})$.

These functions are all clearly multi-valued, so some remarks are in order. For any simple closed curve γ in E which encloses the points ζ_μ, we have that

$$\int_\gamma \sum_\mu \frac{\alpha_\mu F'(\zeta)}{F(\zeta)-F(\zeta_\mu)}\,d\zeta$$

is pure imaginary by the condition $\operatorname{Im}(\alpha_1+\cdots+\alpha_n)=0$. Thus there is a single-valued determination of p in a doubly connected region whose outer boundary is ∂E. Similarly, p^* has a single-valued determination in a doubly connected region one of whose boundaries is ∂E^*, and p_k has a single-valued determination in $\mathbb{C}\backslash(E\cup E^*)$.

We now verify that p, p^*, and p_k have the four properties listed in Theorem 2.

(i) It is clear from the definition of p and p^* that $p(-1/(k\bar\zeta))=-p^*(\zeta)$ (on the annular regions described in the previous paragraph).

(ii) On $\mathbb{C}\backslash(E\cup E^*)$, one can write (using $\operatorname{Re}z=\operatorname{Re}\bar z$ in the second equality)

$$\begin{aligned}
p_k(-1/(k\bar\zeta)) &= \operatorname{Re}\left(\sum \alpha_\nu \log\sqrt{k}\left(-\frac{1}{k\bar\zeta}-\zeta_\nu\right)-\overline{\alpha}_\nu\log\left(1-\frac{\bar\zeta_\nu}{\bar\zeta}\right)\right)\\
&= \operatorname{Re}\left(\sum \overline{\alpha}_\nu \log(1+k\bar\zeta_\nu\zeta)-\alpha_\nu\log\sqrt{k}(\zeta-\zeta_\nu)+\alpha_\nu\log\left(-\frac{\zeta}{\bar\zeta}\right)\right)\\
&= p_k(\zeta).
\end{aligned}$$

In the last step we have again used the condition on the parameters α_μ.

(iii) Again using $\operatorname{Re}z=\operatorname{Re}\bar z$, we have that

$$p(\zeta)=\operatorname{Re}\left(\sum \alpha_\nu \log\left(\frac{F(\zeta)-F(\zeta_\nu)}{1-\overline{F(\zeta_\nu)}F(\zeta)}\right)+\overline{\alpha}_\nu\log\left(\frac{1-F(\zeta_\nu)\overline{F(\zeta)}}{1-\overline{F(\zeta_\nu)}F(\zeta)}\right)\right).$$

Since both arguments of log are of unit modulus when $|\zeta|=1$, the inside of the brackets is pure imaginary by the condition on the parameters α_μ; thus $p(\zeta)$ is zero on ∂D.

(iv) Finally, since

$$\frac{F(\zeta)-F(\zeta_\nu)}{\zeta-\zeta_\nu}\neq 0$$

for all $\zeta\in E$, we get that $p-p_k$ and p^*-p_k are non-singular on E and E^* respectively.

Define the function

$$\epsilon(\zeta)=\begin{cases} p(\zeta)-p_k(\zeta) & \text{if } \zeta\in E\\ -p_k(\zeta) & \text{if } \zeta\in\overline{\mathbb{C}}\backslash(E\cup E^*)\\ p^*(\zeta)-p_k(\zeta) & \text{if } \zeta\in E^*\end{cases}$$

which is harmonic on each piece, and continuous. Also define, for a constant α,

$$\epsilon'(\zeta) = \begin{cases} \epsilon(\zeta) + \alpha & \text{if } \zeta \in E, \\ \epsilon(\zeta) + \alpha\omega(\zeta) & \text{if } \zeta \in \overline{\mathbb{C}} \backslash (E \cup E^*), \\ \epsilon(\zeta) & \text{if } \zeta \in E^*, \end{cases}$$

where ω is the harmonic measure of $\overline{\mathbb{C}} \backslash (E \cup E^*)$ given by

$$\omega(\zeta) = \begin{cases} 1 & \text{if } \zeta \in \partial E, \\ 0 & \text{if } \zeta \in \partial E^*. \end{cases}$$

Then the inequality

$$\iint_{\mathbb{C}} |\nabla \epsilon'|^2 \, dA \geq 0$$

leads to

$$\begin{aligned} 0 \ \leq \ & \iint_{\mathbb{C}} \nabla \epsilon \cdot \nabla \epsilon \, dA - 2\alpha \iint_{\overline{\mathbb{C}} \backslash (E \cup E^*)} \nabla p_k \cdot \nabla \omega \, dA \\ & + \alpha^2 \iint_{\overline{\mathbb{C}} \backslash (E \cup E^*)} \nabla \omega \cdot \nabla \omega \, dA. \end{aligned} \tag{3}$$

We will first show that

$$\begin{aligned} \iint_{\mathbb{C}} \nabla \epsilon \cdot \nabla \epsilon \, dA \ = \ & 2\pi \operatorname{Re} \Bigg(\sum_{\mu,\nu} \overline{\alpha_\mu} \alpha_\nu \log \frac{1 + k \overline{f(z_\mu)} f(z_\nu)}{1 - \overline{z_\mu} z_\nu} \\ & - \sum_{\mu,\nu} \alpha_\mu \alpha_\nu \log \frac{\sqrt{k}(f(z_\mu) - f(z_\nu))}{z_\mu - z_\nu} \Bigg) \end{aligned} \tag{4}$$

As in Theorem 2, the Dirichlet energy of (4) can be computed to be

$$\iint_{\overline{\mathbb{C}}} |\nabla \epsilon|^2 \, dA = 2 \operatorname{Re} \left(\frac{2}{i} \int_{\partial E} (p - p_k) \frac{\partial p}{\partial \zeta} \, d\zeta \right).$$

On ∂E,

$$\frac{2}{i} \frac{\partial p}{\partial \zeta} \, d\zeta$$

is real, so

$$\frac{2}{i} \int_{\partial E} (p - p_k) \frac{\partial p}{\partial \zeta} \, d\zeta = \operatorname{Re} \left(\int_E (q - q_k) \frac{2}{i} \frac{\partial p}{\partial \zeta} \, d\zeta \right)$$

where

$$q(\zeta) = \left[\sum_\nu \alpha_\nu \log \left(F(\zeta) - F(\zeta_\nu) \right) - \overline{\alpha}_\nu \log \left(1 - \overline{F(\zeta_\nu)} F(\zeta) \right) \right],$$

and

$$q_k(\zeta) = \left[\sum_\nu \alpha_\nu \log \sqrt{k} (\zeta - \zeta_\nu) - \overline{\alpha}_\nu \log \left(1 + k \overline{\zeta}_\nu \zeta \right) \right].$$

By the Cauchy-Riemann equations, we also have that

$$2\frac{\partial p}{\partial \zeta} = \frac{\partial q}{\partial \zeta}.$$

Since $q - \sum \alpha_\nu \log(\zeta - \zeta_\nu)$ is holomorphic and single-valued on E, we then have that

$$\begin{aligned}
\int_{\overline{\mathbb{C}}} |\nabla \epsilon|^2 dA &= \mathrm{Re}\left(\int_{\partial D_1} (q - q_k)\left(\frac{1}{i}\frac{\partial}{\partial \zeta}\left(\sum \alpha_\nu \log(\zeta - \zeta_\nu)\right)\right) d\zeta\right) \\
&= 2\pi\,\mathrm{Re}\left(\sum \alpha_\nu \lim_{\zeta \to \zeta_\nu} (q(\zeta) - q_k(\zeta))\right) \\
&= 2\pi\,\mathrm{Re}\left(\sum_{\mu,\nu} \alpha_\mu \alpha_\nu \log \frac{F(\zeta_\mu) - F(\zeta_\nu)}{\sqrt{k}(\zeta_\mu - \zeta_\nu)} - \overline{\alpha}_\mu \alpha_\nu \log \frac{1 - \overline{F(\zeta_\mu)}F(\zeta_\nu)}{1 + k\overline{\zeta}_\mu \zeta_\nu}\right).
\end{aligned}$$

Substituting $f = F^{-1}$ gives the expression (4).

We now return to the computation of (3). The best inequality is attained when one chooses

$$\alpha = \left(\iint_{\mathbb{C}\setminus(E\cup E^*)} \nabla\omega \cdot \nabla\omega \, dA\right)^{-1} \iint_{\mathbb{C}\setminus(E\cup E^*)} \nabla p_k \cdot \nabla\omega \, dA,$$

which results in

$$0 \leq J - \left(\iint_{\mathbb{C}\setminus(E\cup E^*)} \nabla\omega \cdot \nabla\omega \, dA\right)^{-1} \left(\iint_{\mathbb{C}\setminus(E\cup E^*)} \nabla p_k \cdot \nabla\omega \, dA\right)^2.$$

Now as is well-known,

$$\iint_{\mathbb{C}\setminus(E\cup E^*)} \nabla\omega \cdot \nabla\omega \, dA = \frac{1}{d(E, E^*)},$$

where d is the extremal distance between ∂E and ∂E^*. Also, by Green's identity,

$$\iint_{\mathbb{C}\setminus(E\cup E^*)} \nabla p_k \cdot \nabla\omega \, dA = -\int_{\partial E} \frac{\partial p_k}{\partial n} \, ds$$

where n is the unit outward normal on ∂E. This is easily evaluated once we recognize that

$$\int \mathrm{Re}\left(2\frac{\partial p_k}{\partial z} \, dz\right) = 0$$

and

$$\mathrm{Im}\left(2\frac{\partial p_k}{\partial z} \, dz\right) = \frac{\partial p_k}{\partial n} \, ds.$$

So

$$\begin{aligned}\int_{\partial E} \frac{\partial p_k}{\partial n}\, ds &= \frac{2}{i}\int_{\partial E} \frac{\partial p_k}{\partial z}\, dz \\ &= \frac{1}{i}\int_{\partial E} \sum_{\mu} \left(\frac{\alpha_\nu}{z - z_\nu} - \frac{\overline{\alpha}_\nu k \overline{z}_\nu}{1 + k\overline{z}_\nu z} \right) dz \\ &= 2\pi \left(\sum_{\nu} \alpha_\nu \right).\end{aligned}$$

We thus have the desired inequality. ∎

Remark 4. Kühnau's Theorem is the inequality of Theorem 3 with the right hand side replaced by 0. Theorem 2, when written in terms of the mapping function f, can be seen as a kind of higher-order version of Kühnau's Theorem ('order' refers to the number of derivatives.) As mentioned before, the case $k = -1$ in Kühnau's Theorem is a Theorem of Nehari [11]. Finally, Theorem 1 is a higher-order version of this Theorem of Nehari to higher orders of differentiation.

We stress, however, that Theorem 1 and Theorem 2 are not direct generalizations of Nehari's and Kühnau's Theorems respectively. In the special case that all the constants α_ν are real, the relation between the theorems is particularly nice: for example, Kühnau's Theorem (i.e. set the right hand side to zero in Theorem 3) becomes

$$\mathrm{Re}\left(\sum_{\mu,\nu} \alpha_\mu \alpha_\nu \left(g(\zeta_\mu, \zeta_\nu) - g_k(\zeta_\mu, \zeta_\nu) \right) \right) \geq 0. \tag{5}$$

Compare this with Theorems 1 and 2.

This completes a 'rectangle' of theorems, summarized in the table below, in which l denotes the order of differentiation:

	hyperbolic case	elliptic case
$l = 0$	Nehari's Theorem	Kühnau's Theorem
$l \geq 2$	Theorem 1	Theorem 2

Equation (5) is closely related to the Schwarz Lemma. Choosing a single point z and parameter $\alpha = 1$ results in the inequality

$$\frac{\sqrt{k}|f'(z)|}{1 + k|f(z)|^2} \leq \frac{1}{1 - |z|^2},$$

see Pommerenke [12, p. 98]. Denoting the pull-back of a metric $\rho(w)\,|dw|$ under a map $w = f(z)$ by $f^*(\rho(w)\,|dw|) = \rho(f(z))|f'(z)|\,|dz|$, we can write this in the form

$$f^*(\lambda_k(w)|dw|) \leq \lambda_{-1}(z)\,|dz|.$$

In other words, elliptically k-schlicht functions satisfy a 'Schwarz Lemma' comparing the hyperbolic metric on the disc to the metric of constant curvature k on the image.

One can sharpen this 'Schwarz Lemma' using information about $\mathbb{C}\backslash(E \cup E^*)$. Assume all α_ν are real in Theorem 3. We get that

$$d(E, E^*) \left(\sum_\nu \alpha_\nu \right)^2 \leq \sum_{\mu,\nu} \alpha_\mu \alpha_\nu \left(g(\zeta_\mu, \zeta_\nu) - g_k(\zeta_\mu, \zeta_\nu) \right).$$

In particular, choosing a single point ζ_0 and constant $\alpha = 1$, we have that

$$d(E, E^*) \leq \lim_{\zeta \to \zeta_0} \left(g(\zeta, \zeta_0) - g_k(\zeta, \zeta_0) \right). \tag{6}$$

The above inequality has an interesting interpretation, if one extends the definition of capacity and reduced module to the elliptic case, which we will now do.

We first define the elliptic reduced module.

Definition. Let E be a planar domain. Let D_r be a disc of radius r in the metric λ_k. Let $m(E\backslash D_r)$ denote the module of the ring domain with respect to the family of curves separating the boundary. The 'elliptic reduced k-module' of E at a is

$$m_k(E, a) = \lim_{r \to 0} m(E\backslash D_r) + \frac{1}{2\pi} \log r.$$

For $k = 0$, this is just the ordinary reduced module (for which the convenient notation m_0 will be used.) If we allow k to be negative, this is the 'hyperbolic reduced module' defined by Barnard, Hadjicostas and Solynin [1]. In what follows we will deal only with the special case $k = 1$.

In the case that E is simply connected and we have a canonical mapping function from the disc, we can easily compute the elliptic reduced module.

Proposition 1. *If E is simply connected, and f is a conformal map from the unit disc onto E with $f(z_0) = a$, and $k > 0$, then*

$$m_1(E, a) = \frac{1}{2\pi} \log \frac{|f'(z_0)|(1 - |z_0|^2)}{1 + |f(z_0)|^2}.$$

Proof. Consider the inner conformal radius $R(E, a)$ of E at a, defined to be $|g'(0)|$ where g is a conformal mapping of the unit disc onto E with $g(0) = a$. The Euclidean reduced module and the inner conformal radius $R(E, a)$ are related by the following formula (as given in [6])

$$\frac{1}{2\pi} \log R(E, a) = m_0(E, a). \tag{7}$$

So in particular, letting $g(z) = f \circ T_1$ with

$$T_1(z) = \frac{z + z_0}{1 + \bar{z}_0 z},$$

we get

$$m_0(E, a) = \frac{1}{2\pi} \log |f'(z_0)|(1 - |z_0|^2). \tag{8}$$

Let

$$T(z) = \frac{z - a}{1 + \bar{a} z}.$$

By (7), the Euclidean reduced module transforms according to

$$\begin{aligned} m_0(T(E), 0) &= m_0(E, a) + \frac{1}{2\pi} \log |T'(a)| \\ &= m_0(E, a) - \frac{1}{2\pi} \log (1 + |a|^2). \end{aligned} \tag{9}$$

On the other hand, since T is an isometry of λ_1,

$$m_1(T(E), 0) = m_1(E, a). \tag{10}$$

Now λ_1-circles about the origin are Euclidean circles. Furthermore if t denotes the Euclidean radius and $r(t)$ denotes the λ_1-radius,

$$\lim_{r \to 0} \frac{r(t)}{t} = 1,$$

and so

$$m_1(T(E), 0) = m_0(T(E), 0). \tag{11}$$

Combining (8), (9), (10), and (11) results in the desired formula. ∎

Proposition 1 implies that

$$m_1(E, z_0) = \frac{1}{2\pi} \lim_{z \to z_0} [g(z, z_0) - g_1(z, z_0)],$$

which provides an interpretation for the right side of (6).

The left side of (6) relates to elliptic capacity. The notion of elliptic capacity is due to Duren and Kühnau [3]. We briefly describe the idea here. In the case of elliptically schlicht domains E, for a positive unit measure μ one considers the function

$$U(z) = \iint_E \log \frac{|1 + \bar{\zeta} z|}{|z - \zeta|} \, d\mu(\zeta),$$

and the energy integral

$$I(\mu) = \iint_E \iint_E \log \frac{|1 + \bar{\zeta} z|}{|z - \zeta|} \, d\mu(z) \, d\mu(\zeta).$$

Let $V(\mu) = \sup_{z \in E} U(z)$, and $V = \inf_\mu V(\mu)$. It can then be shown, as in the hyperbolic and Euclidean cases ([3], [9]), that the measure μ attaining V also attains $\inf_\mu I(\mu)$, and furthermore $V = I(\mu)$ on E except on a set of zero capacity. One can also show that this relates nicely to an elliptic transfinite diameter. One then defines the 'elliptic capacity' of E to be $C = e^{-V}$.

Duren and Kühnau also show that, quite generally, when one can speak of a domain Ω 'between' the domains E and E^*, that for the extremal distance $d(E, E^*)$ between E and E^*, measured from within the domain Ω,

$$d(E, E^*) = \frac{V}{\pi}.$$

In our case $\Omega = \mathbb{C}\backslash(E \cup E^*)$.

We now can interpret the inequality (6) as one relating elliptic k-capacity to elliptic reduced k-module:

$$-\frac{1}{\pi}\log C(E, z_0) \leq m_1(E, z_0).$$

The notion of elliptic capacity, as defined above, only applies to elliptically 1-schlicht functions. One can add the parameter k to Duren and Kühnau's definition in a natural way using a simple scaling argument. A similar inequality holds for the general case.

The hyperbolic case of this inequality, apparently due to Nehari, has a similar interpretation in terms of hyperbolic capacity and the 'hyperbolic reduced module' of Barnard, Hadjicostas and Solynin [1]. One can prove this exactly as above using results of Duren and Pfaltzgraff [4].

References

1. R. Barnard, P. Hadjicostas, and A. Yu. Solynin, The Poincaré metric and isoperimetric inequalities for hyperbolic polygons, preprint.
2. S. Bergman, and M. Schiffer, Kernel functions and conformal mapping, *Compositio Math.* **8** (1951), 205–249.
3. P. Duren, and R. Kühnau, Elliptic capacity and its distortion under conformal mapping, *J. Anal. Math.* **89** (2003), 317–335.
4. P. Duren, and J. Pfaltzgraff, Hyperbolic capacity and its distortion under conformal mapping, *J. Anal. Math.* **78** (1999), 205–218.
5. H. Grunsky, Einige Analoga zum Schwarzschen Lemma, *Math. Ann.* **108** (1933), 190–196.
6. J. A. Jenkins, *Univalent Functions and Conformal Mappings*, Ergebnisse der Mathematik und ihrer Grenzgebiete v 18, Springer, 1958.
7. ______, On Bieberbach-Eilenberg functions III, *Trans. Amer. Math. Soc.* **119** (1965), 195–215.
8. R. Kühnau, Über vier Klassen schlichter Funktionen, *Math. Nachr.* **50** (1971), 17–26.
9. ______, Transfiniter Durchmesser, Kapazität und Tschebyschewsche Konstante in der euklidischen, hyperbolischen und elliptischen Geometrie, *J. Reine. Angew. Math.* **234** (1969), 216–220.
10. D. Minda, unpublished notes.
11. Z. Nehari, Some inequalities in the theory of functions, *Trans. Amer. Math. Soc.* **75** (1953), 256–286.
12. C. Pommerenke, *Univalent Functions*, Vandenhoeck & Ruprecht, Göttingen 1975.

13. E. Schippers, Conformal invariants and higher-order Schwarz lemmas, *J. Anal. Math.* **90** (2003), 217–242.

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